TOPIC 1: ALGEBRA

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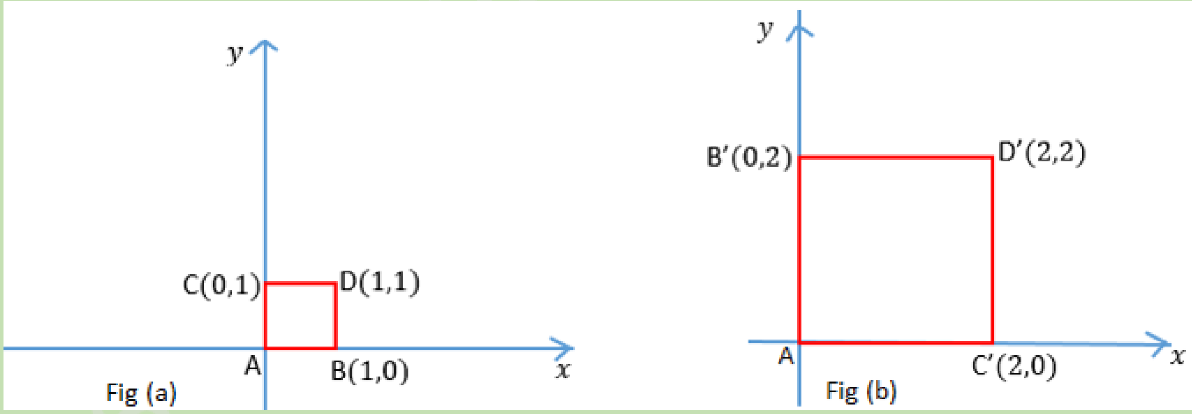
1.2 MATRICES

Learning Objectives

After completing this section you should be able to:

- carry out basic operations of matrices
- calculate determinant of a square matrix (up to 3 x 3)
- identify null matrix, identity matrix, singular and non-singular matrix
- find inverse of a 3 x 3 non- singular matrix
- apply the result $(AB)^{-1} = B^{-1}A^{-1}$ for non-singular matrices to solve problems
- construct 2 x 2 matrices to represent enlargement, rotation, reflection, stretch and shear transformations
- use a 2 x 2 matrix to represent certain geometrical transformation in the x – y plane
- use the composite transformation AB and A^n in problem solving
- derive the relationship between the area scale-factor of a transformation and the determinant of the corresponding matrix
- solve simultaneous equations in 2 or 3 unknowns by reducing them to the matrix equation form (A**X** =**b**)



A matrix is a rectangular array of numbers, e.g. $\begin{pmatrix} 2 & -2 & 0 \\ 8 & -5 & 1 \\ 0 & 4 & 7 \end{pmatrix}$

1.2.1 Types of matrices

Square matrix

A square matrix is a matrix with the same number of rows and columns. An n x n matrix is known as a square matrix of order n. Eg 2x2 matrix, 3x3 matrix.

Identity matrix/Unit matrix

Is a square matrix in which all the elements of the principal diagonal are ones and all other elements are zeros. The effect of multiplying a given matrix by an identity matrix is to leave the given matrix unchanged.

Examples of identity matrices are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

An identity matrix is denoted by letter **I**.

Null matrix/Zero matrix

A null matrix is a matrix, whose all elements are zero. Examples are $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

A null matrix id denoted by letter **O**.

1.2.2 Matrix operations

Addition, subtraction and scalar multiplication

We can only add or subtract matrices of the same size. Two matrices **A** and **B** are said to be of the same size if the number of rows and columns in **A** is the same as the number of rows and columns in **B**. Matrices are added (or subtracted) by adding corresponding entries.

Example

Given that $\mathbf{A} = \begin{pmatrix} 5 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & 2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 & -2 \\ 6 & 15 & 7 \\ 0 & 5 & 9 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & 4 & 6 \\ 1 & -2 & 5 \\ 1 & 3 & 0 \end{pmatrix}$

Find $\mathbf{A} + \mathbf{B} - 2\mathbf{C}$

.

Solution

$$\mathbf{A}+\mathbf{B}-2\mathbf{C}=\begin{pmatrix}5 & -1 & 3\\ 1 & 0 & 2\\ -3 & 2 & 4\end{pmatrix}+\begin{pmatrix}1 & 0 & -2\\ 6 & 15 & 7\\ 0 & 5 & 9\end{pmatrix}-2\begin{pmatrix}-2 & 4 & 6\\ 1 & -2 & 5\\ 1 & 3 & 0\end{pmatrix}$$

Every entry of C is multiplied by the scalar 2

$$=\begin{pmatrix}5+1 & -1+0 & 3+(-2)\\ 1+6 & 0+15 & 2+7\\ -3+0 & 2+5 & 4+9\end{pmatrix}-\begin{pmatrix}2(-2) & 2(4) & 2(6)\\ 2(1) & 2(-2) & 2(5)\\ 2(1) & 2(3) & 2(0)\end{pmatrix}$$

$$=\begin{pmatrix}6 & -1 & 1\\ 7 & 15 & 9\\ -3 & 7 & 13\end{pmatrix}-\begin{pmatrix}-4 & 8 & 12\\ 2 & -4 & 10\\ 2 & 6 & 0\end{pmatrix}$$

$$=\begin{pmatrix}10 & -9 & -11\\ 5 & 19 & -1\\ -5 & 1 & 13\end{pmatrix}$$

For any 3 matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ that are compatible for addition, we find that
$\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$.
This is known as the **associative law** of addition and it means that for the sum of three matrices, we can choose to begin by adding either the first pair or the second pair.

Multiplication of matrices

The product **AB** is only possible if and only if the number of columns in matrix A is the same as the number of rows in matrix **B.**

For any 3 matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ that are compatible for multiplication, we find that

$$\mathbf{A}\text{x}(\mathbf{B}\text{x}\mathbf{C})=(\mathbf{A}\text{x}\mathbf{B})\text{x}\mathbf{C}.$$

This is known as the **associative law** of multiplication and it means that for a product of three matrices, provided that their order is not changed, we can choose to begin by multiplying either the first pair or the second pair.

In general, matrix multiplication is not commutative, ie for two matrices **A** and **B**, **AB≠BA**

Exercise 1.2.2

1 Arrange the matrices $(3\ 1\ 0\ 2)$, $\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\begin{pmatrix}-1 & 1 & 0\\2 & 1 & -2\end{pmatrix}$ so that they are conformable for multiplication and evaluate their product.

2 If $\mathbf{A} = \begin{pmatrix}1 & 3\\-1 & 2\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}2 & 1 & 0\\3 & -1 & 4\end{pmatrix}$, $\mathbf{C} = \begin{pmatrix}-1 & 2 & 1\\3 & 1 & 0\\-1 & 0 & 3\end{pmatrix}$

Prove the result that **A** x (**B** x **C**)=(**A** x **B**) x **C**.

3 Prove that for any 3 matrices **A**, **B** and **C** that are compatible for addition

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}.$$

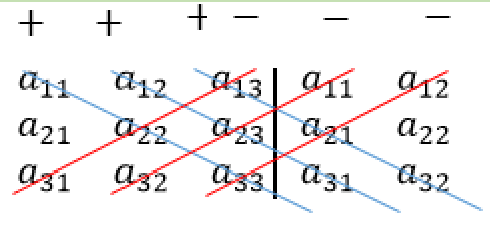
1.2.3 Determinant and inverse of a 3x3 matrix

Determinant of a 3x3 matrix

Each square matrix **A** is assigned a special scalar called the **determinant**, denoted det**A**, or $|\mathbf{A}|$.

Method 1: Using the **Sarrus' Rule**

Consider the following 3 x 3 matrix



$$\mathbf{A} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

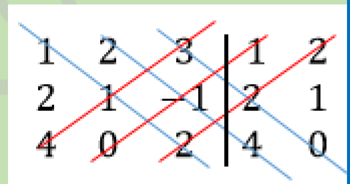
Example

Find the determinant of the matrix $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 4 & 0 & 2 \end{pmatrix}$

Solution

Using Method 1

$$\begin{aligned} \det \mathbf{B} &= 1(1)2 + 2(-1)(4) + 3(2)(0) - 3(1)(4) - 1(-1)(0) - 2(2)(2) \\ &= 2 - 8 + 0 - 12 - 0 - 8 \\ &= -26 \end{aligned}$$



Method 2: Using **cofactors**

Taking $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 4 & 0 & 2 \end{pmatrix}$

The value of $|\mathbf{B}|$ is found by extracting 2 x 2 determinants (called minors) from the 3 x 3 determinant. The minors are assigned signs, + or – depending on the position in the determinant.

$$|\mathbf{B}| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 4 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= 1(2) - 2(8) + 3(-4)$$

$$= -26$$

A matrix whose determinant is zero is known as a singular matrix. Such a matrix is non-invertible (has no inverse).

Inverse of a 3 x 3 matrix

To find the inverse of a 3 x 3 matrix we need

(i) the determinant,

(ii) matrix of minors,

(iii) matrix of cofactors,

(iv) Adjoint matrix.

Inverse of a matrix $\mathbf{A}$ is given by $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}$ x $\text{adj}(\mathbf{A})$

Example 1

Find the inverse of the matrix $\mathbf{C} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$\det(\mathbf{C}) = 1(3) + 1(2) + 1(-4) = 1$

Matrix of minors $\begin{pmatrix} \begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ -2 & -3 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ -3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -2 & -3 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & 2 & -4 \\ 3 & 2 & -5 \\ 1 & 1 & -2 \end{pmatrix}$

To find the cofactors matrix we affect the matrix of minors by the signs $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Matrix of cofactors $= \begin{pmatrix} 3 & -2 & -4 \\ -3 & 2 & 5 \\ 1 & -1 & -2 \end{pmatrix}$

Adjoint matrix

The adjoint matrix is found by transposing the matrix of cofactors. To transpose is to make each row become a column, in their order.

$Adj(\mathbf{C}) = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$

$\mathbf{C}^{-1} = \frac{1}{\det \mathbf{C}}$ x $Adj(\mathbf{C}) = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$

For any n x n invertible matrices **A** and **B**,

(i) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

(ii) $\mathbf{A^{-1}A = AA^{-1} = I}$

(iii) **if** $\mathbf{AB = M}$ then $\mathbf{A = MB^{-1}}$ and also $\mathbf{B = A^{-1}M}$.

Example 1

(a) Find the matrix **B** such that $\begin{pmatrix} -1 & 1 \\ -6 & 5 \end{pmatrix}\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$

(b) The matrices **A** and **C** are given by **A**= $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ **C**= $\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Find the matrix **D** satisfying **DA**=**C**

Solution

(a) $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ -6 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ -6 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix}$$

$$\therefore \mathbf{B} = \begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 10 & -8 \\ 12 & -9 \end{pmatrix}$$

(b) $\boldsymbol{A}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 & 0 \\ 1 & 1 & -1 \\ -3 & 1 & 1 \end{pmatrix}$

$$\mathbf{DA = C}$$

$$\mathbf{D} = \mathbf{CA^{-1}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 & 0 \\ 1 & 1 & -1 \\ -3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 6\frac{1}{2} & -2\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

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Exercise 1.2.3

1 Find the inverses of the matrices

(a) $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 0 & 1 \\ 3 & -4 & 5 \\ -7 & -3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} -2 & 1 & 4 \\ 3 & -2 & 5 \\ 0 & 1 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 4 \\ 2 & 6 & -2 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & -2 & 1 \\ 3 & -1 & 5 \\ -1 & 4 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 4 & -2 \\ 2 & -1 & 5 \\ -3 & 4 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

2 Given that $\mathbf{A}^{-1} = \begin{pmatrix} -1 & -2 & 2 \\ 2 & 5 & -4 \\ 1 & 1 & -1 \end{pmatrix}$, $\mathbf{B}^{-1} = \begin{pmatrix} 3 & -3 & 3 \\ -1 & 1 & 1 \\ 2 & 4 & -2 \end{pmatrix}$ and that

$\mathbf{AB}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$, find x, y and z (Zimsec J2014 p2)

3 Given that $\mathbf{M} = \begin{pmatrix} 3 & 2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 4 & -2 \\ 1 & 0 & 2 \end{pmatrix}$

Find (i) **MN**, (ii) $(\mathbf{MN})^{-1}$

Hence find the values of x, y and z for which $\mathbf{MN}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}$ (Zimsec J2012p2)

1.2.4 Solving simultaneous equations

Example

Solve the simultaneous

$$x - y + 2z = 6$$

$$2x + y + z = 3$$

$$3x - y + z = 6$$

Solution

The system of equations can be represented by the matrix equation

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}$$

Let the above 3x3 matrix be **A.**

$$\mathbf{A}^{-1} = -\frac{1}{9}\begin{pmatrix} 2 & -1 & -3 \\ 1 & -5 & 3 \\ -5 & -2 & 3 \end{pmatrix}$$

If we pre-multiply both sides of the equation by $\mathbf{A}^{-1}$ we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} 2 & -1 & -3 \\ 1 & -5 & 3 \\ -5 & -2 & 3 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -9 \\ 9 \\ -18 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$x = 1$, $y = -1$ and $z = 2$.

Exercise 1.2.4

Solving simultaneous equations

1 $x - y + z = -2$
$2x - y - 2z = -9$
$3x + y - z = -2$

2 $2x - 3y + z = 13$
$x + y - 2z = -1$
$3x - y + 2z = 17$

3 $2x + 5y + 4z = 1$
$3x + 12y + 8z = 1$
$x + 8y + 5z = 1$

4 $x - y \quad = -4$
$5x - 4y + z = -12$
$2x + 2y + z = 11$

5 $2x + 2y + z = 4$
$2x + 3y + z = 1$
$3x + 2y + z = 3$

6 $3a + 2b + c = 5$
$4a - 2b + 3c = 4$
$a - b + c = 1$

1.2.5 TRANSFORMATION MATRICES

Note the following facts when working with transformation matrices

1. When a shape with area A is transformed by matrix M the area of the image will be: $|\det \mathrm{M}|$ x A
2. If matrix M transforms point A to its image A′ then M^{-1}(The inverse of M) will transform A′ back to A. M^{-1} must exist.
3. Points are always pre-multiplied by the transformation matrix. Order of multiplication is crucial. Be careful when applying this to multiple transformations and their inverses.

4. **Composite Transformations**
$(p') = \mathbf{MR}(p)$ means p' is the image of p when **R is** done first followed by **M**.
If **M** and **R** are matrices the combined transformation can be found by performing the matrix multiplication **M** x **R**

Example 1

Find the transformation matrix that represents the composite transformation of a reflection in the y-axis followed by a stretch parallel to y-axis by stretch factor 2.

Solution

The matrix of reflection in the y-axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

The matrix of a stretch parallel to y-axis by stretch factor -2 is $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

Hence the composite transformation matrix is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Example 2

(a) Find the image of the point $(-1, 3)$ under a transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$.

(b) The point **A'** with coordinates $(5, -2)$ is the image of a point **A** under a transformation represented by the matrix $\mathbf{N} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$Find the coordinates of **A**.

Solution

(a) Transformation matrix x Object coordinates = Image.

$$\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Hence the image has coordinates $(3, 3)$.

(b) The inverse of **N** maps the point $(5, -2)$ back to its original position.

$$\mathbf{N}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ gives the coordinates of **A** as $(2, 7)$.

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Exercise 1.2.5

1 Find the images of the following points under a transformation represented by the given matrix.

(i) **A**$(0,1)$, $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ (ii) **B**$(2,4)$, $\begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}$ (iii) **C**$(1,2)$, $\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}$

2 A linear transformation **T** has matrix $\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}$. Find the coordinates of the point having an image of (7, 2) under **T**.

3 Find the transformation matrix **M** which maps the triangle ABC with vertices A$(1,1)$, B$(2,1)$ and C$(1,2)$ onto the image $A_1(1,2)$, $B_1(3,3)$ and $C_1(0,3)$

4 A triangle ABC has its vertices at the points A$(1,1)$, B$(2,1)$, C(24)
Find the coordinates of the image of triangle ABC under the transformation represented by each of the following matrices. Show the object and image on a graph and describe the transformation represented by each matrix.

(i) $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$ (ii) $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ (iii) $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ (iv) $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

5 Find the matrix of the transformation **T** which maps $(1,3)$ and $(2,1)$ onto $(5,11)$ and $(5,7)$ respectively.

6 A transformation T is represented by the matrix **M**$=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$. The point $(2,8)$ is the image of a point (x,y) under the transformation T. Find the coordinates (x,y).

7 A triangle ABC of area 11 square units is mapped into triangle A'B'C' by **M** $=\begin{pmatrix}3 & 2\\ 1 & 5\end{pmatrix}$. Find the area of triangle A'B'C'.

8 The square ABCD is transformed to $A_1B_1C_1D_1$ by the transformation matrix $\begin{pmatrix}5 & -1\\ -3 & 1\end{pmatrix}$
If A, B, C and D have coordinates (1, 0), (3, 0), (3, 2) and (1, 2) respectively find

(i) the coordinates of A_1, B_1, C_1 and D_1,

(ii) the area of $A_1B_1C_1D_1$,

(i) the matrix that will transform $A_1B_1C_1D_1$ back to ABCD.

9 Shape A is transformed to a second shape B by the transformation matrix $\begin{pmatrix}2 & 1\\ 4 & 3\end{pmatrix}$ and a shape C is obtained from shape B using the transformation matrix $\begin{pmatrix}-1 & 5\\ -1 & 2\end{pmatrix}$. Find the single matrix that would transform shape A onto shape C direct. If shape A has an area of 5 units^2, find the areas of the shapes B and C.

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1.2.6 Transformation by a singular matrix

What happens if the determinant of the transformation matrix **M** is zero, i.e. a singular matrix? If **M** is a singular matrix the images of A, B and C will lie in a straight line.

Consider a general point (x, y) and a singular matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + 4y \end{pmatrix}$$

$\therefore \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2(x + 2y) \end{pmatrix}$ i.e. all image points lie on the line $y = 2x$.

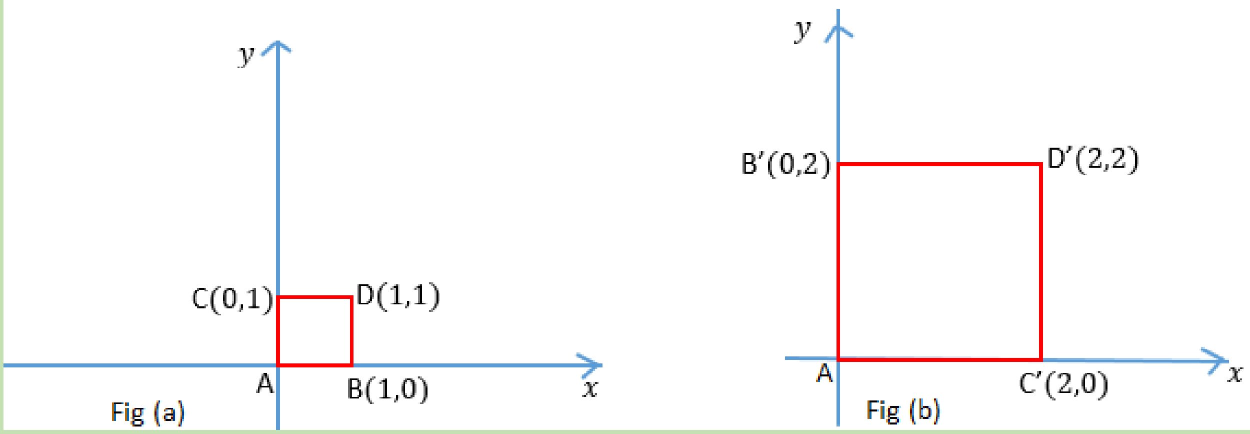
An alternative method would be to consider any shape (e.g. triangle) with known vertices. Transform each vertex by the given matrix and obtain points in a straight line. Then find the equation of the line.

Example

Describe fully the transformation represented by the matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

Solution

To be able to determine the transformation, consider the unit square shown below



If each vertex of the square is transformed by $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ we would get

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \end{array}$$

The image of ABCD is shown on fig(b) above.

Clearly the transformation is an enlargement centre (0, 0) by factor 2 and a reflection in the line $y = x$.

Exercise 1.2.6

1 Find the equation of the straight line onto which all points of the $x - y$ plane are mapped by the transformation matrix

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$ (d) $\begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix}$

2 Describe fully the transformations represented by the following matrices

(i) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (v) $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ (vi) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(vii) $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ (viii) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ (ix) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$

(x) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (xi) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (xii) $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

(xiii) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (xiv) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (xv) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(xvi) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

3 Find the transformation matrix that represents the following composite transformations:

(a) Stretch parallel to x-axis by factor -2 followed by Shear parallel to x-axis by factor 2.

(b) reflection in the line $y = -x$ followed by a shear parallel to y-axis, by factor -3 .

1.2.7 Transformation of a line

Example 1

Find the equation of the image of the line $y = 3x + 4$ under a transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$.

Solution

Method 1

Find any two points on the line $y = 3x + 4$, find their images under operator **M** and construct the equation of the image line.

Any two points on $y = 3x + 4$ are $(0,4)$ and $(-1,1)$

Their images are:

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \Longrightarrow (0,-12)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \Longrightarrow (-1,-3)$$

$\therefore$ the image line is $\frac{y-(-12)}{x-0} = \frac{-3-(-12)}{-1-0}$

$$\Longrightarrow y = -9x - 12$$

Method 2

Any general point on $y = 3x + 4$ has coordinates of the form $(k, 3k + 4)$ and image (x', y')

Thus,

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} k \\ 3k + 4 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k \\ -9k - 12 \end{pmatrix}$$

$x' = k$……..(i)

$y' = -9k - 12$……(ii)

Eliminating k from the two equations gives
$y' = -9x' - 12$
Thus the image line has equation $y = -9x - 12$.

Example 2

The transformation matrix $\begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}$ maps the line y onto the line $y' = 6x + 8$. Find the equation of the line y giving your answer in the form $ax + by = c$, where a, b, and c. are integers. (Zimsec J2012 p2)

Solution

$\begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}^{-1}$ maps $y' = 6x + 8$ back to y.

$$\begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = -\frac{1}{7}\begin{pmatrix} 2 & -1 \\ -3 & -2 \end{pmatrix}$$

Therefore $\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} 2 & -1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} k \\ 6k + 8 \end{pmatrix}$

$x = -\frac{1}{7}(-4k - 8) \Longrightarrow 7x = 4k + 8 \ldots\ldots$ (i)

$y = -\frac{1}{7}(-15k - 16) \Longrightarrow 7y = 15k + 16 \ldots\ldots$ (ii)

Eliminating k gives $7x = 4\left(\frac{7y-16}{15}\right) + 8$

$\Longrightarrow 105x = 28y - 64 + 120$

$\Longrightarrow 105x - 28y = 56$

$\Longrightarrow 15x - 4y = 8$

Exercise 1.2.7

1 Find the equations of the images of the following lines under a transformation represented by the matrix $\begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$

(a) $2y = x$ (b) $2x + y + 4 = 0$ (c) $y = 2$ (d) $x = 3$

2 Prove that the line $y = 2x - 1$ is mapped onto itself under the transformation $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$

3 Prove that the transformation matrix $\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$ maps all points on the line $y = 3x$ onto themselves.

4 The image of the point (x, y) is (x', y') under the transformation

$$\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Find the equations of the images of the following lines under the given transformation.

(a) $y = 4x - 1$ (b) $3y = 2x + 1$ (c) $x = 5$

5 The point (x, y) is transformed to the point (x', y') by means of the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Find

(a) the image of the point $(1,2)$,
(b) the point whose image is $(-8,9)$
(c) the image of line $y = 2x$ under this transformation.

6 Find the equation of a straight line whose image is $4y' - 3x' = 6$ under the transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$.

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1.2.8 INVARIANT POINTS

An **invariant** (or fixed) **point** is a point which is mapped onto itself, that is, it is its own image. Thus if (x, y) is an invariant point and its image is (x', y') then $x = x'$ and $y = y'$.

Example 1

Find the invariant points of the transformation defined by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

Solution

For invariant points $x' = x$ and $y' = y$

Thus $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$x = 4x - 3y + 6 \Longrightarrow x = y - 2$….. (i)

$y = -x + 3y + 1$ …. (ii)

Solving (i) and (ii) simultaneously gives

$y = -y + 2 + 3y + 1$

$y = -3$ and $x = -5$

$(-5, -3)$ is the invariant point.

Example 2

Prove that all points on the line $y + 3x + 2 = 0$ are mapped onto a single point under the transformation that maps (x, y) onto (x', y') according to the relationship

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and find the coordinates of this single point.

Solution

$y + 3x + 2 = 0 \Longrightarrow y = -3x - 2$. Any point on the line $y = -3x - 2$ has coordinates of the form $(k, -3k - 2)$

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Hence the transformation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} k \\ -3k-2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$x' = -1$ and $y' = 1$

Hence all points are mapped onto a single point with coordinates $(-1, 1)$.

Example 3

Find the invariant points under the transformations represented by the matrix $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$

Solution

The transformation is written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$x' = x$ and $y' = y$ for invariant points

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Longrightarrow x = 4x + y \Longrightarrow y = -3x$ ……… (i)

and $y = 6x + 3y \Longrightarrow y = -3x$ ……. (ii)

Equations (i) and (ii) are the same hence all points on the line $y = -3x$ are invariant.

Exercise 1.2.8

1 Find the invariant points under the transformations represented by following matrices

(a) $\begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}$ (b) $\begin{pmatrix}-1 & -1\\2 & 2\end{pmatrix}$ (c) $\begin{pmatrix}7 & -4\\3 & -1\end{pmatrix}$ (d) $\begin{pmatrix}0 & 1\\-1 & 2\end{pmatrix}$

2 Find the invariant points of the following transformations:

(a) $\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}8 & -15\\-7 & 16\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ (b) $\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x+y-1\\1-2x\end{pmatrix}$

(c) $\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}2\\0\end{pmatrix}$ (d) $5x' = 9x + 8y - 12$, $5y' = 8x + 21y - 24$

(e) $x' = 2x + 1$, $y' = 3 - 2y$ (g) $x' = 1 - 2y$, $y' = 2x - 3$

3 A transformation T is defined algebraically by $x' = y - \sqrt{2}$, $y' = x + \sqrt{2}$. Find the invariant points of T.

4 A plane transformation T is defined by $x' = 7x - 24y + 12$, $y' = -24x - 7y + 56$.

Show that T has just one invariant point P, and find its coordinates.

5 The point (x, y) is transformed to the point (x', y') by means of the transformation

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}3 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}$$

Find the coordinates of the point that is mapped onto itself by the transformation.

1.2.9 INVARIANT LINES

An **invariant** (or fixed) **line** is a line all of whose points have image points on the same line, not necessarily on the same points. When points on a line are mapped back to themselves, the line is known as a **line of invariant points**. A line of invariant points is a special type of an invariant line.

Example 1

Prove that the transformation matrix $\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$ maps all points on the line $y = 3x$ onto themselves.

Solution

Points on the line $y = 3x$ have coordinates of the form $(k, 3k)$

Thus the transformation is written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} k \\ 3k \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k \\ 3k \end{pmatrix}$$

$x' = k$ …….. (i)

$y' = 3k$ ……. (ii)

Eliminating k gives $y' = 3x'$

Thus the image line is $y = 3x$ same as original line.

Considering a point which lies on the line, eg (1,3)

$$\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Hence all points on the line $y = 3x$ are mapped onto themselves.

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Example 2

If the line $y = 4x$ is mapped onto itself under the transformation $\begin{pmatrix} 3 & -1 \\ a & -3 \end{pmatrix}$, find the value of a and the equation of the other line that passes through the origin and is mapped onto itself by the transformation.

Solution

A general point on $y = 4x$ has coordinates of the form $(k, 4k)$. Thus the transformation is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ a & -3 \end{pmatrix}\begin{pmatrix} k \\ 4k \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -k \\ ak - 12k \end{pmatrix}$$

$x' = -k$…….. (i)

$y' = ak - 12k$…….. (ii)

Eliminating k gives,

$y' = -ax' + 12x'$

The image line is $y = (12 - a)x$

Thus if the line $y = 4x$ is mapped onto itself $4 = 12 - a$

$\Longrightarrow a = 8.$

Every line which passes through the origin is of the form $y = mx$ and has coordinates of the form (k, mk). Thus

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 8 & -3 \end{pmatrix}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} 3k - mk \\ 8k - 3mk \end{pmatrix}$$

The point $(3k - mk, 8k - 3mk)$ must lie on $y = mx$, thus

$8k - 3mk = m(3k - mk)$

$\Longrightarrow m^2 - 6m + 8 = 0$

$\Longrightarrow (m - 4)(m - 2) = 0$

$\Longrightarrow m = 4$ or 2

Hence the other line is $y = 2x$.

Exercise 1.2.9

1 Prove that the following lines are mapped onto themselves using the transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -9 \end{pmatrix}$

(a) $y = x + 5$ (b) $y + 3x - 1 = 0$

2 Prove that the line $y = 2x - 1$ is mapped onto itself under the transformation $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$.

3 A transformation T assigns any point (x, y) an image (x', y') according to the rule

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

Prove that the line $y = 3x + 1$ maps onto itself under the transformation T.

4 Prove that all lines of the form $y = x + a$ are mapped onto themselves under the transformation given by the matrix $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$.

5 A transformation matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

(a) Show that under this transformation

(i) all points on the line $y = x$ are invariant points.

(ii) any point on the line $y = x + c$ where $c \neq 0$, is transformed to another point on that line.

1.2.10 Finding invariant lines

Example 1

Find the invariant line of the transformation represented by the following matrix $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

Solution

In this example we use the line $y = mx + c$

All points on $y = mx + c$ have coordinates of the form $(k, mk + c)$

The transformation is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} k \\ mk + c \end{pmatrix} = \begin{pmatrix} mk + c \\ -k + 2mk + 2c \end{pmatrix}$$

If $y = mx + c$ maps onto itself then the point $(mk + c, -k + 2mk + 2c)$ must also lie on the line. Thus

$-k + 2mk + 2c = m(mk + c) + c$

This is an identity in k which gives the equations

$-1 + 2m = m^2$ and $2c = cm + c$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m - 1)^2 = 0$

$m = 1$ twice

Also, $2c = cm + c$

$\Rightarrow c - cm = 0$

$\Rightarrow c(1 - m) = 0$

$c = 0$ or $m = 1$

Thus the invariant line has equation $y = x$.

Example 2

A transformation matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$. Show that the transformation has no invariant line using the equation $y = mx$.

Solution

All points on $y = mx$ have coordinates of the form (k, mk)

The transformation is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} k \\ mk \end{pmatrix} = \begin{pmatrix} k - 2mk \\ 2k + 3mk \end{pmatrix}$$

Substituting into $y = mx$

$2k + 3mk = m(k - 2mk)$

This is an identity in k, thus

$\Rightarrow 2 + 3m = m - 2m^2$

$\Longrightarrow 2m^2 + 2m + 2 = 0$

$\Longrightarrow m^2 + m + 1 = 0$ which does not have real roots.

Hence the transformation does not have invariant lines.

Example 3

Find the equation of the two invariant lines under the transformation represented by the matrix $\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$. Which of the two lines is a line of invariant points?

Solution

In this example we use the line $y = mx + c$

All points on $y = mx + c$ have coordinates of the form $(k, mk + c)$

The transformation is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} k \\ mk+c \end{pmatrix} = \begin{pmatrix} 2k-2mk-2c \\ -k+3mk+3c \end{pmatrix}$$

Using $y = mx + c$

$-k + 3mk + 3c = m(2k - 2mk - 2c) + c$

This is an identity in k and so

$-1 + 3m = 2m - 2m^2$ and $3c = -2cm + c$

$2m^2 + m - 1 = 0$

$\Longrightarrow (2m-1)(m+1) = 0$

$\Longrightarrow m = \frac{1}{2}$ or -1

Taking $3c = -2cm + c$

$\Longrightarrow 2c(1+m) = 0$

$\Longrightarrow c = 0$ or $m = -1$.

Thus the invariant lines are $y = \frac{1}{2}x$ and $y = -x$

Considering point $(2, 1)$ which lies on $y = \frac{1}{2}x$,

$\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ hence line $y = \frac{1}{2}x$ is a line of invariant points.

Considering point $(1, -1)$ which lies on $y = -x$

$\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ hence line $y = -x$ is an invariant line.

Note: **A line of invariant points has each point mapping onto itself*

Exercise 1.2.10

1 A transformation of a plane is given by the matrix $\begin{pmatrix}1 & 4\\ 4 & 1\end{pmatrix}$. Find the invariant lines of the transformation.

2 A transformation of a plane is given by the matrix $\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$. Find the invariant lines of the transformation.

3 Determine the fixed lines of the transformation $x' = \frac{1}{2}x + \frac{1}{4}y, \quad y' = x + \frac{1}{2}y$.

4 The transformation T is represented by the matrix $\begin{pmatrix}-\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{pmatrix}$. Find the equation of the line of invariant points.

5 Find in the form $y = mx + c$ the equations of all invariant lines of the transformation: $\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}7 & 24\\ 24 & -7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$.

6 Find the equations of any straight lines that pass through the origin and that map onto themselves under the transformation whose matrix is:

(a) $\begin{pmatrix}-5 & 2\\ -4 & 1\end{pmatrix}$ (b) $\begin{pmatrix}-3 & 3\\ 1 & -5\end{pmatrix}$ (c) $\begin{pmatrix}a & 1\\ 8 & a\end{pmatrix}$ (d) $\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$ (e) $\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$

(f) $\begin{pmatrix}3 & 2\\ 3 & 4\end{pmatrix}$

7 Find the equations of any straight lines that are invariant under the transformation defined by:

$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}-6 & 3\\ 4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}4\\ -1\end{pmatrix}$.

8 The point $\begin{pmatrix}x\\ y\end{pmatrix}$ is transformed by $\begin{pmatrix}x\\ y\end{pmatrix} :\longmapsto \begin{pmatrix}-2 & 2\\ -2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}X\\ Y\end{pmatrix}$.

Find the equations of lines which are mapped onto themselves.

9 A transformation T of a plane is given by $x' = 5 - 2y, \quad y' = 4 - 2x$. Find the invariant point and the invariant lines of T.

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10 A transformation of a plane is given by $x' = 2 - 2y$, $y' = 7 - 2x$. Find the invariant point and give that Cartesian equations of the two invariant lines.

11 Explain the difference between an invariant (fixed) line and a line of invariant points.

A transformation of a plane is given by the equations $x' = 7 - 2y$, $y' = 5 - 2x$.

(a) Calculate the coordinate of the invariant point.

(b) Determine the equations of the invariant lines of the transformation

12 Find the invariant lines of the transformation represented by the following matrices

(a) $\begin{pmatrix} 4 & 11 \\ 11 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$

Miscellaneous Exercise 1.2

1 The matrix $\begin{pmatrix} 3 & -2 \\ 4 & -6 \end{pmatrix}$ defines a transformation **M** of the (x, y) plane.

A triangle S has area 3 square units, and **M** transforms S to a triangle T.

(i) Find the area of T.

(ii) Find the matrix which transforms T to S.

(iii) Find the image of the line $y = 2x + 1$ under the transformation **M**.

(iv) Find the two values of m for which $y = mx$ is an invariant line under the transformation **M**. OCR

2 A matrix **D** is given by $\begin{pmatrix} 2 & a \\ b & -2 \end{pmatrix}$, where a and b are constants.

The matrix **D** maps the point P$(2, 5)$ onto the point Q$(-1, 2)$.

(a) Find the value of a and the value of b.

A triangle T_1 with an area of 9 square units is transformed by **D** into the triangle T_2.

(b) Find the area of T_2.

3 The 3x3 matrix **A** is defined in terms of the scalar constant c by

$$\begin{pmatrix} 2 & -1 & 3 \\ c & 2 & 4 \\ c-2 & 3 & c+7 \end{pmatrix}.$$

Given that $|\mathbf{A}| = 8$, find the possible values of c.

4 The 2 x 2 matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}$.

Find the 2 x 2 matrix **X** which satisfies the equation $\mathbf{AX} = \mathbf{B}$.

5 A matrix **M** is given by $\begin{pmatrix} a & 1 & 2 \\ 2 & -1 & a \\ 3 & a & 4 \end{pmatrix}$. Find the values of a for which **M** is singular.

6 The triangle T_1 is mapped by the 2 x 2 matrix $\begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}$ onto another triangle T_2, whose vertices have coordinates A_2 $(-1, 2)$, $B_2(10, 15)$ and C_2 $(-18, -14)$.

Find the coordinates of the vertices of T_1.

7 Find the image of the line $y = x + c$ under a transformation given by the matrix $\begin{pmatrix} 1 & 2 \\ 2 & -4 \end{pmatrix}$.

8 The 2× 2 matrices A and B are given below $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$.

The matrix C represents the combined effect of the transformation represented by **B**, followed by the transformation represented by **A** .

(i) Determine the elements of C.

(ii) Describe geometrically the transformation represented by C.

9 Give a geometrical description for the transformation represented by the matrix.
$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, stating the equation of the line of invariant points under this transformation.

10 Find the inverse of the matrix $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & -4 & 2 \\ 3 & 11 & 13 \end{pmatrix}$. Hence solve the following system of equations

$$x + 3y + 5z = -2$$
$$3 - 4y + 2z = 7$$
$$3x + 11y + 13z = 2.$$

11 The 2 x 2 matrices **X** and **B** are given by $\mathbf{X} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 19 & 36 \\ 8 & 15 \end{pmatrix}$.

Find the 2 x 2 matrix **A** which satisfies the equation $\mathbf{AX} = \mathbf{B}$.

12 A transformation T is represented by the matrix $\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}$. Find the equation of the line of invariant points which passes through the origin under the transformation T.

13 Find the image of the straight line with equation $2x + 3y = 10$, under the transformation represented by the 2× 2 matrix $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$.

14 Given matrix $\mathbf{C} = \begin{pmatrix} 1 & -2 & 2 \\ k & 1 & k-1 \\ 2 & 2k-1 & 2-k \end{pmatrix}$. Show that det **C** is independent of k.

15 Under the transformation represented by the 2 x 2 matrix $\begin{pmatrix} 1 & 2 \\ 4 & -7 \end{pmatrix}$ the straight line with equation $y = mx$ is reflected about the x-axis. Find the possible values of m.

16 Matrices **A** and **AB** are given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 4 & 2 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} -8 & 11 & 9 \\ -7 & 10 & 8 \\ -13 & 18 & 15 \end{pmatrix}$.

(i) Find the inverse of **AB**.

(ii) Hence determine the inverse of **B**.

17 The 2x2 matrix **A** maps $\mathbb{R}^2 \longmapsto \mathbb{R}^2$ and is given by $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$.

Determine the equation of any invariant straight line(s) under **A**.

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18 The straight line M is transformed by matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ into the straight line M' with equation $11x + 6y = 4$. Find a Cartesian equation of M.

19 The three by three matrices **A** and **B** are given by

$\mathbf{A} = \begin{pmatrix} 5 & 2 & 4 \\ 7 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 14 & -8 \\ -13 & -1 & 18 \\ 23 & -17 & 1 \end{pmatrix}$. Find an expression for **AB** and use it to solve the following system of linear equations.

$$5x + 2y + 4z = 10$$
$$7x + 3y + 2z = 21$$
$$4x + 5y + 3z = 5.$$

20 A plane transformation maps the general point (x, y) onto the general point (X , Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(i) Find the area scale factor of the transformation.

(ii) Determine the equation of the straight line of invariant points under this transformation.

21 Find the equation of the straight line of the invariant points under the transformation represented by $\mathbf{A} = \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$.

22 Find in the form $y = mx$ the equation of the line of invariant points under the transformation given by the matrix $\begin{pmatrix} 25 & 8 \\ 6 & 1 \end{pmatrix}$.

23 A transformation T, maps the general point (x, y) onto the general point (X , Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(i) Find the area scale factor of the transformation.

(ii) Determine the equation of the line of invariant points under this transformation.

24 Show that the transformation represented by the matrix $\begin{pmatrix} -2 & 2 \\ -2 & -3 \end{pmatrix}$ has no invariant lines.

25 The point $\begin{pmatrix} x \\ y \end{pmatrix}$ is mapped onto $\begin{pmatrix} X \\ Y \end{pmatrix}$ by the transformation $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$. Find the equations of the lines which pass through the origin which are mapped onto themselves when **M** is

(i) $\begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 \\ 5 & -4 \end{pmatrix}$

"Your mind is like a computer. What you put in is what you get out"

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