TOPIC 1: ALGEBRA

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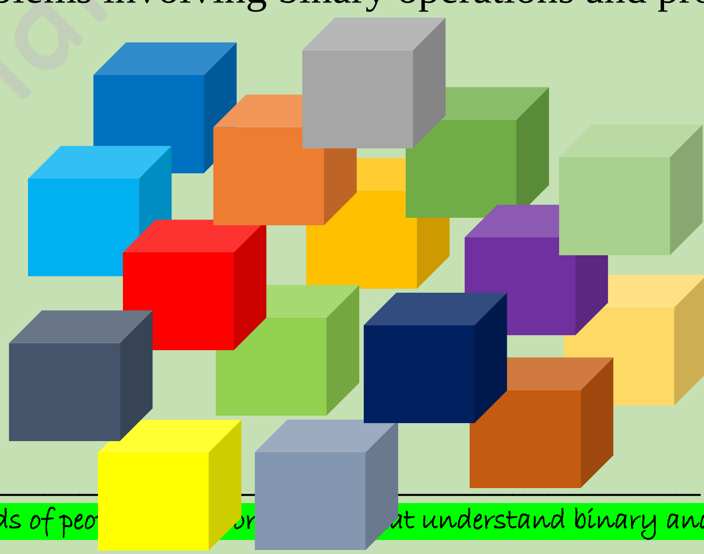
1.2 GROUPS

Learning Objectives

After completing this topic you should be able to:

- define a binary operation
- define closure, commutation, association, distribution, identity and inverse element
- define a group
- use the basic properties to show that a given structure is, or is not, a group
- solve problems involving binary operations and properties of a group



"There are 10 kinds of peo[illegible] or [illegible] at understand binary and the other 9" Jonathan Kariv

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1.2.1 Binary operations

- Binary means composed of two parts.
- A binary is a rule for combining two values to create a new value.
- Binary operation on a set is a calculation involving two elements of a set to produce another element of the set.
- A binary operation on set S is a function $*$ such that $a * b = y$ where a, b and y are elements of set S.
- The notation $(S,*)$ can be used to indicate that set S has a binary operation $*$.
- The symbol $*$ can be used to represent a law of binary composition.
- Basic binary operations are $\times;\ \div;\ +;\ -$.
- A binary operation on a finite set can be displayed on a Cayley table or Latin square which shows how operations are to be performed.

Properties of binary operation

(a) Closure

The set S is closed under a binary operation $*$ if, for every pair of elements a and b of set S, $a * b$ is also an element of set S.

The binary operation $*$ is said to be a closed binary operation on set S.

Example 1

Determine whether the following set is closed under the given law of binary operation.

(i) Integers, multiplication.

(ii) Natural numbers, subtraction.

Solution

(i) The set of integers is given by $\mathbb{Z} = \{\ldots -3; -2; -1; 0; 1; 2; 3; \ldots\}$

$(\mathbb{Z}; \text{x})$

It is clear that $\forall a, b \in \mathbb{Z}$, $a\text{x}b \in \mathbb{Z}$. E.g. if $a = -3$ $b = 4$, $-3\text{x}4 = -12 \in \mathbb{Z}$. Hence the set of integers is closed under multiplication.

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(ii) The set of natural numbers is given by $\mathbb{N} = \{1; 2; 3; \dots\}$

$(\mathbb{N}; -)$

We will use a counter example to determine this.
let $a, b \in \mathbb{N}$ where $a = 2$ and $b = 3$.

$2 - 3 = -1 \notin \mathbb{N}$.

Hence the set of natural numbers is not closed under subtraction.

Example 2

Determine whether the binary operation $*$ of multiplication is closed on set G, where $G = \{-1; 1; -i; i\}$ is a set of complex numbers of unit moduli.

Solution

Set G is a finite set and so we can use the following operation table to show all the possible outcomes.

x	-1	1	$-i$	i
-1	1	-1	i	$-i$
1	-1	1	$-i$	i
$-i$	i	$-i$	-1	1
i	$-i$	i	1	-1

Every element in the table is a member of the set G. Hence the binary operation $*$ is closed.

(b) Commutativity

A binary operation $*$ on a set S is said to be commutative if for all $a, b \in S$, the equality $a * b = b * a$ holds, ie, the order of the pair does not affect the result.

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Example 1

(i) Show that multiplication is commutative on the set $\mathbb{Z}$ of integers.

(ii) The operation $\circ$ on real numbers is defined by $a \circ \mathrm{b} = a|b|$.

Show that $\circ$ is not commutative.

(iii) A set $M = \{a, b, c, d\}$ under a binary operation $*$ has an operation table as follows:

$*$	a	b	c	d
a	c	d	a	b
b	d	c	b	a
c	a	b	c	d
d	b	a	d	c

Show that $*$ is commutative on the set M.

Solution

(i) $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

$(\mathbb{Z}; \mathrm{x})$

It is clear that $\forall a, b \in \mathbb{Z}\ a \text{ x } b = b \text{ x } a$. Eg if $a = -2$ and $b = 7$

$-2 \text{ x } 7 = 7 \text{ x } -2 = -14$.

Hence multiplication is commutative on the set $\mathbb{Z}$ of positive integers.

(ii) $a \circ \mathrm{b} = a|b|\ \forall a, b \in \mathbb{R}$

For commutativity, $a \circ \mathrm{b} = b \circ a$

LHS $a \circ \mathrm{b} = a|b|$ RHS $b \circ a = b|a|$

$a|b| \neq b|a|$ hence $\circ$ is not commutative.

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(iii)

$*$	a	b	c	d
a	c	d	a	b
b	d	c	b	a
c	a	b	c	d
d	b	a	d	c

The table is symmetrical about the main diagonal (as shown by the red line) hence $*$ is commutative on M.

(c) Associativity

The binary operation $*$ on a set S is said to be associative if for for every three elements $a, b, c \in S$ then $a * (b * c) = (a * b) * c$.

Example 1

The operation $\circ$ on real numbers is defined by $a \circ \mathrm{b} = a|b|$.

Determine whether $\circ$ is associative.

Solution

$a \circ \mathrm{b} = a|b|\ \forall a, b \in \mathbb{R}$

Let $a, b, c \in \mathbb{R}$.

We want to show that $a \circ (b \circ c) = (a \circ b) \circ c$

LHS $a \circ (b \circ \mathrm{c}) = a \circ (\mathrm{b}|\mathrm{c}|) = a\big|b|c|\big| = a|bc|$

RHS $(a \circ \mathrm{b}) \circ \mathrm{c} = (a|b|) \circ \mathrm{c} = a|b||c| = a|bc|$

LHS = RHS hence $\circ$ is associative.

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Example 2

Given that $*$ is a binary operation on the set $\mathbb{Z}$, of integers defined by

$a * b = a + b + ab$

(i) Evaluate $3 * 4$

(ii) Prove that $*$ is both commutative and associative and $\mathbb{Z}$.

Solution

(i) $3 * 4 = 3 + 4 + 3(4) = 19$

(ii) Commutativity: to prove that $a * b = b * a \ \forall a, b \in \mathbb{Z}$

LHS $\Longrightarrow a * b = a + b + ab$

RHS $\Longrightarrow b * a = b + a + ba$

$= a + b + ab.$

LHS=RHS Hence $*$ is commutative

Associativity: to prove that $\forall a, b, c \in \mathbb{Z}, a * (b * c) = (a * b) * c$

Now, $b * c = b + c + bc$

LHS $\Longrightarrow a * (b * c) = a * (b + c + bc) = a + b + c + bc + ab + ac + abc$

Also,

RHS$\Longrightarrow (a * b) * c = (a + b + ab) * c = a + b + ab + c + ac + bc + abc$

$= a + b + c + bc + ab + ac + abc$

LHS=RHS Hence $*$ is associative

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(d) Identity element (Neutral element)

Given a set S defined on a binary operation $*$, the identity element (if it exists) is a neutral element e in S such that for all $a \in S$, $a * e = e * a = a$.

The identity element, if it exists, is unique. That is to say, the set S has only one identity element.

The identity element when multiplying real numbers is 1.

For any real number a,

$1\text{x}a = a\text{x}1 = a$

The identity element when multiplying 2x2 matrices is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

For any 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Zero (0) is the identity element when adding real numbers.

For any real number a,

$0 + a = a + 0 = a$

Example 1

The binary operation $*$ is defined on set $\mathbb{R}$ by $m * n = m + n + 6$ for all $m, n \in \mathbb{R}$. Find the identity element of $\mathbb{R}$ under the binary operation $*$.

Solution

$m * n = m + n + 6 \;\forall\; m, n \in \mathbb{R}.$

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We want to find an element $e \in \mathbb{R}$, which satisfies $m * e = e * m = m$.

Taking the equation $m * e = m$

$$\Longrightarrow m + e + 6 = m$$

$$\Longrightarrow e = m - m - 6$$

$$\Longrightarrow e = -6$$

Thus the identity element is -6.

Example 2

The binary operation $*$ is defined on set M of positive numbers is given $a * b = \frac{a+b}{3ab}$, for all $a, b \in M$. Find, if it exists, the identity element of M under the binary operation $*$.

Solution

$a * b = \frac{a+b}{3ab}$ for all $a, b \in M$

Looking for an element e of M which satisfies $e * a = a * e = a$

Taking the equation $e * a = a$

$$\Longrightarrow \frac{e+a}{3ea} = a$$

$$\Longrightarrow e + a = 3ea^2$$

$$\Longrightarrow 3ea^2 - e = a$$

$$\Longrightarrow e(3a^2 - 1) = a$$

$$\Longrightarrow e = \frac{a}{3a^2-1}$$

$\frac{a}{3a^2-1}$ can not be an identity element since the identity is a unique element.

Thus M does not have an identity element.

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Example 3

The binary operation $*$ is defined on set $\mathbb{R}$ by $p * q = p + q - 4p^2q^2$ for all $p, q \in \mathbb{R}$. Find the identity element of $\mathbb{R}$ under the binary operation $*$.

Solution

Looking for an element e of $\mathbb{R}$ which satisfies $e * a = a * e = a$ for all $a \in \mathbb{R}$

Taking $a * e = a$

$\Longrightarrow a + e - 4a^2e^2 = a$

$\Longrightarrow e - 4a^2e^2 = 0$

$\Longrightarrow e(1 - 4a^2e) = 0$

$\Longrightarrow e = 0$ or $\frac{1}{4a^2}$

The identity element is unique if it exists. $e = 0$ is the identity element under $*$.

Example 4

The binary operation $*$ is defined on the set R of ordered pairs of real numbers by $(x, y) * (p, q) = (x + p, yq)$ for all $x, y, p, q \in \mathbb{R}$.

(a) Show that

(i) R is commutative under the binary operation $*$.

(ii) R is associative under the binary operation $*$.

(b) Find the identity element of R under the binary operation $*$.

Solution

(a) (i) Let $x, y, p, q \in \mathbb{R}$

For commutativity we want to prove that $(x, y) * (p, q) = (p, q) * (x, y)$

LHS $(x, y) * (p, q) = (x + p, yq)$

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RHS $(p,q)*(x,y)=(p+x,qy)=(x+p,yq)$

LHS = RHS hence R is commutative under the binary operation $*$..

(ii) Let $x,y,p,q,m,n\in\mathbb{R}$

For associativity we want to prove that
$(x,y)*[(p,q)*(m,n)]=[(x,y)*(p,q)]*(m,n)$

Now $(p,q)*(m,n)=(p+m,qn)$

$\therefore$ LHS $(x,y)*[(p,q)*(m,n)]=(x,y)*(p+m,qn)$

$=(x+p+m,yqn)$

RHS $[(x,y)*(p,q)]*(m,n)=(x+p,yq)*(m,n)$

$=(x+p+m,yqn)$

LHS = RHS hence R is associative with respect to the binary operation $*$.

(b) Let $x,y,a,b\in\mathbb{R}$

We want to find an identity element (a,b) such that

$(x,y)*(a,b)=(a,b)*(x,y)=(x,y)$

Taking the equation $(x,y)*(a,b)=(x,y)$

$\Longrightarrow (x+a,yb)=(x,y)$

Thus $x+a=x$ and $yb=y$

$\Longrightarrow a=0$ and $y=y$

Hence the identity element is $(0,y)$

(e) Inverse Element

Given a set S defined under a binary operation $*$. For each element a in S, there exists a unique inverse for each element, a^{-1}, which is such that $a*a^{-1}=a^{-1}*a=e$ where $a^{-1},e\in S$. e is the identity element.

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The inverse of an element, if it exists, under given binary operation is unique for each element.

Example 1

A binary operation $*$ is defined on the set of real numbers, $\mathbb{R}$, by $m * n = m + n + 2mn$, for all $m, n \in \mathbb{R}$. Find

(i) the identity element e of $\mathbb{R}$ under $*$.

(ii) the inverse x^{-1} of x, where $x \in \mathbb{R}$.

(iii) for what value of x is x^{-1} not defined?

Solution

(i) Looking for an $e \in \mathbb{R}$ such that $m * e = e * m = m$ for all $m \in \mathbb{R}$

Taking the equation $m * e = m$

$$\Longrightarrow m + e + 2me = m$$

$$\Longrightarrow e(1 + 2m) = 0$$

$$\Longrightarrow e = 0$$

(ii) x^{-1}, the inverse of x satisfies $x * x^{-1} = x^{-1} * x = e$ where $e = 0$.

Taking the equation $x * x^{-1} = 0$

$$\Longrightarrow x + x^{-1} + 2xx^{-1} = 0$$

$$\Longrightarrow x^{-1} + 2xx^{-1} = -x$$

$$\Longrightarrow x^{-1}(1 + 2x) = -x$$

$$\Longrightarrow x^{-1} = \frac{-x}{1+2x}$$

(iii) x^{-1} is undefined when the denominator is zero that is when $x = -\frac{1}{2}$.

Example 2

The binary operation $*$ is defined on the set $\mathbb{R}$ of real numbers by $m * n = m + n + 3$, for all $m, n \in \mathbb{R}$.

(i) Find the inverse of a under the binary operation $*$ of real numbers.

(ii) Find the inverse of 4.

Solution

(i) We first find the identity element e under $*$

Let $m, e \in \mathbb{R}$.

$m * e = e * m = m$

Taking $m * e = m$

$\Longrightarrow m + e + 3 = m$

$\Longrightarrow e = -3$

We now find a real number a^{-1}such that $a * a^{-1} = a^{-1} * a = -3$

Taking $a^{-1} * a = 3$

$\Longrightarrow a^{-1} + a + 3 = -3$

$\Longrightarrow a^{-1} = -a - 6$

(ii) $4^{-1} = -4 - 6 = -10$

Modular Arithmetic

Modular arithmetic is a system of arithmetic for integers, which considers the remainder after dividing two integers.

In general:

If $a \div n = q + R$ where q is the quotient and R is the remainder, we say that $a \text{ mod } n = R$ where $0 \leq R < n$. The focus is on the remainder after dividing a by n.

For instance $18 mod 5 = 3$, $21 mod 4 = 1$ and $16 mod 4 = 0$

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Example 3

The set $G = \{0; 1; 2; 3; 4; 5\}$, is defined under the binary operation $*$ of addition (mod 6). Complete a Cayley table for the operation. State the identity element for set and the inverse for each element. Determine whether $*$ is associative.

Solution

$+_6$	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Clearly, the identity element is 0.

Element	0	1	2	3	4	5
Inverse	0	5	4	3	2	1

For associativity, consider any three elements, 0, 1 and 2 say.

$0 * (1 * 2) = (0 * 1) * 2 = 3$ Hence in general, $*$ is associative.

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Example 4

The table below defines the binary operation $*$ on the set M, where $M = \{a, b, c, d\}$.

(a) Find, giving reasons, whether or not

(i) M is closed with respect to $*$.

(ii) the operation $*$ is commutative.

(iii) There is an identity element.

(b) Find, where possible, the inverse of each element a, b, c and d.

$*$	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

Solution

(a) (i) M is closed under $*$ because all outcomes in the table are elements of set M.
(ii) The table is symmetrical about the main diagonal, hence $*$ is commutative.
(ii) c is the identity element because for each element x in M, $x * c = c * x = x$.
(b)

Element	a	b	c	d
Inverse	d	b	c	a

(f) Distributive

The distributive property uses two binary operations on the same set.
If the binary operations $*$ and $\otimes$ are defined on the set S such that;

$a * (b \otimes c) = (a * b) \otimes (a * c)$.

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Then the operation $*$ is said to be distributive over the operation $\otimes$.

The distributive property is simple. It is just a matter of one operation having authority over another operation.
If an operation overrules another operation, we say the overruling operation is distributive over the other.

Example 1

The binary operations of addition " + " and multiplication "x" are defined on the set $\mathbb{R}$.

Determine whether or not

(a) "x" is distributive over " + ",

(b) " + " is distributive over "x".

Solution

(a) If $a, b, c \in \mathbb{R}$ $\quad a \text{ x}(b + c) = (a \text{ x } b) + (a \text{ x } c)$, hence "x" is distributive over " + ".

(b) If $a, b, c \in \mathbb{R}$ $\quad a$ +(b x c)$\neq$ $(a + b)\text{x}(a + c)$, hence "+" is not distributive over "x".

Example 2

The binary operations $*$ and $\otimes$ are defined on the set $\mathbb{R}$ by $x * y = x^2 - y^2$ and $x \otimes y = x + 2y$, for all $x, y \in \mathbb{R}$.

(a) Evaluate (i) $1 * (2 \otimes 3)$ (ii) $(1 * 2) \otimes (1 * 3)$

(b) Comment on your results for part (i) and (ii)?

Solution

(a) (i) $1 * (2 \otimes 3)$

$2 \otimes 3 = 2 + 2(3) = 8$

$\therefore 1 * (2 \otimes 3) = 1 * 8 = 1^2 - 8^2 = -63$

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(ii) $(1*2)\otimes(1*3) = (1^2 - 2^2)\otimes(1^2 - 3^2)$

$= -3 \otimes -8$

$= -3 + 2(-8)$

$= -19$

(b) $1*(2\otimes 3) \neq (1*2)\otimes(1*3)$ hence $*$ is not distributive over $\otimes$.

Exercise 1.2.1

1 Determine whether the binary operation on $\mathbb{R}$ defined by $a * b = a + b$ is closed, associative and commutative. Find the identity element and the inverse of each element.

2 Find out whether the binary operation on the set of rational numbers $\mathbb{Q}$ defined by $a * b = \frac{a}{b}$ is closed, associative or commutative.

3 Prove that the operation $*$ defined by $a * b = a^b$ on the set $\mathbb{N}$ is not associative or commutative. You may use counterexamples.

4 Prove that the binary operation on $\mathbb{R}$ defined by $a * b = a + b - 1$ is both associative and commutative.

5 Show that the binary operation on $\mathbb{R}$ defined by $a * b = 1 + ab$ is commutative but not associative.

6 Find the identity element of the operation $a * b = a + b - 1$ on the set of integers, $\mathbb{Z}$.

7 Show that the operation $a * b = 1 + ab$ on the set of integers $\mathbb{Z}$ has no identity element.

8 Consider the operation $*$ on the set of integers defined by $a * b = a + b - 1$. Show that each integer has an inverse under this operation.

9 If $*$is defined on $\mathbb{Z}$ by $x * y = x + y + 1$. Find the identity element and the inverse of each element.

10 The operation $*$ is defined on the elements (a, b), where $a, b \in \mathbb{R}$ by

$$(a, b) * (c, d) = (ac, ad + b).$$

It is given that the identity element is $(1,0)$.

(a) Prove that $*$ is associative.

(b) Find all the elements which commute with $(1,1)$.

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(c) It is given that the particular element (m,n) has an inverse denoted by (p,q), where $(m,n) * (p,q) = (p,q) * (m,n) = (1,0)$

Find (p,q) in terms of m and n.

(d) Find all self-inverse elements.

11 (a) The operation $*$ is defined by $x * y = x + y - a$, where x and y are real numbers and a is a real constant.

(i) Prove that the set of real numbers under the operation $*$ is closed and associative.

(ii) Find the identity and inverse element.

(iii) Determine whether the set of real numbers is commutative under the given binary operation.

(b) The binary operation $\circ$ given by $x \circ y = x + y - 5$ is defined over a set of **positive** real numbers. Show, by giving a counterexample or otherwise, that the properties of closure and inverse are not satisfied.

1.2.2 Groups

A group $(G;*)$ is a non-empty set G with binary operation $*$ satisfying the following basic properties(axioms):

(i) Closure – For every pair of elements a and b in set G then $a * b \in G$.

(ii) Associativity-For any three elements a, b and c in G, the equality $a * (b * c) = (a * b) * c$ must hold.

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(iii) Identity element-There is a group element e in G, the identity element, such that $e * a = a * e = a$ for all $a \in G$.

(iv) Inverse - Every element a in G has an inverse, a^{-1}, such that $a * a^{-1} = a^{-1} * a = e$ where a, a^{-1} and e are elements of set G.

Abelian group

A group (G;$*$) which satisfies the commutative property is called an **Abelian group**.
The commutative property says $\forall a, b \in G$ then $a * b = b * a$.
Check for commutative in order to determine whether the given structure is an Abelian group or not. All basic properties of a group are still valid.

Subgroup

A subset H of a group G is a subgroup of G if H is itself a group under the operation in G.

Note: Every group G has at least two subgroups: G itself and the subgroup $\{e\}$, containing only the identity element. All other subgroups are said to be **proper** subgroups.

Definition. Let $(G; *)$ be a group. A subset H of G is called a subgroup if it satisfies the following conditions:

(i) $e \in H$

(ii) H is closed under $*$, that is if $x, y \in H$, then $x * y \in H$

(iii) H is closed under inversion, that is if $\in H$, then $x^{-1} \in$ H

Example 1

The composition table for a group G of order 8 is given below

	a	b	c	d	e	f	g	h
a	c	e	b	f	a	h	d	g
b	e	c	a	g	b	d	h	f
c	b	a	e	h	c	g	f	d
d	f	g	h	a	d	c	e	b
e	a	b	c	d	e	f	g	h
f	h	d	g	c	f	b	a	e
g	d	h	f	e	g	a	b	c
h	g	f	d	b	h	e	c	a

State which is the identity element and give the inverse of each element of G.

Solution

The identity element is e.

Element	a	b	c	d	e	f	g	h
Inverse	b	a	c	g	e	h	d	f

Example 2

Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40.

Solution

	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35
35	15	5	35	25

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Closure

All elements in the table are members of the set $\{5, 15, 25, 35\}$, hence the set is closed under multiplication modulo 40.

Associativity

Since we have $a(bc) = a(bc)$ for every a, b, c in $\mathbb{Z}$, we get $a(bc) = (ab)c$ for every a, b, c in $\{5, 15, 25, 35\}$. Hence associativity is satisfied.

Identity

25 is the identity element. Observe that $25 \cdot a = a \cdot 25 = a$ for all a in $\{5, 15, 25, 35\}$.

Inverse

Each element is its own inverse.

Since the set $\{5, 15, 25, 35\}$ with multiplication modulo 40 satisfies all requirements given in the definition of a group, $\{5, 15, 25, 35\}$ is a group under the multiplication modulo 40.

Example 3

(a) Show that the set $S = \{0, 1, 2, 3\}$ forms a group under addition (mod 4).

(b) Show that the set S in (a) is an Abelian group.

(c) State all subgroups of S.

Solution

(a)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Line of symmetry

Closure

All outcomes in the table are members of the set S, hence S is closed under addition (mod4).

Associativity

The set S is associative since for any three elements $a, b, c \in S$, $a + (b + c) = (a + b) + c$

Identity

0 is the identity element

Inverse

Every element has an inverse

Element	0	1	2	3
Inverse	0	3	2	1

Since the set S satisfies all the 4 group axioms, set S forms a group under addition (mod4).

(b) The table is symmetrical about the main diagonal hence the set S is commutative under addition (mod 4). Thus set S is an Abelian group.

(c) Subgroups are $\{0\}$, $\{0,1,2,3\}$, $\{0, 2\}$.

Notice that the sets $\{0, 1\}$, $\{0, 3\}$, $\{0, 1, 2\}$,$\{0, 2, 3\}$ *and* $\{0, 1, 3\}$ *and cannot be subgroups of S because each of these sets violet either closure or inverse property.*

Example 4

Let G be the set of all 3x3 real matrices of the form $\begin{bmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix}$ where $p, q, r \in \mathbb{R}$. Prove that G forms a group under matrix multiplication.

Solution

Closure

Let $a, b, c, d, e, f \in \mathbb{R}$. Then $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+d & e+af+b \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix} \in G$

Hence the set G is closed under matrix multiplication.

Associativity

For any three matrices A, B and C in G, $(AB)C = A(BC)$ (matrix multiplication is associative).

Identity

The identity 3 x 3 matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is an element of G.

Inverse

For all $A \epsilon G$ such that $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \in G$

Thus every element has an inverse

Since G satisfies all the properties of a group, G forms a group under matrix multiplication.

"There are 10 kinds of people in the world, those that understand binary and the other 9" Jonathan Kariv

Example 4

Let H be the set of all matrices of the form $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$, where $y \in \mathbb{R}$.

Show that

(i) H does not form a group under matrix addition.

(ii) H forms an abelian group under matrix multiplication. [Assume associativity]

(Zimsec Specimen paper 2018)

Solution

Closure

(i) Let y_1 and y_2 be real numbers.

$\begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & y_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & y_1 + y_2 \\ 0 & 2 \end{bmatrix} \notin$ H since 2 is a new entry.

Hence H is not closed under matrix addition.

Thus H does not form a group under matrix addition.

(ii) **Closure**

Let y_1 and y_2 be real numbers.

$\begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y_1 + y_2 \\ 0 & 1 \end{bmatrix} \in$ H, $y_1 + y_2 \in \mathbb{R}$

Hence H is closed under matrix multiplication.

Assuming associativity.

Identity

The identity element is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since

$\begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix} \in$ H, where $y_1 \in \mathbb{R}$

Inverse

The inverse element is $\begin{bmatrix}1 & -y_1\\ 0 & 1\end{bmatrix}$ since

$$\begin{bmatrix}1 & y_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -y_1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -y_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & y_1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Commutativity

For commutativity, we want to show that $a * b = b * a$

Let $y_1, y_2 \in \mathbb{R}$

LHS: $a * b = \begin{bmatrix}1 & y_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & y_2\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & y_2 + y_1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & y_1 + y_2\\ 0 & 1\end{bmatrix}$

RHS: $b * a = \begin{bmatrix}1 & y_2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & y_1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & y_1 + y_2\\ 0 & 1\end{bmatrix}$

LHS=RHS
Commutativity is satisfied hence H forms an Abelian group under matrix multiplication.

Example

To prove that the set S of all 2x2 invertible matrices with real entries forms a group under matrix multiplication.

Solution

Closure

Let $a,b,c,d,e,f,g,h \in \mathbb{R}$

$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}e & f\\ g & h\end{pmatrix} = \begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix} \in S$ Hence S is closed under matrix multiplication

Associativity

For any three invertible 2x2 matrices A, B and C, it is known that A(BC) = (AB)C, thus the set S is associative with respect to multiplication.

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Identity

The identity element when multiplying 2x2 matrices is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \forall a, b, c, d \in \mathbb{R}$$

Inverse

Every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of S has inverse $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$= \begin{pmatrix} \frac{d}{ad-bc} & \frac{b}{bc-ad} \\ \frac{c}{bc-ad} & \frac{a}{ad-bc} \end{pmatrix} \in S$$

Since the set S satisfies all the four group properties, it forms a group under matrix multiplication.

Commutativity

Let $a, b, c, d, e, f, g, h \in \mathbb{R}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \neq \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Hence the group is not commutative.

Exercise 1.2.2

1 A binary operation $\oplus$ is defined on the set $\mathbb{R}$ of real numbers by $x \oplus y = x - 5xy + y$. Determine if $\mathbb{R}$ forms a group under $\oplus$.

2 A set $G = \{i, -i, 1, -1\}$ is for complex numbers of unity moduli

(i) Construct a multiplication table for the elements.

(ii) Show that the set G forms a group under multiplication.

(iii) Explain briefly how the table shows that G is an Abelian group under multiplication

3 The set M consists of matrices of the form $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ where $n \in \mathbb{R}$. Determine whether $(M, *)$ forms a commutative group under matrix multiplication.

4 The elements of a group G are complex numbers $a + bi$ where $a, b \in \{0,1,2,3,4\}$. These elements are combined under the operation of addition modulo 5.

(a) State the identity element of G.

(b) Write down the inverse of $2 + 4i$.

5 The function f is defined by $f: x \mapsto \frac{1}{2-2x}$ for $x \in \mathbb{R}, x \neq 0, x \neq \frac{1}{2}, x \neq 1$. The function g is defined by $g(x) = ff(x)$.

(a) Show that $g(x) = \frac{1-x}{1-2x}$ and that $gg(x) = x$.

It is given that f and g are elements of a group K under the operation of composition of functions.

The element e is the identity, where $e: x \mapsto x$ for $x \in \mathbb{R}, x \neq 0, x \neq \frac{1}{2}, x \neq 1$.

(b) The inverse of the element f is denoted by h. Find $h(x)$.

(c) Construct the operation table for the elements e, f, g and h of the group K.

Miscellaneous Exercise 1.2

1 A binary operation $*$ is defined on the set $Q = \{a, b, c\}$ by the table below

$*$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$\boldsymbol{a}$	b	a	c
$\boldsymbol{b}$	a	b	c
$\boldsymbol{c}$	c	c	b

Determine (a) whether or not Q is closed with respect to $*$.

(b) whether the operation is commutative,

(c) the identity element e, under the operation $*$,

(d) the inverse of each element.

2 Determine whether the given set S is, or is not a group under the given law of binary operation.

(a) Natural numbers: Multiplication

(b) Integers: addition

(c) Non-singular 2 x 2 matrices, Multiplication.

3 The elements of the set $P = \{1, 3, 9, 11\}$ are combined under the binary operation, $*$, defined as multiplication modulo 16.

(a) Demonstrate associativity for the elements $3, 9, 11$ in that order.

Assuming associativity holds in general, show that P forms a group under the binary operation $*$.

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(b) Write down all subgroups of P.

4 Define Q^+as the set of positive rational numbers. Prove that Q^+ forms an abelian group under the composition $*$ given as $a * b = \frac{a}{10}$, where $a, b \in Q^+$.

5 The binary operation $*$ is defined on the set $\mathbb{R}$ by $a * b = a + b + 4a^2b^2 \ \forall a, b \in \mathbb{R}$

Determine whether $*$ is (i) commutative, associative,

(ii) distributive over ordinary addition "+" of real numbers.

6 Consider the four matrices $\mathbf{E}, \mathbf{F}, \mathbf{G}$ and $\mathbf{H}$ where

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

Show that the set $\{\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}\}$ forms a group under the binary operation matrix multiplication.

State with a reason whether the set $\{\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}\}$ is an Albelian group.

7 A binary operation $*$ is defined on the set T= {0,1,2,3,4,5}

by $a * b = a + b - ab$, for all $a, b \in$ T on the table.
If $*$ denotes addition modulo 6,

$*$	**0**	**1**	**2**	**3**	**4**	**5**
0						
1						
2						
3						
4						
5						

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(i) copy and complete the table above.

(ii) Is the operation $*$ closed on T?

(iii) Find the identity element e under $*$.

Find the inverse of the element 2

8 A binary operation $\diamond$ is defined on a set M= $\{(p,q)$ such that $p,q \in \mathbb{R}\}$ by

$(p,q) \diamond (m,n) = (pm + 2qn, 2pq - q)$, for $m,n \in \mathbb{R}$.

(a) Find (i) $(2,3) \diamond (-4,1)$ (ii) $(-4,1) \diamond (2,3)$

What conclusion can you draw from the result of (i) and (ii).

(b) Find the values of x and y if $(x,1) \diamond (2,y) = (1,1) \diamond (2,1)$.

9 M is the set os all 2x2 matrices of the form $\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$, $a \neq 0$, where a and b are rational numbers.

(a) Show that under matrix multiplication M is a group. You may assume associativity of matrix multiplication.

(b) Determine whether the group is commutative.

10 The binary operations $*$ and $\otimes$ are defined on the set $\mathbb{R}$ of real numbers by

$a * b = a + 2b + 4, \forall a, b \in \mathbb{R}$ and $a \otimes b = a + 3b - ab, \forall a, b \in \mathbb{R}$

Find the value of

(a) $2 * (3 \otimes 5)$ (b) $(2 * 3) \otimes (2 * 5)$

(c) What conclusion can you draw from the result of (a) and (b).

11 Show that the set $G = \{1, 5, 7, 11\}$ under the binary operation of multiplication modulo 12, is a group. State whether or not the set is an Abelian group.

12 A binary operation $*$ is defined on the set $\mathbb{R}$, of real numbers by $p * q = pq - 3\sqrt{3}$, where $p, q \in \mathbb{R}$.

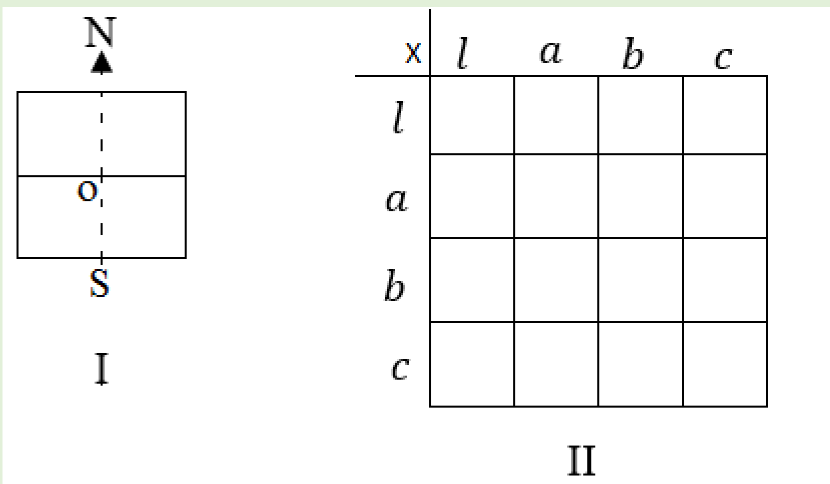
(a) Evaluate $(\sqrt{3}+1) * (\sqrt{3}-1)$

(b) Find (i) the identity element e of $\mathbb{R}$ under the operation $*$,

(ii) the inverse p^{-1} of p,

(iii) The inverse of $3\sqrt{3}$.

13



(a) The operations below are performed on the square above to produce elements of the set $= \{l, a, b, c\}$, where

$l =$ leave the square where it is,

$a =$ rotate it through 90° anticlockwise about O,

$b =$ rotate it through 180° anticlockwise about O,

$c =$ rotate it through 90° clockwise about O.

The x sign means "followed by" binary operation on the set S elements.

Make use of square I and the meanings of the elements, to complete table II. Copy and fill up the table above. State the inverse of a.

(b) The set G = {0, 1, 2, 3}. Show that the set G forms a group under the operation of addition modulo 4.

14 A binary operation $\odot$ defined on a set $Q = \{2,3,4,5\}$ has operation table shown.

$\odot$	**2**	**3**	**4**	**5**
2	2	3	4	5
3	3	4	5	2
4	4	5	1	3
5	5	1	3	4

(a) Determine whether or not

(i) Q is closed with respect to the binary operation $\odot$,

(ii) the operation $\odot$ is commutative.

(b) What is the identity element e, under the operation $\odot$.

15 Find the matrices for the clockwise rotations about the origin through the angles $0°, 90°, 180°, 270°$.

By constructing a table of compositions show that this set denoted by M forms a group under matrix multiplication.

16 The binary operation $*$ is defined on the set $\mathbb{R}$ of real numbers by $a * b = a + b + 3ab, \forall a, b \in \mathbb{R}$.

Find

(a) $3a * 5b$ (b) $(x^2 + x + 1) * x$ (c) $(a * b) * c$ (d) $a * (b * c)$

(e) what can you conclude about your answers for part (c) and (d)?

17 Show that the set $P = \{1, 3, 4, 5, 9\}$ under the binary operation of multiplication modulo 11, is a group.

18 A binary operation $*$ is defined on the set $\mathbb{R}$ of real numbers by $p * q = p + q - 3pq$ where $p, q \in \mathbb{R}$.

Find (a) the identity element e under the operation (b) the inverse of an element $x \in \mathbb{R}$

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19 A set G = {0,1,2,3,4}, where the elements are defined from the turning of a minute hand of a clock as follows:

0 = a turn through 0 minutes or multiples of 60 minutes

1 = turn through 12 minutes

2 = turn through 24 minutes

3 = turn through 36 minutes

4 = turn through 48 minutes

Defining $a * b$ as "a turn of the element a followed by a turn of the element b".

(i) Construct a multiplication table for all the elements of the set G.

(ii) Show that the set G is a group.

(iii) Establish whether the set G forms an Abelian group. Give a reason for your answer.

20 The binary operation $*$ is defined on the set $\mathbb{R}$ of real numbers by $a * b = a + ab^2 + 3, \forall a, b \in \mathbb{R}$.

Find (a) $(5 * 3) * 7$ (b) $5 * (3 * 7)$

(c) What conclusion can you draw from the results of (a) and (b).

21 Consider the set $H = \{1, 3, 5, 7\}$ under the binary operation of multiplication modulo 8.

(i) Construct the operation table and, assuming associativity, show that H forms a group.

(ii) Find all the proper subgroups of G.

22 The binary operation $*$ is defined on the set $\mathbb{R}$ of real numbers by $a * b = \frac{a+b}{2ab}$. $\forall a, b \in \mathbb{R}$. Find

(a) $a * (b * c)$ (b) $(a * b) * c$ (c) $(2x + 1) * (3x + 2)$ (d) $\frac{3}{3a*5b}$

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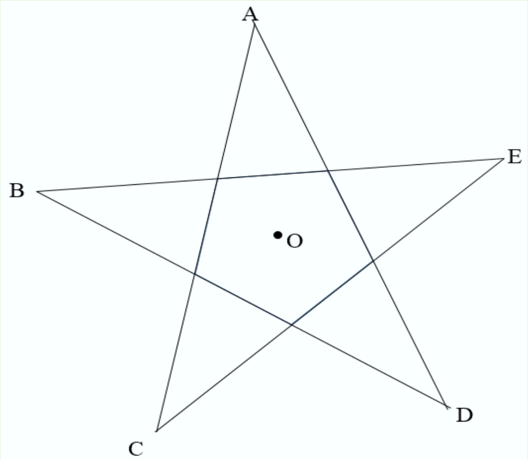
23



The diagram shows a regular pentagon ABCDE with centre O. Five transforms of the figure are defined as follows:

e is the volentity transformation "do nothing".

r_1, r_2, r_3, r_4 are anticlockwise rotations about O through $\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$, respectively.

Given that the five transformations form a group, construct the corresponding group table. State the inverse of each transformation.

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