TOPIC 1: ALGEBRA Prepared by G. Manuwere

1.2 MATHEMATICAL INDUCTION

Learning Objectives

After completing this section you should be able to:

- describe the process of mathematical induction
- prove by mathematical induction to establish a given result
- use the strategy of conducting limited trials to formulate a conjecture and proving it by the method of induction

The Principle of Mathematical Induction

The principle of *mathematical induction* is an important property of the positive integers. It is especially useful in proving statements involving all positive integers when it is known, for example, that the statements are valid for $n = 1, 2, 3$ but it is *suspected* or *conjectured* that they hold for all positive integers. The method of proof consists of the following steps:

1. Prove the statement for $n = 1$ or some other positive integer. (Initial step)

2. Assume the statement is true for $n = k$, where k is any positive integer. (Inductive Hypothesis/Assumption)

3. From the assumption in 2, prove that the statement must be true for $n = k + 1$. (Proof stage)

4. Since the statement is true for $n = 1$, and if it is true for $n = k$ implies it is true for $n = k + 1$, it must be true for all positive integers of n. (Conclusion)

<u>SUMMATIONS</u>

Example 1

Prove by Mathematical Induction that

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ for all natural values of n.

Solution

For $n = 1$

LHS $\sum_{r=1}^{k} r^2 = 1^2 = 1$

RHS $\frac{1}{6}(1)(1+1)(2(1)+1) = 1$

$\therefore$ LHS = RHS , hence true for $n = 1$

Assuming that the formula is true for $n = k$ ie $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$

We use this to show that the result is true for $n = k + 1$, ie to show that
$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+1+1)(2(k+1)+1)$

Proof

Taking the LHS

$$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^{th}\ term$$

$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

$$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$$

$$= \frac{1}{6}(k+1)[2k^2 + 7k + 6]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \frac{1}{6}(k+1)(k+1+1)(2(k+1)+1)$$
$$= \text{RHS}$$

$\therefore$ true for $n = k + 1$

Since the formula is true for $n = 1$, and if true for $n = k$ implies it is true for $n = k + 1$, it follows for the Principle of Mathematical Induction that it is true for all natural values of n.

Example 2
Prove by induction that $1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$, for $n \in \mathbb{N}$.

Solution

For $n = 1$: LHS: $1^2 = 1$

RHS: $\frac{1}{3}(1)(4(1)^2-1)=\frac{1}{3}(3)=1,$
LHS = RHS hence true for $n=1$.

Assume the result is true for $n=k$, $k\in\mathbb{N}$, ie assume that

$$1^2+3^2+5^2+7^2+\cdots+(2k-1)^2=\frac{1}{3}k(4k^2-1)$$

We want to prove that the result is true for $n=k+1$ ie we will prove that

$$1^2+3^2+5^2+7^2+\cdots+(2(k+1)-1)^2=\frac{1}{3}(k+1)(4(k+1)^2-1)$$

Proof

$$1^2+3^2+5^2+7^2+\cdots+(2(k+1)-1)^2=1^2+3^2+5^2+7^2+\cdots+(2k-1)^2+(k+1)^{th}\ term$$

$$=\frac{1}{3}k(4k^2-1)+(2(k+1)-1)^2$$
$$=\frac{1}{3}k(4k^2-1)+(2k+1)^2$$
$$=\frac{1}{3}k(2k+1)(2k-1)+(2k+1)^2$$
$$=\frac{1}{3}(2k+1)[k(2k-1)+3(2k+1)]$$
$$=\frac{1}{3}(2k+1)(2k^2+5k+3)$$
$$=\frac{1}{3}(2k+1)(2k+3)(k+1)$$
$$=\frac{1}{3}(k+1)(2k+1)(2k+3)$$
$$=\frac{1}{3}(k+1)(4k^2+8k+3)$$
$$=\frac{1}{3}(k+1)(4k^2+8k+4-1)$$
$$=\frac{1}{3}(k+1)(4(k^2+2k+1)-1)$$
$$=\frac{1}{3}(k+1)(4(k+1)^2-1)$$

Hence true for $n=k+1$

Since the result is true for $n=1$, $n=k$ and $n=k+1$, it should be true for all $n\in\mathbb{N}$.

Example 3

Prove by Mathematical Induction that $\sum_{r=1}^{n} \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}$

for $n \geq 1, n \in \mathbb{N}$.

Solution

For $n = 1$ LHS $\frac{2(1)^2-1}{1^2(1+1)^2} = \frac{1}{4}$

RHS $\frac{1^2}{(1+1)^2} = \frac{1}{4}$

LHS = RHS, hence true for $n = 1$.

Assuming that $\sum_{r=1}^{k} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2}$ for $k \geq 1, k \in \mathbb{N}$.

We want to show that $\sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+1+1)^2}$

Proof

$$\sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2} + (k+1)^{th}\ term$$

$$= \frac{k^2}{(k+1)^2} + \frac{2(k+1)^2-1}{(k+1)^2(k+1+1)^2}$$

$$= \frac{k^2}{(k+1)^2} + \frac{2k^2+4k+1}{(k+1)^2(k+2)^2}$$

$$= \frac{k^2(k+2)^2+2k^2+4k+1}{(k+1)^2(k+2)^2}$$

$$= \frac{k^4+4k^3+4k^2+2k^2+4k+1}{(k+1)^2(k+2)^2}$$

$$= \frac{k^4+4k^3+6k^2+4k+1}{(k+1)^2(k+2)^2}$$

By inspection $(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$

$$\text{Thus } \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^4}{(k+1)^2(k+2)^2}$$

$$= \frac{(k+1)^2}{(k+2)^2}$$

$$= \frac{(k+1)^2}{(k+1+1)^2}$$

Hence true for $n = k + 1$

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it should be true for all $n \in \mathbb{N}$.

Example 4

Prove by Mathematical Induction that $\sum_{r=1}^{n}(r \text{ x } r!) = (n+1)! - 1$ for $n \geq 1$, $n \in \mathbb{N}$

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Solution

For $n = 1$ LHS $1 \text{ x } 1! = 1 \text{ x } 1 = 1$

RHS $(1 + 1)! - 1 = 2! - 1 = 2 - 1 = 1$

Hence true when $n = 1$

Assuming that $\sum_{r=1}^{k}(r \text{ x } r!) = (k+1)! - 1$

We want to show that $\sum_{r=1}^{k+1}(r \text{ x } r!) = (k+1+1)! - 1$

Proof

$$\sum_{r=1}^{k+1}(r \text{ x } r!) = (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! + (k+1)(k+1)! - 1$$

$$= (1+k+1)(k+1)! - 1$$

$$= (k+2)(k+1)! - 1$$

$$= (k+2)! - 1$$

$$= (k+1+1)! - 1$$

Hence true for $n = k + 1$

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it should be true for all $n \in \mathbb{N}$.

Exercise

Prove the following by Mathematical Induction

1 $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{Z}^+$.

2 $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$ for all $n \in \mathbb{N}$

3 $\sum_{r=1}^{n} 8r - 5 = 4n^2 - n \quad \forall n \in \mathbb{N}$

4 $\sum_{r=1}^{n} (r-1)(3r-2) = n^2(n-1)$ for all positive integers n.

5 $\sum_{r=1}^{n} (r^3 - r) = \frac{1}{4}n(n-1)(n+1)(n+2)$ for all natural values of n.

6 $1+2+\ldots+n = \frac{n(n+1)}{2}$, where n is any positive integer.

7 $\sum_{r=1}^{n} r(2^{r-1}) = 1+(n-1)2^n \quad \forall n \in \mathbb{Z}^+$

8 $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

9 $a + ar + ar^2 + \cdots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$

10 The geometric progression $1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$, for $n \geq 0$, $n \in \mathbb{Z}^+$.

DIVISIBILLITY

Example 1

For any positive integer n, $n^3 + 2n$ is divisible by 3.

Solution

METHOD 1

For $n = 1$

$n^3 + 2n = 1^3 + 2(1) = 3$ which is divisible by 3.

$\therefore$ True for $n = 1$

Assuming that the statement is true for $n = k$, i.e.

$k^3 + 2k = 3M$, for some natural number M.

We use this to show that the statement is also true for $n = k + 1$

Now, when $n = k + 1$ we have

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 3k^2 + 5k + 3 \\ &= k^3 + 2k + 3k + 3k^2 + 3 \\ &= 3M + 3k + 3k^2 + 3 \\ &= 3(M + k + k^2 + 1)\end{aligned}$$

Which is divisible by 3.

Hence true for $n = k + 1$.

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it should be true for all $n \in \mathbb{N}$

METHOD 2

Let $f(n) = n^3 + 2n$

$f(1) = 1^3 + 2(1) = 3$ which is divisible by 3

$\therefore$ True for $n = 1$

Assuming that the result is true for $n = k$, $i.e$ assuming that

$f(k) = k^3 + 2k = 3M$ for $M \in \mathbb{N}$

We want to show that $f(k+1)$ is divisible by 3

Taking $f(k+1) - f(k) = (k+1)^3 + 2(k+1) - (k^3 + 2k)$

$$f(k+1) - 3M = k^3 + 3k^2 + 3k + 1 + 2k + 2 - k^3 - 2k$$

$$f(k+1) - 3M = 3k^2 + 3k + 3$$
$$f(k+1) = 3k^2 + 3k + 3 + 3M$$

$$= 3(k^2 + k + 1 + M)$$
Which is divisible by 3.

Hence true for $n = k + 1$.

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it should be true for all $n \in \mathbb{N}$

Example 2

Prove by induction that $4\text{x}7^n + 3\text{x}5^n + 5$ is divisible by 12, for all $n \in \mathbb{N}$.

Solution

Let $f(n) = 4\text{x}7^n + 3\text{x}5^n + 5$

$f(1) = 4\text{x}7^1 + 3\text{x}5^1 + 5 = 48$ which is divisible by 12

Assuming that the result is true for $n = k$, where $k \in \mathbb{N}$ ie $f(k) = 12M$ for $M \in \mathbb{N}$

We want to show that $f(k+1)$ is divisible by 12.
Taking $f(k+1) - f(k) = 4\text{x}7^{k+1} + 3\text{x}5^{k+1} + 5 - 4\text{x}7^k - 3\text{x}5^k - 5$

$$f(k+1) - 12M = 28\text{x}7^k + 15\text{x}5^k + 5 - 4\text{x}7^k - 3\text{x}5^k - 5$$

$$f(k+1) - 12M = 24\text{x}7^k + 12\text{x}5^k$$

$$f(k+1) = 24\text{x}7^k + 12\text{x}5^k + 12M$$

$$f(k+1) = 12(2\text{x}7^k + 5^k + M)$$

Which is divisible by 12

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Hence true for $n = k + 1$.

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it should be true for all $n \in \mathbb{N}$

Exercise

Prove by induction that:

1 all numbers of the form $8^n - 2^n$ are divisible by 6 for all $n \in \mathbb{Z}^+$.

2 $f(n) = 7^{2n} - 48n - 1$ is divisible by 2304 for all $n \in \mathbb{Z}^+$.

3 $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{Z}^+$.

4 $f(n) = 24\text{x}2^{4n} + 3^{4n}$ is a multiple of 5 for every $n \in \mathbb{N}$.

5 $13^n - 4^n$ is divisible by 9 for every integer $n \geq 1$.

6 $2^{2n} - 1$ is divisible by 3 for all integers $n \geq 0$.

7 $P(n) = 5^n + 8n + 3$ is a multiple of 4 for $n \in \mathbb{N}$

8 8 is a factor of $5^{2n} - 3^{2n}$ for all $n \in \mathbb{Z}^+$.

9 $P(n) = 4^n + 6n - 1$ is a multiple of 9 for $n \in \mathbb{N}$

10 $11^n - 6$ is divisible by 5 for every positive integer n.

11 $3^{2n+4} - 2^{2n}$ is divisible by 5 for every positive integer n.

12 $4^{2n+1} + 3^{n+2}$ is a multiple of 13 for $n \in \mathbb{N}$

MATRICES

Example 1

Let A be the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. Prove that $A^n = \begin{pmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{pmatrix}$ for $n \geq 1$

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Solution

For $n = 1$

$$A^1 = \begin{pmatrix} 3^1 & 3^1 - 2^1 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = A$$

$\therefore$ true for $n = 1$.

Assuming that the statement is true for $n = k$, i.e

$$A^k = \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix}$$

We use this to show that the statement is also true for $n = k + 1$, i.e. to show that

$$\boxed{A^{k+1} = \begin{pmatrix} 3^{k+1} & 3^{k+1} - 2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}}$$

Now,

$$A^{k+1} = A^k . A^1$$

$$= \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3^k . 3 & 3^k + 2.3^k - 2.2^k \\ 0 & 2.2^k \end{pmatrix}$$

$$= \begin{pmatrix} 3^k . 3 & 3.3^k - 2.2^k \\ 0 & 2.2^k \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & 3^{k+1} - 2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}$$

which is true for $n = k + 1$

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it follows that it is true for all $n \geq 1$.

Example 2

A transformation is defined by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(a) Find the matrices (i) $\mathbf{M}^2$ (ii) $\mathbf{M}^3$

(b) Write down a suitable form for $\mathbf{M}^n$ and use the method of proof by induction to prove it.

Solution

(a) (i) $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{M^2} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(ii) $\mathbf{M^3} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$.

(b) $\mathbf{M}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$.

We want to prove that if $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then $\mathbf{M}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$ where $n \in \mathbb{N}$.

For $n = 1$, $\mathbf{M}^1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ which is true.

Assuming that the result holds for $n = k$, ie $\mathbf{M}^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$

We want to prove that $\mathbf{M}^{k+1} = \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix}$

Now, $\mathbf{M}^{k+1} = \mathbf{M}.\mathbf{M}^k$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2k+2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix}$$

Hence true for $n = k + 1$

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it follows that it is true for all $n \geq 1$.

Exercise

1 The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

(i) Find , $(\mathbf{AB})^2$, and $(\mathbf{AB})^3$

(ii) Hence suggest a suitable form for the matrix $(\mathbf{AB})^{\mathrm{n}}$.

(iii) Use induction to prove that your answer to part (ii) is correct.

(iv) Describe fully the single geometrical transformation represented by $(\mathbf{AB})^8$

2 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(i) Find $\mathbf{M}^2$ and $\mathbf{M}^3$.

(ii) Hence suggest a suitable form for the matrix $\mathbf{M}^n$.

(iii) Use induction to prove that your answer to part (ii) is correct.

(iv) Describe fully the single geometrical transformation represented by $\mathbf{M}^{10}$.

3 Prove by induction that $\begin{pmatrix} 1 & a \\ 0 & a \end{pmatrix}^n = \begin{pmatrix} 1 & \frac{a^{n+1}-a}{a-1} \\ 0 & a^n \end{pmatrix}$

4 Prove by induction that if $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \mathbf{M}^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$ for $n \in \mathbb{Z}^+$

5 Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. Prove by induction that $A^n = \begin{pmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{pmatrix}$ for $n = 1, 2, \ldots$

6 Prove by induction that $\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^n = \begin{pmatrix} 1+4n & 8n \\ -2n & 1-4n \end{pmatrix}$ for $n \geq 1$

7 If $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, prove that $A^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$

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8 Given that $\mathbf{A}=\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}$, use matrix multiplication to find

(a) $\mathbf{A}^2$ (b) $\mathbf{A}^3$

(c) Prove by induction that $\mathbf{A}^n=\begin{pmatrix}1&0&0\\0&2^n&2^n-1\\0&0&1\end{pmatrix}$ for $n\geq 1$.

9 Given the matrix A= $\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$, show by induction that

$\mathbf{A}^n=\begin{pmatrix}\cos n\theta&-\sin n\theta\\\sin n\theta&\cos n\theta\end{pmatrix}$ for all positive integers n.

10 Prove by induction that $\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}^n=\begin{pmatrix}1&n&\frac{1}{2}n(n+1)\\0&1&n\\0&0&1\end{pmatrix}$ for every positive integer n.

DERIVATIVES

Example 1

Prove by induction that $\frac{d^n}{dx^n}(xe^{2x})=2^{n-1}(2x+n)e^{2x}$ for every integer $n\geq 1$.

Solution

For $n=1$

LHS: $\frac{d}{dx}(xe^{2x})=2xe^{2x}+e^{2x}=(2x+1)e^{2x}$

RHS: $2^{1-1}(2x+1)e^{2x}=(2x+1)e^{2x}$

LHS=RHS hence the statement is true for $n=1$.

We assume that the formula is true for $n = k$, i.e.

$$\frac{d^k}{dx^k}(xe^{2x}) = 2^{k-1}(2x + k)e^{2x}$$

We show that it is also true for $n = k + 1$ i.e. we want to show that

$$\boxed{\frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) = 2^{k+1-1}(2x + k + 1)e^{2x}}$$

Now,

$$\frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) = \frac{d}{dx}.\frac{d^k}{dx^k}(xe^{2x})$$

$$= \frac{d}{dx}2^{k-1}(2x + k)e^{2x}$$

$$= 2^{k-1}\frac{d}{dx}(2x + k)e^{2x}$$

$$= 2^{k-1}(2e^{2x} + 4xe^{2x} + 2ke^{2x})$$

$$= 2^{k-1}.2e^{2x}(2x + k + 1)$$

$$= 2^{k+1-1}(2x + k + 1)e^{2x}$$

which is true for $n = k + 1$.

Since the result is true for $n = 1$, $n = k$ and $n = k + 1$, it follows that it is true for every integer $n \geq 1$.

OTHER WORKED EXAMPLES

Example 1

The function $f(x)$ is defined by $f(x) = 2 - \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$.

(a) Prove by induction that $f^n(x) = \frac{(n+1)x-n}{nx-(n-1)}$, $\quad n \geq 1$

where $f^n(x)$ denotes the nth composition of $f(x)$ by itself.

(b) State an expression for the domain of $f^n(x)$.

Solution

(a) $f^1(x) = \frac{(1+1)x-1}{1x-(1-1)} = \frac{2x-1}{x} = 2 - \frac{1}{x} = f(x)$

Hence true for $n = 1$

Assuming the result is true for some integer $n = k$, where $k \geq 1$ ie

$f^k(x) = \frac{(k+1)x-k}{kx-(k-1)}$.

We now show the result is true for $n = k+1$, that is to show that

$f^{k+1}(x) = \frac{(k+1+1)x-(k+1)}{(k+1)x-(k+1-1)}$.

Proof

$$f^{k+1}(x) = f^1\left(f^k(x)\right)$$

$$= f\left(f^k(x)\right)$$

$$= f\left(\frac{(k+1)x-k}{kx-(k-1)}\right) \qquad \text{by assumption}$$

$$= 2 - \frac{1}{\frac{(k+1)x-k}{kx-(k-1)}}$$

$$= 2 - \frac{kx-(k-1)}{(k+1)x-k}$$

$$= \frac{2[(k+1)x-k]-[kx-(k-1)]}{(k+1)x-k}$$

$$= \frac{2kx+2x-2k-kx+k-1}{(k+1)x-k}$$

$$= \frac{2kx-kx+2x-2k+k-1}{(k+1)x-k}$$

$$= \frac{kx+2x-k-1}{(k+1)x-k}$$

$$= \frac{(k+2)x-(k+1)}{(k+1)x-k}$$

$$= \frac{(k+1+1)x-(k+1)}{(k+1)x-(k+1-1)}$$

Which is true for $n = k + 1$

Since the result holds for $n = 1, n = k$ and $n = k + 1$, it should be true for all $n \geq 1$.

(b) $x \neq \frac{n-1}{n}$

Example 2

Prove by induction that

$$\frac{d^n}{dx^n}\left(e^x \sin(\sqrt{3}x)\right) = 2^n e^x \sin\left(\sqrt{3}x + \frac{1}{3}n\pi\right), n \geq 1, n \in \mathbb{N}$$

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Solution

For $n = 1$:

LHS $\frac{d}{dx}\left(e^x \sin(\sqrt{3}x)\right) = \sqrt{3}e^x \cos(\sqrt{3}x) + e^x \sin(\sqrt{3}x)$

$$= e^x\left(\sqrt{3}\cos(\sqrt{3}x) + \sin(\sqrt{3}x)\right)$$

RHS: $2e^x \sin\left(\sqrt{3}x + \frac{\pi}{3}\right) = 2e^x\left[\sin(\sqrt{3}x)\cos\frac{\pi}{3} + \cos(\sqrt{3}x)\sin\frac{\pi}{3}\right]$

$$= 2e^x\left[\tfrac{1}{2}\sin(\sqrt{3}x) + \tfrac{\sqrt{3}}{2}\cos(\sqrt{3}x)\right]$$

$$= e^x[\sin(\sqrt{3}x) + \sqrt{3}\cos(\sqrt{3}x)]$$

$$= e^x[\sqrt{3}\cos(\sqrt{3}x) + \sin(\sqrt{3}x)]$$

LHS = RHS, hence true for $n = 1$

Assuming the result is true for $n = k$, ie

$$\frac{d^k}{dx^k}(e^x \sin(\sqrt{3}x)) = 2^k e^x \sin\left(\sqrt{3}x + \frac{1}{3}k\pi\right), k \geq 1, k \in \mathbb{N}$$

We want to show that the results is true for $n = k + 1$ ie

$$\frac{d^{k+1}}{dx^{k+1}}(e^x \sin(\sqrt{3}x)) = 2^{k+1} e^x \sin\left(\sqrt{3}x + \frac{1}{3}(k+1)\pi\right)$$

Proof

$\frac{d^{k+1}}{dx^{k+1}}(e^x \sin(\sqrt{3}x)) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(e^x \sin(\sqrt{3}x))\right)$

$$= \frac{d}{dx}\left(2^k e^x \sin\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right)\right) \text{ by assumption}$$

$$= 2^k \frac{d}{dx}\left(e^x \sin\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right)\right)$$

$$= 2^k\left[e^x \sin\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right) + \sqrt{3}e^x \cos\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right)\right]$$

$$= 2^k e^x\left[\sin\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right) + \sqrt{3}\cos\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right)\right]$$

$$= 2^k e^x.2\left[\tfrac{1}{2}\sin\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right) + \tfrac{\sqrt{3}}{2}\cos\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right)\right]$$

$$= 2^{k+1} e^x\left[\cos\tfrac{\pi}{3}\sin\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right) + \sin\tfrac{\pi}{3}\cos\left(\sqrt{3}x + \tfrac{1}{3}k\pi\right)\right]$$

$$= 2^{k+1} e^x\left[\sin\left(\tfrac{\pi}{3} + \left(\sqrt{3}x + \tfrac{1}{3}k\pi\right)\right)\right]$$

$$= 2^{k+1} e^x\left[\sin\left(\sqrt{3}x + \tfrac{\pi}{3} + \tfrac{1}{3}k\pi\right)\right]$$

$$= 2^{k+1}e^{x}\left[\sin\left(\sqrt{3}x + \frac{k\pi+\pi}{3}\right)\right]$$

$$= 2^{k+1}e^{x}\sin\left(\sqrt{3}x + \frac{1}{3}(k+1)\pi\right)$$

Which is true for $n = k + 1$

Since the result holds for $n = 1, n = k$ and $n = k + 1$, it should be true for all $n \geq 1$.

Example 3

If $n \geq 1$, $n \in \mathbb{N}$, prove by induction that

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1.$

Solution

For $n = 1$

LHS: $1 \times 1! = 1$

RHS: $(1 + 1)! - 1 = 2! - 1 = 2 - 1 = 1$

LHS = RHS Hence true for $n = 1$

Assuming that the result holds for $n = k$, $k \in \mathbb{N}$ i.e.

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! = (k + 1)! - 1$

We want to show that the result also holds for $n = k + 1$, ie

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + (k + 1) \times (k + 1)! = (k + 1 + 1)! - 1$

Proof

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + (k + 1) \times (k + 1)! = (k + 1)! - 1 + (k + 1) \times (k + 1)!$$

$$= (k + 1)! + (k + 1) \times (k + 1)! - 1$$

$$= (k+1)!\,[1+(k+1)] - 1$$

$$= (k+1)!\,(k+1+1) - 1$$

$$= (k+1\ +1)!\ - 1$$

Which is true for $n = k+1$

Since the result holds for $n = 1, n = k$ and $n = k+1$, it should be true for all $n \geq 1$

Exercise

1 Prove by Mathematical induction that if $y = xe^x$ then $\frac{d^n y}{dx^n} = (x+n)e^x, \forall n \in \mathbb{Z}^+$.

2 If $y = xe^x$ find

(i) $\frac{dy}{dx}$,

(ii) $\frac{d^2 y}{dx^2}$

(iii) $\frac{d^3 y}{dx^3}$

Hence deduce a formula for $\frac{d^n y}{dx^n}$

Prove you result for $\frac{d^n y}{dx^n}$ by Mathematical induction.

3 For any integer $n \geq 0$ we have $\int_0^\infty x^n e^{-x} dx = n!$

4 Prove the proposition $P(n)$: $\frac{d^{2n}}{dx^{2n}} \sin x = (-1)^n \sin x$

5 Given the equation $xy + 3y = x$,

Prove by induction that a $(x+3)\frac{d^n y}{dx^n} + (n+1)\frac{d^{n-1} y}{dx^{n-1}} = 0$

Miscellaneous Questions

1 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geq 1$,

$\mathbf{M}^n = \begin{pmatrix} 2^n & 2^{n+1} - 2 \\ 0 & 1 \end{pmatrix}$.

2 Prove by induction that $\frac{d}{dx}(x^n) = nx^{n-1}$, for $n > 0$.

3 $\sum_{r=1}^{n}(2r-1)(2r+3) = \frac{n}{3}(4n^2+12n-1)$ for all positive integer value of n.

4 Prove by induction that $2^n > 2n$ for every positive integer $n > 2$.

5 Prove by Mathematical induction that $3^n - 2n - 1$ is divisible by 4, for $n \in \mathbb{N}$.

6 Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for $n \geq 1$.

7 Prove by induction that $\sum_{r=1}^{n} r\left(2^{r-1}\right) = 1 + (n-1)2^n$ for all positive integral values of n.

(Zimsec N2011)

8 The n^{th} term of a sequence is given by $U_n = n(2n+1)$, for $n \geq 1$. The sum of the first n terms is denoted by S_n. Use the method of mathematical induction to prove that $S_n = \frac{1}{6}n(n+1)(4n+5)$.

9 Prove by mathematical induction that $\sum_{r=1}^{n}(r+3)(2r+1) = \frac{n}{6}(4n^2+27n+41)$. [7]

(Zimsec J2012)

10 Prove that $1 + 8 + 27 + 64 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for $n \in \mathbb{N}$

11 Prove by mathematical induction that $4^{n+1} + 5^{2n-1}$ is a multiple of 21 $\forall n \in \mathbb{N}$

12 Prove by induction that for all even natural numbers n

$$\frac{d^n}{dx^n}(\sin 3x) = (-1)^{\frac{n}{2}} \text{ x } 3^n \text{ x } \sin 3x$$

13 The matrix $\mathbf{D}$ is given by $\mathbf{D} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geq 1$,

$\mathbf{D}^n = \begin{pmatrix} 3^n & \frac{1}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$.

14 Prove by induction that $\frac{d^n}{dx^n}(e^x \sin x) = 2^{\frac{1}{2}n} e^x \sin\left(x + \frac{1}{4}n\pi\right), n \geq 1, n \in \mathbb{N}$

15 Prove by Mathematical induction that every term of the sequence $a_n = 12^{n+1} + 2\text{x}5^n$ is a multiple of 7 for $n \in \mathbb{N}$

16 The sequence $u_1, u_2, u_3, \ldots$ is given by $u_1 = 2$ and $u_{n+1} = \frac{u_n}{1+u_n}$ for $n \geq 1$.

(i) Find u_2 and u_3, and show that $u_4 = \frac{2}{7}$.

(ii) Hence suggest an expression for u_n.

(iii) Use induction to prove that your answer to part (ii) is correct.