TOPIC 2: VECTORS

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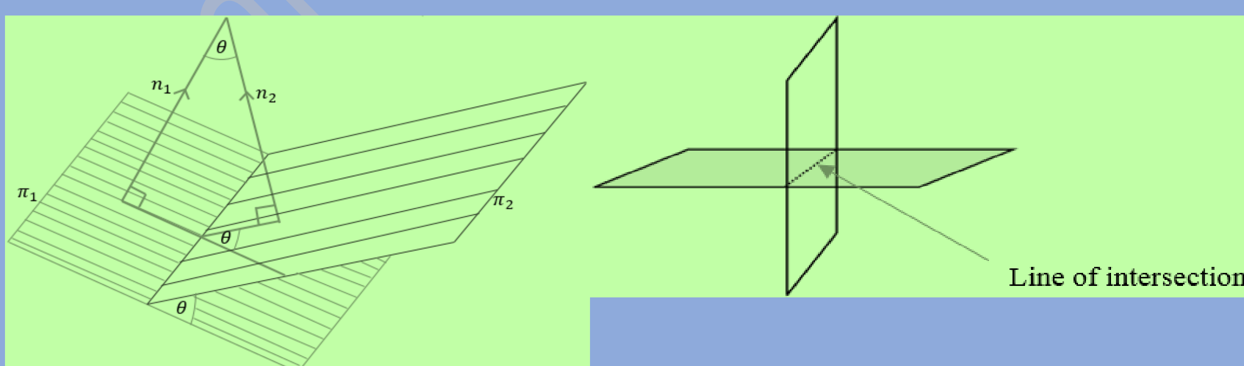
2.2 VECTORS

Learning Objectives

After completing this section you should be able to:

- find vector and Cartesian equations of a straight line in 3D
- find cross product of given vectors
- use cross product to find area of plane shapes
- find vector, parametric and Cartesian equations of a plane
- find the angle between a line and a plane, two lines and two planes
- determine whether two lines are parallel, intersecting or skewed
- determine whether two lines intersect
- find the perpendicular distance from a point to a straight line and the coordinates of the foot of the perpendicular
- determine whether a line lies in, is parallel to or intersects a plane
- find the point of intersection of a line and a plane
- find the line of intersection of two planes
- find the perpendicular distance from a point to a plane
- solve problems involving lines and planes



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2.2.1 Vector and Cartesian equation of line

The vector equation of a line is of the form

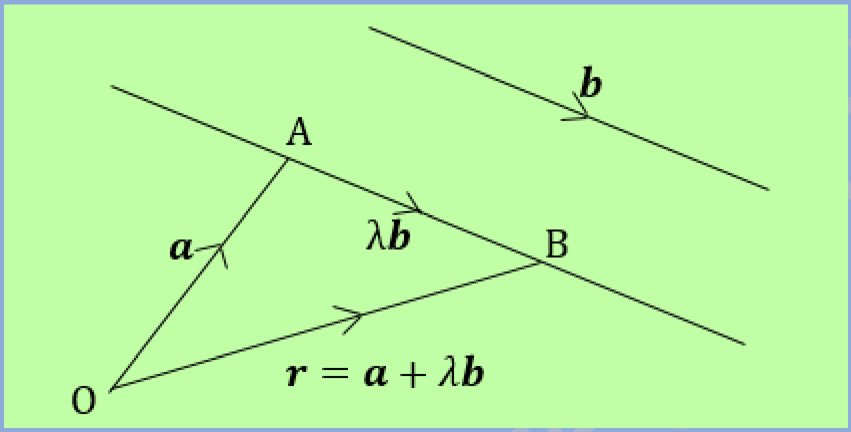
$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ where:

$\boldsymbol{a}$ is the position vector of a fixed point on the line

$\boldsymbol{b}$ is any vector parallel to the line (a direction vector) and λ is a parameter / free variable

Therefore the equation of a line passing through points A and B is

$$r = \overrightarrow{OA} + \lambda\overrightarrow{AB}$$



Example 1

Find the vector equation of a straight line which passes through the point $(1, 0, 3)$ and is parallel to the straight line through A$(-1,1,3)$ and B$(2, -3, 1)$.

Solution

Direction vector $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$ (NB can use $\overrightarrow{BA}$ instead)

Vector equation of the line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$

The above equation can also be written in cartesian form by eliminating λ as follows:

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$$\boldsymbol{r} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\-4\\-2\end{pmatrix} \quad \Longrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+3\lambda\\-4\lambda\\3-2\lambda\end{pmatrix}$$

$x = 1 + 3\lambda \;\Longrightarrow\; \lambda = \frac{x-1}{3}$

$y = -4\lambda \;\Longrightarrow\; \lambda = \frac{y}{-4}$

$z = 3 - 2\lambda \Longrightarrow \lambda = \frac{z-3}{-2}$

Thus the cartesian equation of the line is

$\frac{x-1}{3} = \frac{y}{-4} = \frac{z-3}{-2}$ $(=\lambda)$ (Also called symmetric form)

Example 2

A line has cartesian equations $\frac{x-1}{3} = \frac{y+2}{4} = \frac{z-4}{5}$

Find a vector equation for a parallel line passing through the point with position vector $5\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$ and find the coordinates of the point on this line where $y = 0$.

Solution

The direction vector for the line $\frac{x-1}{3} = \frac{y+2}{4} = \frac{z-4}{5}$ is $\begin{pmatrix}3\\4\\5\end{pmatrix}$. The required line has the same direction since the two are parallel.

Therefore the vector equation of the line is $\boldsymbol{r} = \begin{pmatrix}5\\-2\\-4\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\5\end{pmatrix}$.

The above line can be written as $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\-2\\-4\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\5\end{pmatrix}$.

When $y = 0, \Longrightarrow 0 = -2 + 4\lambda$

$\Longrightarrow \lambda = \frac{1}{2}$

$\therefore x = 5 + \frac{1}{2}(3) = \frac{13}{2}$

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$z = -4 + \frac{1}{2}(5) = -\frac{3}{2}$.

Thus the coordinates of the point on this line where $y = 0$ are $\left(\frac{13}{2}, 0, -\frac{3}{2}\right)$

If we want to find an equation of a line, we look for one point on the line and a vector PARALLEL to the line.

Exercise 2.2.1

1 Given the equation of the line in the form $\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2}$. Express the equation in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ and show that the line passes through the point $(8, 14, 11)$.

2 Find the equation of the line through the points A(1, 2, 3) and B(4, 4, 4) and find the coordinates of the point where this line meets the plane $z = 0$.

3 Write down a vector equation for the line through A and B if

(a) $\overrightarrow{OA} = 3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$ and $\overrightarrow{OB} = \boldsymbol{i} + 7\boldsymbol{j} + 8\boldsymbol{k}$

(b) A and B have coordinates $(1, 1, 7)$ and $(3, 4, 1)$

Find in each case the coordinates of the points where the line crosses the xy plane, the yz plane and the zx plane.

4 Convert the following vector equations to cartesian form

(a) $r = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (b) $r = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ (c) $r = \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

5 Convert the following equations to vector form,

(a) $\frac{x-3}{4} = \frac{y-1}{2} = \frac{z-7}{6}$ (b) $\frac{x-3}{2} = \frac{y+2}{3} = \frac{5-z}{7}$

(c) $x = 3\lambda + 2,\ y = \lambda - 5,\ z = 4\lambda + 1$ (d) $\frac{1-x}{3} = \frac{y}{5} = z$

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2.2.2 Parallel Lines

Two lines are parallel if they have the same direction vector.

Example 1

Determine whether the following pairs of lines are parallel

(a) $r_1 = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + \lambda(2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})$ $\quad r_2 = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + \lambda(3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

(b) $r_1 = 2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k} + s(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$ $\quad r_2 = (3+t)\boldsymbol{i} + (t-1)\boldsymbol{j} + (5-t)\boldsymbol{k}$

Solution

(a) The direction vector for r_1 is $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and that of r_2 is $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

The lines are not parallel since they do not have the same direction vector.

(b) $r_1 = 2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k} + s(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$ and the direction vector is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$r_2 = (3+t)\boldsymbol{i} + (t-1)\boldsymbol{j} + (5-t)\boldsymbol{k}$ can be written as

$\boldsymbol{r_2} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and the direction vector is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

The two lines are parallel since they have the same direction vector.

Exercise 2.2.2

Determine whether the following pairs of lines are parallel

1 $r_1 = 2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k} + \lambda(2\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$ $\quad r_2 = \lambda(\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$

2 $r_1 = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \lambda(2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})$ $\quad r_2 = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k} + \lambda(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$

3 $\frac{x-1}{2} = \frac{y-4}{3} = \frac{z+1}{-4}$ $\quad \frac{x}{4} = \frac{y+5}{6} = \frac{3-z}{8}$

4 $r_1 = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k} + \lambda(\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$ $\quad x - 4 = y + 7 = \frac{z}{3}$

5 $r_1 = \lambda(3\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k})$ $\quad r_2 = 4\boldsymbol{j} + \lambda(-\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

6 $r_1 = 3\boldsymbol{i} + \boldsymbol{k} + \lambda(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$ $\quad \frac{x-3}{1} = \frac{y}{1} = \frac{z-1}{2}$

2.2.3 Intersecting lines

Two lines intersect if the value of the parameters is constant.

To find the point of intersection of two line expressed in vector form, equate both vectors and find the value of the parameters and substitute in either equation.

Example 1

Determine whether the lines r_1 and r_2 intersect and find the point of intersection where $r_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ and $r_2 = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$

Solution

At intersection $r_1 = r_2$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 2-\lambda \\ 1-\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 3+2\mu \\ 0 \\ 5+6\mu \end{pmatrix}$$

$2 - \lambda = 3 + 2\mu$ ………..… (i)

$1 - \lambda = 0$ ………………. (ii)

$5 + 6\mu = 5 + 6\mu$…...……. (iii)

The solution to the above equations is $\lambda = 1, \mu = -1$

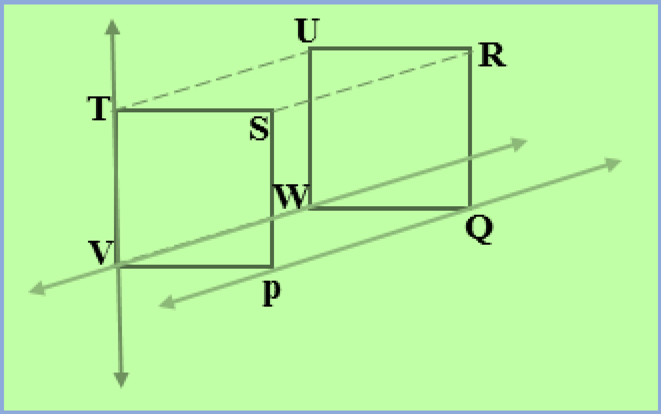
Since there is a constant value for each parameter, the lines do intersect.

Substituting $\mu = -1$ in r_2 gives the point of intersection as $\begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} + -1 \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Which is $(1, 0, -1)$

2.2.4 Skew lines

If lines are neither parallel nor intersecting, they are skew.



TV and PQ are skew lines.

If the lines are not parallel and there are no values of the parameters for which $r_1 = r_2$, then the lines r_1 and r_2 are skew.

Example 1

Show that the lines $r_1 = \boldsymbol{i} + 3\boldsymbol{k} + s(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ and $r_2 = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j})$ are skew.

Solution

First check if the lines are not parallel.

Direction for r_1 is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. And the direction for r_2 is $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ which are different. Hence the lines are not parallel.

$r_1 = \begin{pmatrix} 1+2s \\ s \\ 3+s \end{pmatrix}$ and $r_2 = \begin{pmatrix} 2+t \\ -1-2t \\ 1 \end{pmatrix}$

Taking $r_1 = r_2$

$$\begin{pmatrix} 1+2s \\ s \\ 3+s \end{pmatrix} = \begin{pmatrix} 2+t \\ -1-2t \\ 1 \end{pmatrix}$$

$1 + 2s = 2 + t$………………. (i)

$s = -1 - 2t$ ……………...... (ii)

$3 + s = 1$ …………………... (iii)

From the first two equations, $s = \frac{1}{5}$ and $t = -\frac{3}{5}$

The value $s = \frac{1}{5}$ does not satisfy the third equation and so there is no unique solution for the system of equations. Hence the lines are skew.

Exercise 2.2.4

1 The straight line l_1 passes through the points with coordinates $(5,1,6)$ and $(2,2,1)$.

(a) Find a vector equation of l_1.

A different straight line l_2 passes though the point $\mathrm{C}(6,6,-4)$ and is parallel to the vector $4\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$.

(b) Show clearly that l_1 and l_2 are skew.

2 Find whether the following lines are parallel, intersecting or skew.

(a) $r_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ $\quad r_2 = \mu \begin{pmatrix} -9 \\ 12 \\ -3 \end{pmatrix}$

(b) $\frac{x-4}{1} = \frac{y-8}{2} = \frac{z-3}{1}$ $\quad \frac{x-7}{6} = \frac{y-6}{4} = \frac{z-5}{5}$

3 The straight line l_1 passes through the points $\mathrm{A}(2,5,9)$ and $\mathrm{B}(6,0,10)$.

(a) Find a vector equation for the straight line l_1.

The straight line l_2 has vector equation $\boldsymbol{r} = \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ where μ is a scalar.

(b) Show that the point A is the intersection of l_1 and l_2.

(c) Show further that l_1 and l_2 are perpendicular to each other.

4 Two lines have equation $r_1 = 2\boldsymbol{i} + 9\boldsymbol{j} + 13\boldsymbol{k} + \lambda(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$ and $r_2 = a\boldsymbol{i} + 7\boldsymbol{j} - 2\boldsymbol{k} + \mu(-\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$. Given that they intersect, find the value of a and the position vector of the point of intersection.

5 Relative to a fixed origin O, the points A and B have respective position vectors

$\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}$ and $5\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}$.

(a) Find a vector equation of the straight line l_1 which passes through A and B.

The straight line l_2 has vector equation $r_2 = 5\boldsymbol{i} - 4\boldsymbol{j} + 4\boldsymbol{k} + \mu(\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k})$,

where μ is a scalar parameter.

The point C is the point of intersection between l_1 and l_2.

(b) Find the position vector of C.

(c) Show that C is the midpoint of AB.

6 If the line $r_1 = 5\boldsymbol{i} + t\boldsymbol{k} + \lambda(-\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k})$ intersects the line $r_2 = -4\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + \mu(5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$, find the value of t and the coordinates of the point of intersection.

7 A line L has equation $r_1 = -3\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k} + \lambda(4\boldsymbol{i} + p\boldsymbol{j} - 3\boldsymbol{k}$. Find the value of p if

(a) The line $r_2 = 5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k} + \mu(4\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ is parallel to L.

(b) The line $r_3 = -\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ intersects L.

8 The equations of two lines are $r_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $r_2 = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$.

Show that these lines are skew.

$\overrightarrow{OQ}$ is the unit vector in the direction of the first line and $\overrightarrow{OR}$ is the unit vector in the direction of the second line. Write down the coordinates of Q and R.

9 Show that the lines $r = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + m \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + n \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ do not meet (are skew).

10 Find the point of intersection of the lines
$r = (1 + m)\boldsymbol{i} + (2 + m)\boldsymbol{j} + (4 + 2m)\boldsymbol{k}$ and
$r = (1 + 3n)\boldsymbol{i} + 5n\boldsymbol{j} + (3 + 7n)\boldsymbol{k}$

11 Show that the equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + m \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \\ -3 \end{pmatrix} + n \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

represent the same line.

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2.2.5 Angle between two lines

The angle between lines is the angle between their direction vectors.

Example

Find the angle between lines

$r_1 = \begin{pmatrix}2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$ and $r_2 = \begin{pmatrix}3\\0\\5\end{pmatrix} + \mu\begin{pmatrix}2\\0\\6\end{pmatrix}$

Solution

Let the angle be θ

$$\cos\theta = \frac{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}.\begin{pmatrix}2\\0\\6\end{pmatrix}}{\sqrt{(-1)^2+(-1)^2+(-1)^2}.\sqrt{2^2+6^2}}$$

$$= \frac{-2+0-6}{\sqrt{3}\sqrt{40}}$$

$$= \frac{-8}{\sqrt{120}}$$

$$\theta = cos^{-1}\left(\frac{-8}{\sqrt{120}}\right)$$

$$= 152°$$

Exercise 2.2.5

1 Find the angle between the lines

(a) $r_1 = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k} + \lambda(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$ and $r_2 = 5\boldsymbol{i} - 2\boldsymbol{k} - \mu(3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$

(b) $\frac{x+4}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

(c) $\frac{x-2}{3} = \frac{y+1}{-2} = z = 2$ and $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$

(d) $r_1 = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k} + \lambda(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ and $r_2 = 5\boldsymbol{i} - 2\boldsymbol{k} - \mu(3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$

(e) $\frac{x+4}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

(f) $\frac{x-2}{3} = \frac{y+1}{-2} = z = 2$ and $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$

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(g) $r = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $r = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$

2 With respect to a fixed origin O, the points A$(2,6,5)$ and B$(5,0,-4)$ are given.

(a) Find a vector equation of the straight line l_1, which passes through A and B .The straight line l_2 has a vector equation $r_2 = \begin{pmatrix} -4 \\ 4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, where μ is a scalar parameter.

(b) Show clearly that l_1 and l_2 intersect, and find the coordinates of their point of intersection. The straight line l_3 is in the direction $\begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}$.

(c) Given the acute angle between l_2 and l_3 is 60°, show clearly that $k = \pm\sqrt{6}0$

3 The straight lines l_1 and l_2 have the following vector equations

$l_1 = 4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + \lambda\,(\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$

$l_2 = 8\boldsymbol{i} + 8\boldsymbol{j} + 13\boldsymbol{k} + \mu\,(2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k})$, where λ and μ are scalars.

(a) Show that l_1 and l_2 intersect at some point P and find its coordinates.

(b) Calculate the acute angle between l_1 and l_2.

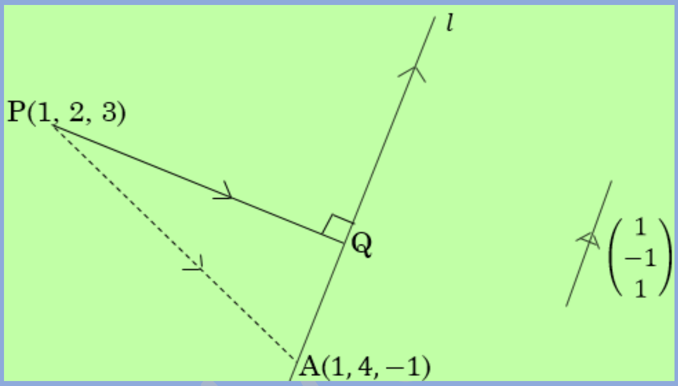
2.2.6 Perpendicular distance from a point to a straight line and the coordinates of the foot of the perpendicular

Example

Find the shortest distance from the point $P(1, 2, 3)$ to the line l with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find the coordinates of the foot of the perpendicular from P to l.

Solution

Consider the following diagram



Let point Q be the foot of the perpendicular from P. We can calculate this point by knowing that the vector $\overrightarrow{PQ}$ is perpendicular to the line.

$$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$$

$$\overrightarrow{PQ} = \begin{pmatrix}0\\2\\-4\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}.$$

$$\overrightarrow{PQ} = \begin{pmatrix}\lambda\\2-\lambda\\-4+\lambda\end{pmatrix}$$

Now, $\begin{pmatrix}1\\-1\\1\end{pmatrix}.\overrightarrow{PQ} = 0$

$$\Rightarrow \begin{pmatrix}1\\-1\\1\end{pmatrix}.\begin{pmatrix}\lambda\\2-\lambda\\-4+\lambda\end{pmatrix} = 0$$

$$\lambda - 2 + \lambda - 4 + \lambda = 0$$

$$\lambda = 2$$

Substituting λ into $\overrightarrow{PQ}$ gives $\overrightarrow{PQ} = \begin{pmatrix}2\\0\\-2\end{pmatrix}$

Shortest distance is given by $\left|\overrightarrow{PQ}\right| = \sqrt{2^2 + 0^2 + 2^2} = 2\sqrt{2}$

It is known that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$\Rightarrow \begin{pmatrix}2\\0\\-2\end{pmatrix} = \overrightarrow{OQ} - \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\Rightarrow \overrightarrow{OQ} = \begin{pmatrix}2\\0\\-2\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

Thus the coordinates of the foot of the perpendicular are $Q(3, 2, 1)$.

Exercise 2.2.6

Angle between lines. Perpendicular distance from a point P to a line. Coordinates of the foot of the perpendicular

4 Find the perpendicular distance from the point P to the given line. Find the coordinates of the foot of the perpendicular from P to the given line.

(a) P$(3,5,2)$ $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ $\left[\lambda = \frac{11}{18}, \frac{1}{35}\sqrt{2358} = 2.70\ units\right]$

(b) P$(1,1,2)$ $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ $\left[t = -\frac{16}{35}, \frac{3}{35}\sqrt{910} = 2.59\ units\ Q(\frac{124}{35}, \frac{22}{35}, \frac{12}{7})\right]$

(c) P$(-1,2,1)$ $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ $\left[\frac{\sqrt{69}}{\sqrt{14}}\right]$

(d) P$(2,-1,1)$ $\boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(e) P$(3,5,6)$ $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$ $\left[\frac{\sqrt{4922}}{107}\right]$

(f) P$(1,0,1)$ $\boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

(g) P$(3,5,6)$ $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$

(h) P$(1,-2,1)$ $x = 1 + 2t,\ y = 3t,\ z = 1 + 2t$

(i) P$(1,0,-1)$ $x = 5 - t,\ y = 2 - 3t,\ z = 4t$

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2.2.7 Vector Cross product

Given two vectors $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$, the vector cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted $\boldsymbol{u}$ x $\boldsymbol{v}$ is given by

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example 1

Find the vector cross product of the following pair of vectors

$\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$

Solution

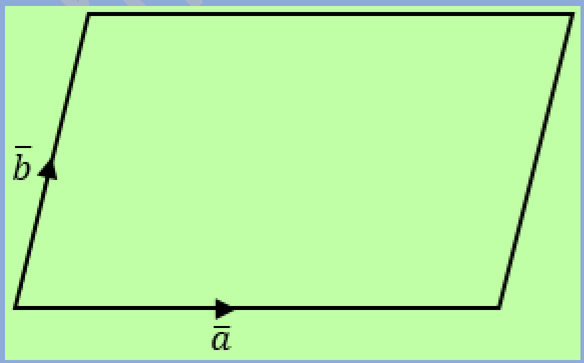
$$\boldsymbol{u} \text{ x } \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -3 & 0 \\ 1 & -1 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 0 \\ -1 & 5 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix}\boldsymbol{k}$$

$$= -15\boldsymbol{i} - 10\boldsymbol{j} + \boldsymbol{k}$$

Using cross product to find area of plane shapes

Two nonzero and nonparallel vectors $\bar{a}$ and $\bar{b}$ can be considered to be the sides of a parallelogram



Area of a parallelogram with sides $\bar{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\bar{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is given by

Area = $|\bar{a} \times \bar{b}|$ where $\bar{a} \times \bar{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Example 2

The points P(1, 2, 1), Q(1, 0, 0) and R(0, 3, 1) are three vertices of a parallelogram.
(a) Find the area of the parallelogram

(b) What is the area of triangle PQR?

Solution

(a) Parallelogram Area = $|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} \qquad \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -2 & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

∴ Parallelogram Area = $|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$ units^2

(b) Area of triangle PQR = $\frac{1}{2}$(Area of parallelogram)

$$= \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$$

$$= \frac{\sqrt{6}}{2} \text{ Units}^2$$

Exercise 2.2.7

1 Find the area of the triangle with vertices at

(a) $\mathrm{A}(-2,-2,0)$, $\mathrm{B}(6,8,6)$, $\mathrm{C}(-6,8,12)$. (b) $\mathrm{A}(1,-1,2)$, $\mathrm{B}(-1,2,1)$, $\mathrm{C}(2,-3,3)$.

(c) $\mathrm{A}(2,1,1)$, $\mathrm{B}(-1,0,4)$, $\mathrm{C}(3,-1,-1)$. (c) $(1,1,3), (4,-1,1), (0,1,8)$

2.2.8 Equation of a plane

The symbol π is usually used to denote a plane. The equation of a plane in Cartesian form is $ax + by + cz = d$ and the vector form is $(\boldsymbol{r} - \boldsymbol{a}).\hat{n} = 0$, where
$\hat{n}$ is a vector normal to the plane. This equation simplifies to $\boldsymbol{r}.\hat{n} = d$ where d is a scalar.

The Cartesian equation form of a plane can also be expressed in vector form and vice versa as follows:

$3x - 2y + z = 4$ can be expressed as $r.\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 4$ and

$r.\begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} = 2$ can be expressed as $5x - 3y + 4z = 2$

If we want an equation of a plane, we look for one point on the plane and a vector PERPENDICULAR to the plane.

The vector cross product can be used to create a vector normal to a plane.

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Example

(a) A plane passes through three points A, B and C with the following position vectors: $\boldsymbol{a} = i + 3k$, $\boldsymbol{b} = 2i + 2j + 5k$, $\boldsymbol{c} = i - j + k$.
Find a vector normal to the plane ABC and hence find the equation of the plane.

(b) Find the equation of the plane through the point $(1,1,1)$ parallel to the vectors $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{j}$

Solution

(a) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = i + 2j + 2k \quad \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = -j - 2k$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 2 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 \\ -1 & -2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \boldsymbol{k}$$

$$= -2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

Thus the normal to the plane is $\hat{n} = -2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

It should be noted that $\hat{n}$ is perpendicular to each of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, meaning that $\overrightarrow{AB}.\hat{n} = \overrightarrow{AC}.\hat{n} = 0$.

The equation of the plane is found using the formula $\quad \boldsymbol{r}.\hat{n} = \boldsymbol{a}.\hat{n}$

$$\boldsymbol{r}.\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$$

$$\boldsymbol{r}.\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = -5$$

In Cartesian form the equation of the plane is $-2x + 2y - z = -5$

(b) We first find the normal vector using the vectors $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{j}$

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$$\text{Normal vector} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -3\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$

Using $\boldsymbol{r}.\hat{n} = \boldsymbol{a}.\hat{n}$, the equation of the plane is

$$\boldsymbol{r}.\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix}$$

$$\boldsymbol{r}.\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} = -1$$

Cartesian equation of the plane $3x - 3y + z = 1$

Exercise 2.2.8

1 Write down the Cartesian equation of the plane whose vector equation is

(a) $\boldsymbol{r}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=2$ (b) $\boldsymbol{r}.(2\boldsymbol{i}+3\boldsymbol{j}-4\boldsymbol{k})=0$ (c) $\boldsymbol{r}.\begin{pmatrix}1\\1\\-1\end{pmatrix}=2$

2 Write down the vector equation of the plane whose Cartesian equation is

(a) $3x-4y+5x=6$ (b) $x-7y-z=5$

3 Find the equation of the plane which passes through the points

(a) $\mathrm{A}(5,2,2)$, $\mathrm{B}(-1,2,1)$ and $\mathrm{C}(3,-2,-2)$ (b) $\mathrm{P}(0,0,-1)$, $\mathrm{Q}(1,3,0)$ and $\mathrm{R}(0,2,1)$

(b) A, B and C with position vectors $\boldsymbol{i}+\boldsymbol{j}$, $\boldsymbol{i}+\boldsymbol{k}$ and $\boldsymbol{j}-3\boldsymbol{k}$ respectively relative to the origin O.
(c) with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ where $\boldsymbol{a}=\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}, \boldsymbol{b}=2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$
$\boldsymbol{c}=-\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$ in scalar product form and in scalar product form.

4 Of the following equations, which represent lines and which represent planes?

(a) $\frac{x-2}{1}=\frac{y-1}{2}=\frac{z-3}{-1}$ (b) $x+2y-z=1$ (c) $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

Describe, or show in a clear diagram, how these lines and planes are related to each other.

5 Determine a cartesian equation of the plane that contains the point $\mathrm{A}(9,-1,0)$ and the straight line with vector equation $r = 5\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} + \mu(\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k})$, where μ is a scalar parameter.

6 A plane goes through the points whose position vectors are $i - 2j + k$ and $2i - j - k$ and is parallel to the line with equation $r = i - j + \lambda(3i + j - 2k)$

Find the equation of the plane.

7 Find the vector equation of the plane which goes through the origin and which contains the line $r = 2\boldsymbol{i} + \lambda(\boldsymbol{j} + \boldsymbol{k})$

8 Find a Cartesian equation of the plane that contains the parallel straight lines with vector equations $r_1 = 2\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ and $r_2 = 3\boldsymbol{i} - \boldsymbol{j} + 6\boldsymbol{k} + \mu(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ where λ and μ are scalar parameters.

9 Two planes π_1 and π_2 have vector equations $r.\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 3$ and $r.\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 9$

Explain why the planes are parallel.

10 Find a Cartesian equation of the plane that contains the intersecting straight lines with vector equations $r_1 = 6\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$ and $r_2 = 6\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k} + \mu(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ where λ and μ are scalar parameters.

11 Find the equation of the plane containing the line $r = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and passing through the point $(1,0,3)$

12 Find the vector equation of the plane which goes through the point $(0, 1, 6)$ and is parallel to the plane $\boldsymbol{r}.(\boldsymbol{i} - 2\boldsymbol{j}) = 3$

14 Find the equation of the line through the points $(2, 3, 7)$ and $(3, 1, 4)$. Find also the equation of the plane perpendicular to this line which passes through the origin.

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2.2.9 Vector Equation of a plane

The vector equation of a plane can also occur in the form

$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$ where

$\boldsymbol{r}$ is any point on the plane
$\boldsymbol{a}$ is a known point on the plane
$\boldsymbol{b}$ and $\boldsymbol{c}$ any two vectors in the plane
λ and μ are parameters/free variables

Notice the difference between a plane equation and a line equation; a plane equation has two parameters while a line equation only has one.
If we introduce the normal vector $\hat{n}$ of the plane into the plane equation (as it is perpendicular to all vectors in the plane)

$\boldsymbol{r}.\hat{n} = (\boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}).\hat{n} = \boldsymbol{a}.\hat{n} + \lambda(\boldsymbol{b}.\hat{n}) + \mu(\boldsymbol{c}.\hat{n})$
Since $\hat{n}$ is perpendicular to all vectors in the plane $\boldsymbol{b}.\hat{n} = \boldsymbol{c}.\hat{n} = 0$, so we have

$\boldsymbol{r}.\hat{n} = \boldsymbol{a}.\hat{n}$
$\Longrightarrow \boldsymbol{r}.\hat{n} = d$ where d is a scalar.

Example
Convert the following vector equation of a plane to Cartesian form

$\boldsymbol{r} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) + \mu(2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$

Solution

$$\hat{n} = (\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \text{ x } (2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}) = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 5\boldsymbol{i} - 6\boldsymbol{j} - \boldsymbol{k}$$

$$r.\hat{n} = a.\hat{n} \Longrightarrow \boldsymbol{r}.\begin{pmatrix} 5 \\ -6 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}.\begin{pmatrix} 5 \\ -6 \\ -1 \end{pmatrix}$$

$$\boldsymbol{r}.\begin{pmatrix} 5 \\ -6 \\ -1 \end{pmatrix} = 29$$

Cartesian form of the plane is $5x - 6y - z = 29$

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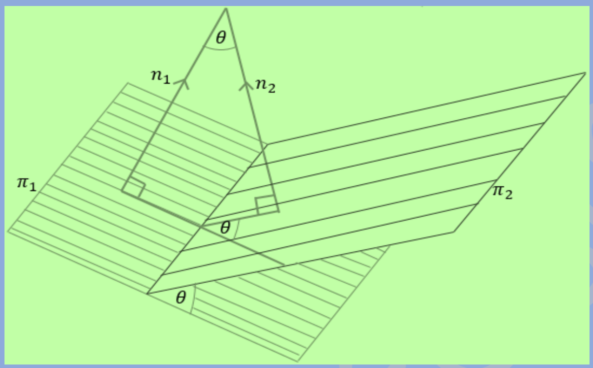
Finding the angle between two planes, and angle between a line and a plane.

Angle between two planes

The angle between two planes is the same as the angle between their normals.

Thus if $r.n_1 = a_1$ and $r.n_2 = a_2$ are equations of two planes, then the angle θ between them is given by

$$\boxed{\cos\theta = \frac{n_1.n_2}{|n_1||n_2|}}$$



Example

Find the acute angle between the two planes

(i) $2x + 2 - 3z = 3$ and $x + 3y - 4z = 6$

(ii) $r.\begin{pmatrix}1\\1\\-2\end{pmatrix} = 3$ and $r.\begin{pmatrix}2\\-2\\1\end{pmatrix} = 2$

Solution

(i) Let the angle be θ

$$\cos\theta = \frac{\begin{pmatrix}2\\2\\-3\end{pmatrix}.\begin{pmatrix}1\\3\\-4\end{pmatrix}}{\sqrt{2^2+2^2+(-3)^2}\sqrt{1^2+3^2+(-4)^2}} = \frac{20}{\sqrt{442}}$$

$$\theta = cos^{-1}\left(\frac{20}{\sqrt{442}}\right) = 18.0°$$

(ii) Let the angle between the planes be β

$$\cos\beta = \frac{\begin{pmatrix}1\\1\\-2\end{pmatrix}.\begin{pmatrix}2\\-2\\1\end{pmatrix}}{\sqrt{1^2+1^2+(-2)^2}\sqrt{2^2+(-2)^2+1^2}} = \frac{-2}{\sqrt{54}} = -\frac{\sqrt{6}}{9}$$

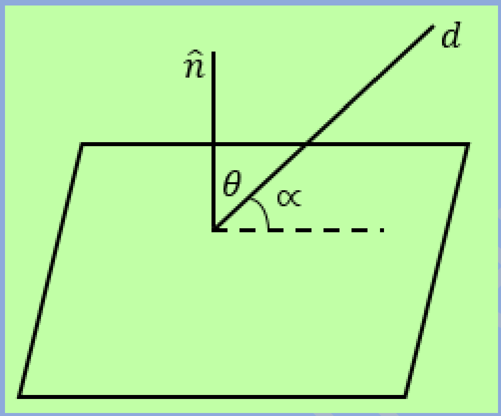
The negative sign indicates that we have found the obtuse angle between the planes.

$$\beta = cos^{-1}\left(-\frac{\sqrt{6}}{9}\right) = 105.8°$$

Thus the acute angle between the planes is $180° - 105.8° = 74.2°$

Angle between a line and a plane

To find the angle between a line and a plane we have to first find the angle between the normal of the plane and the direction vector of the line.



$$\cos\theta = \frac{\hat{n}.d}{|\hat{n}||d|}$$

The angle between the line and the plane is given by $\propto = 90° - \theta$

Alternatively we can directly find the angle $\propto$ by using

$$\sin\ \propto = \frac{\hat{n}.d}{|\hat{n}||d|}$$

Example 1

The plane π and the line l have equations

$$\boldsymbol{r}.\begin{pmatrix}2\\-1\\-1\end{pmatrix}=6 \text{ and } \frac{x-2}{1}=\frac{y-3}{2}=\frac{z+1}{-2} \text{ respectively.}$$

Find the angle between π and l. (Zimsec J2018 p2)

Solution

Let the angle be θ

$$\sin\theta=\frac{\begin{pmatrix}2\\-1\\-1\end{pmatrix}.\begin{pmatrix}1\\2\\-2\end{pmatrix}}{\sqrt{2^2+(-1)^2+(-1)^2}\sqrt{1^2+2^2+(-2)^2}}$$

$$=\frac{1}{3}$$

$$\theta=sin^{-1}\left(\frac{1}{3}\right)=19.5°$$

Example 2

Find the acute angle between the line l with equation $\boldsymbol{r}=\boldsymbol{i}+5\boldsymbol{j}-\boldsymbol{k}+t(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$ and the plane π with equation
$\boldsymbol{r}=5\boldsymbol{i}+5\boldsymbol{k}+\boldsymbol{\lambda}(2\boldsymbol{i}+\boldsymbol{j})+\mu(2\boldsymbol{i}+\boldsymbol{j}+4\boldsymbol{k})$. (Zimsec J2014)

Solution

The normal to the plane is given by $(2\boldsymbol{i}+\boldsymbol{j})$ x $(2\boldsymbol{i}+\boldsymbol{j}+4\boldsymbol{k})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&1&0\\2&1&4\end{vmatrix}=4\boldsymbol{i}-8\boldsymbol{j}$ which can be reduced to $2\boldsymbol{i}-4\boldsymbol{j}$.

Thus the angle between the line and the plane is given by

$$sin^{-1}\left(\frac{\begin{pmatrix}1\\-1\\1\end{pmatrix}.\begin{pmatrix}2\\-4\\0\end{pmatrix}}{\sqrt{1^2+(-1)^2+1^2}\sqrt{2^2+4^2}}\right)=sin^{-1}\left(\frac{6}{\sqrt{60}}\right)=50.8°$$

Exercise 2.2.9

1 Express the following vector equations of planes in cartesian form

(a) $\boldsymbol{r} = \boldsymbol{i} - \boldsymbol{j} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \mu(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$

(b) $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) + t(-\boldsymbol{i} + 2\boldsymbol{k})$

(c) $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{k} + \lambda(\boldsymbol{i}) + \mu(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$

(d) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + m\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + n\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

(e) $\boldsymbol{r} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

(f) $\boldsymbol{r} = (1 + s - t)\boldsymbol{i} + (2 - s)\boldsymbol{j} + (3 - 2s + 2t)\boldsymbol{k})$

(g) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix}$

2 Write the parametric and scalar equations of the plane whose vector equation is

$$r = \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}.$$

3 Find the acute angle between the following pairs of planes

(a) $2x - 4y + 6z = 3 \qquad 8x + 10y + 3z = 4$

(b) $3x - 4y + 4z = 5 \qquad 2x + 2y + 7z = 8$

(c) $r.\begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} = 4 \qquad r.\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 7$

(d) $r.\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 10 \qquad r.\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 0$

(e) $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) + t(-\boldsymbol{i} + 2\boldsymbol{k}) \qquad \boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{k} + \lambda(\boldsymbol{i}) + \mu(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$

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4 Find the acute angle between

(a) the line $\frac{x+2}{2} = \frac{y}{1} = \frac{z-3}{-1}$ and the plane $r.\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = 5$,

(b) the line $r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the plane $r.\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 4$,

(c) the line $x - 2 = \frac{1}{2}(y+1) = \frac{1}{2}(z+3)$ and the plane $2x - y - 2z = 4$.

(d) the line $\boldsymbol{r} = \boldsymbol{i} - \boldsymbol{j} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ and the plane $\boldsymbol{r}.(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) = 4$

5 Find the value of the constant a if the angle between the plane $\boldsymbol{r}.(\boldsymbol{i} + \boldsymbol{j}) = 7$ and the line $\boldsymbol{r} = (3\boldsymbol{i} - \boldsymbol{k}) + \mu(a\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$ is

(a) $60°$ (b) $45°$ (c) $30°$

6 Find the sine of the angle between the line and the plane given by

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

7 Show that the cosine of the angle between the line $\frac{x-2}{2} = \frac{y+2}{2} = z - 5$ and the plane $3x + 2y - 6z = 8$ is $\frac{5\sqrt{17}}{21}$.

8 Find the size of the acute angle between the plane and the straight line with equations

(a) $2x - 2y + z = 12$ and $r = 7\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} + \lambda(2\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k})$,

(b) $2x - 2y - z = 2$ and $r = 2\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(3\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k})$

(c) $3x - 2y + z = 5$ and $r = -3\boldsymbol{i} - \boldsymbol{2j} + \boldsymbol{3k} + \lambda(2i - 3\boldsymbol{j} + 4\boldsymbol{k})$

9 Find the size of the acute angle formed by the planes with Cartesian equations

(a) $4x + 4y - 7z = 13$ and $7x - 4y + 4z = 6$

(b) $4x + 5y + 3z = 82$ and $-2x + 5y + 6z = 124$.

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2.2.10 Determining whether a line lies in, is parallel to or intersects a plane

How to show that a line lies in a plane

If the equation of a line satisfies the equation of a plane, irrespective of the value of the parameter, the line is said to lie on the plane.

Example

(a) Show that the line l having equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ lies on the plane π with equation $\boldsymbol{r}.\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5$

(b) The plane p has equation $3x + 2y - z + 1 = 0$ and the line l has Cartesian equation $\frac{x-4}{-1} = \frac{y+3}{2} = \frac{z-7}{1}$. Show that l lies in p

Solution

(a) l can be written as $\boldsymbol{r} = \begin{pmatrix} 2 + 3\lambda \\ 1 \\ 2\lambda \end{pmatrix}$

Substituting in the equation of the plane we have

$$\begin{pmatrix} 2 + 3\lambda \\ 1 \\ 2\lambda \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 4 + 6\lambda + 1 - 6\lambda$$

$$= 5 \text{ (RHS)}$$

Hence l lies in π.

(b) The scalar product equation of the plane p is $r.\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = -1$

The vector equation of line l is $r = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ ie $r = \begin{pmatrix} 4 - \lambda \\ -3 + 2\lambda \\ 7 + \lambda \end{pmatrix}$

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Substituting in p we have

$$\begin{pmatrix} 4-\lambda \\ -3+2\lambda \\ 7+\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 12 - 3\lambda - 6 + 4\lambda - 7 - \lambda$$

$$= -1 \text{ (RHS)}$$

Hence l lies in p.

How to show that a line is parallel to a plane?

If a line l is parallel to a plane p then the direction vector d of l is perpendicular to the normal, $\hat{n}$ of p.

$$\boxed{d.\hat{n} = 0}$$

Example

(a) The points A, B and C have position vectors

$\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$, $2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, and $2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, respectively, with respect to the origin O.

Show that BC is parallel to the plane π having equation $r.\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 9$.

(b) Find the value of a for which the line $r = 3\boldsymbol{i} - \boldsymbol{k} + \lambda(a\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$ is parallel to the plane $r.(\boldsymbol{i} - 4\boldsymbol{j}) = 7$

Solution

(a) Direction of line BC is $\overrightarrow{BC} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Now, $\begin{pmatrix}0\\1\\-1\end{pmatrix}.\begin{pmatrix}1\\2\\2\end{pmatrix} = 0(1) + 1(2) + (-1)(2)$

$= 0.$

Thus BC is parallel to the plane π.

(b) If the line and the plane are parallel then $\begin{pmatrix}a\\1\\-1\end{pmatrix}.\begin{pmatrix}1\\-4\\0\end{pmatrix} = 0$

$\Longrightarrow a(1) + 1(-4) + (-1)(0) = 0$

$a - 4 = 0$

$a = 4$

Finding the point of intersection of a line and a plane.

Example

The plane p has equation $3x - y - z = 2$ and the line l_2 has equation

$r = \begin{pmatrix}1\\0\\8\end{pmatrix} + \lambda\begin{pmatrix}1\\-3\\-1\end{pmatrix}$. Show that p and l_2 intersect and find the coordinates of the point of intersection.

Solution

The plane p has vector equation $r.\begin{pmatrix}3\\-1\\-1\end{pmatrix} = 2$

$l_2 : r = \begin{pmatrix}1+\lambda\\-3\lambda\\8-\lambda\end{pmatrix}$

Substituting into the equation of the plane:

$\begin{pmatrix}1+\lambda\\-3\lambda\\8-\lambda\end{pmatrix}.\begin{pmatrix}3\\-1\\-1\end{pmatrix} = 2$

$\Longrightarrow 3 + 3\lambda + 3\lambda - 8 + \lambda = 2$

$7\lambda - 5 = 2$

$\lambda = 1$

Since there is a constant value of λ, the plane and the line do intersect.

Substituting the value of λ in l_2 gives the point of intersection $\begin{pmatrix} 1+1 \\ -3(1) \\ 8-1 \end{pmatrix} \Longrightarrow (2, -3, 7)$

Example

Determine whether the given lines are parallel to, contained in, or intersect the plane $\boldsymbol{r}.\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5$.

(a) $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$

(b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

(c) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

(d) $x = y = z$

Solution

(a) $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \neq 0$, hence the line is neither parallel nor contained in the plane

$\begin{pmatrix} 3-2\lambda \\ -1+\lambda \\ 1-3\lambda \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 6\lambda + 2$

$$6\lambda + 2 = 5 \Longrightarrow \lambda = \frac{1}{2}$$

Since there is a constant value of λ, the plane and the line do intersect.

Point of intersection $\left(2, -\frac{1}{2}, -\frac{1}{2}\right)$

(b) $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \neq 0$, hence the line is neither parallel nor contained in the plane

$$\begin{pmatrix} 1+2\mu \\ -1+\mu \\ -3\mu \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 14\mu + 1$$

$$14\mu + 1 = 5 \Longrightarrow \mu = \frac{1}{14}$$

Since there is a constant value of μ, the plane and the line do intersect.

(c) $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 0$ Hence the line is either parallel or contained in the plane

$$\begin{pmatrix} 2+3s \\ 1 \\ 2s \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5$$

Hence the line is contained in the plane

(d) $x = y = z = \lambda$

The vector equation of the line is $r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 0$, hence the line and the plane are parallel.

Exercise 2.2.10

Determining whether a line lies in, is parallel to or intersects a plane

1 In each case, determine if the line and the plane intersect. If so, find the point of intersection.

(a) $\pi: 5x - 2y + 4z = 23$ $\quad l: \quad r = \begin{pmatrix} -17 \\ 7 \\ -6 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$

(b) $\pi: x + 4y + 3z = 11$ $\quad l: \quad r = \begin{pmatrix} -1 \\ -9 \\ 16 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$

(c) $\pi: 9x + 13y - 2z = 29$ $\quad l: \quad x = 5 + 2t,\ y = -5 - 5t,\ z = 2 + 3t$

(d) $\pi: x + 3y - 4z = 10$ $\quad l: \quad x = 4 + 6t,\ y = -7 + 2t,\ z = 1 + 3t$

(e) $\pi: 3x - y + z = -6$ $\quad l: \quad r = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(f) $\pi: 4x - 2y = 11$ $\quad l: \quad r = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

(g) $\pi: \quad r = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + m \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} + n \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ $\quad l: \quad x = 4 + t, y = 2 - 2t, z = 6 + 3t$

(h) $\pi: \quad 4x + 5y - 2z = 18.$ $\quad l: \quad x = 2 + 3t;\ y = -4t;\ z = 5 + t$

2 Find the coordinates of the point of intersection of

(a) the line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and the plane $\boldsymbol{r}.\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 2$

(b) the line $x - 2 = 2y + 1 = 3 - z$ and the plane $x + 2y + z = 3$.

(c) the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and the plane $x - 2y + 3z = 26$.

(d) the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ and the plane $3x + 2y - 2z + 7 = 0$.

3 Show that the line $r = \begin{pmatrix} 10 \\ 5 \\ 16 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ lies in the plane

(a) $9x - 2y - 5z = 0$ $\quad$ (b) $x + 2y - z = 4$

4 Write the equation of the line perpendicular to the plane $7x - 8y + 5z = 1$ and passing through the point $\mathrm{P}(6, 1, -2)$.

5 Show that the following line and the plane are parallel

(a) $x = 2 + t, y = 3 - 2t, z = 5 - t$ $\quad 2x + 3y - 4z = 7$

6 Show that the following line and the plane are not parallel

$x = 4 + 2t, y = 3 + 6t, z = 5 + 9t$ and $2x + 3y - 4z = 7$

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7 Determine the intersection of each line and plane. Interpret the solution.

(a) $r = \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$, $2x + 3y - 5z = 3$

(b) $r = \begin{pmatrix} 8 \\ -2 \\ -2 \end{pmatrix} + s\begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$, $x + 3y - 2z = 6$

8 Given the plane with equation $2x + 3y + 4z = 24$ and the lines

$$r_1 = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}, r_2 = \begin{pmatrix} 14 \\ -12 \\ 8 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, r_3 = \begin{pmatrix} 9 \\ -10 \\ 9 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix}.$$

(a) Show that each line is contained in the plane.

(b) Find the vertices of the triangle formed by the lines.

(c) Find the perimeter of the triangle formed by the lines.

(d) Find the area of the triangle formed by the lines.

9 Determine whether the line with equation $r = \begin{pmatrix} -3 \\ -6 \\ -11 \end{pmatrix} + t\begin{pmatrix} 22 \\ 1 \\ -11 \end{pmatrix}$ lies in the plane that contains the points $A(2, 5, 6)$, $B(-7, 1, 4)$, and $C(6, -2, -9)$.

10 Write the scalar equation of a plane that is parallel to the xz-plane and contains the line

$$r = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + m\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}.$$

11 Show that the line l having equation $r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies on the plane with equation $2x + 3y - 5z = -7$.

12 Show that the line $x + 1 = y = \frac{1}{2}(z - 3)$ is parallel to the plane $r.\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 3$.

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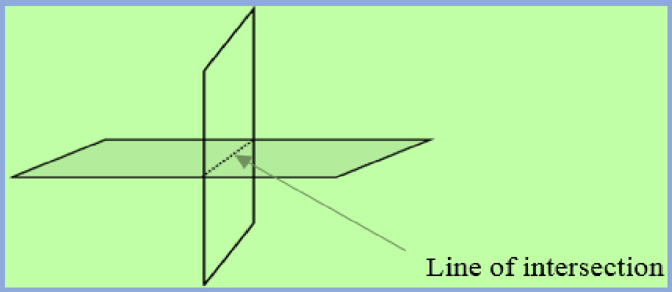
2.2.11 Find the line of intersection of two planes



Example

Find the vector equation of the line of intersection of the following planes

$$\pi_1 : r.\begin{pmatrix}1\\-1\\3\end{pmatrix} = 4 \qquad \pi_2 : r.\begin{pmatrix}2\\1\\-2\end{pmatrix} = 5$$

Solution

Cartesian of π_1 and π_2 are

$x - y + 3z = 4$

$2x + y - 2z = 5$

Adding the equations to eliminate y

$3x + z = 9$

When $x = 0, z = 9 \therefore y = 23$. Hence $(0,23,9)$ is a possible solution to the equations

When $x = 3, z = 0 \therefore y = -1$. Hence $(3,-1,0)$ is also a possible solution.

Direction vector $\begin{pmatrix}0\\23\\9\end{pmatrix} - \begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}-3\\24\\9\end{pmatrix} = -3\begin{pmatrix}1\\-8\\-3\end{pmatrix}$

Equation of line $r = \begin{pmatrix}0\\23\\9\end{pmatrix} + \lambda\begin{pmatrix}1\\-8\\-3\end{pmatrix}$

Exercise 2.2.11

1 Find the equation of the line of intersection of the following planes

(a) $2x - 3y + z = 3 \quad 3x - 5y + z = 8$

(b) $x + y + z = 6 \quad 4x + 3y + z = 10$

(c) $2x - y + z = 0 \quad x + y + z = 6$

(d) $x - 2y - 4z = 9 \quad 2x + y + 6 = 7$

(e) $5x + 4y - 2z = 31 \quad 3x + y + z = 10$

2 Show that the three planes whose equations are

$x + y + z = 10 \quad 3x + 5y + z = 6 \quad 2x + 3y + z = 8$
contain a common line.

3 Prove that the planes $x - 2y + z = 5 \quad 3x - 3y + z = 3 \quad 5x - 4y + z = 1$ meet in a common line and find the coordinates of the point where this line meets the plane $z = 0$.

4 Given two planes π_1: $x + y + z = 1$ and π_2: $x - 2y + 3z = 1$.

Find (a) the line of intersection, and (b) the angle between two planes.

5 Find the intersection of the planes with Cartesian equations $2x - 2y - z = 2$ and $x - 3y + z = 5$, giving the answer in the form $r = a + \lambda b$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors and λ is a scalar parameter.

4 Show that the planes with Cartesian equations $4x + 5y + 3z = 82$ and $-2x + 5y + 6z = 124$ intersect along the straight line with equation
$\boldsymbol{r} = (\lambda - 6)\boldsymbol{i} + (20 - 2\lambda)\boldsymbol{j} + (2\lambda + 2)\boldsymbol{k}$, where λ is a scalar parameter.

5 The planes π_1 and π_2 have Cartesian equations:

$\pi_1 : x - 2y + 2z = 0$

$\pi_1 : 3x - 2y - z = 5$

Show that the two planes intersect along the straight line with Cartesian equation
$\frac{x-4}{6} = \frac{y-3}{7} = \frac{z-1}{4}$

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2.2.12 Finding the perpendicular distance from a point to a plane

Example

Find the perpendicular distance from the point $P(1, 2, 3)$ to the plane with equation

$$r.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 2$$

Solution

Method 1

The vector equation of the line from P to the plane is

$$r = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\2\end{pmatrix} \Rightarrow r = \begin{pmatrix}1-\lambda\\2+2\lambda\\3+2\lambda\end{pmatrix}$$

At intersection with the plane:

$$\begin{pmatrix}1-\lambda\\2+2\lambda\\3+2\lambda\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 2 \Rightarrow 9 + 9\lambda = 2 \Rightarrow \lambda = -\frac{7}{9}$$

Thus the foot of the perpendicular from P to the plane has coordinates $\begin{pmatrix}1 - -\frac{7}{9}\\2+2\left(-\frac{7}{9}\right)\\3+2\left(-\frac{7}{9}\right)\end{pmatrix} =$ $\left(\frac{16}{9}, \frac{4}{9}, \frac{13}{9}\right)$

Perpendicular distance $d = \sqrt{\left(1-\frac{16}{9}\right)^2 + \left(2-\frac{4}{9}\right)^2 + \left(3-\frac{13}{9}\right)^2} = \sqrt{\frac{49}{9}} = 7/3$

Method 2

The Cartesian equation of the plane is $-x + 2y + 2z = 2$.

Formula for the perpendicular distance

$D = \left|\frac{ax+by+cz-d}{\sqrt{a^2+b^2+c^2}}\right|$ where

D is the perpendicular distance, $ax + by + cz - d = 0$ is the Cartesian equation of the plane

From the above question

$d = \left|\frac{-x+2y+2z-2}{\sqrt{(-1)^2+2^2+2^2}}\right|$

At $P(1,2,3)$, $d = \left|\frac{-(1)+2(2)+2(3)-2}{\sqrt{1^2+2^2+2^2}}\right| = \left|\frac{7}{\sqrt{9}}\right| = 7/3$

**An advantage of method 1 is that it gives coordinates of the foot of the perpendicular*

Exercise 2.2.12

Finding the perpendicular distance from a point to a plane

1 Find the distance between the plane $4x + 2y + z - 16 = 0$ and each point.

(a) $\text{A}(10, 3, -8)$ (b) $\text{B}(2, 2, 4)$

2 Find the shortest distance from each given point to the plane. Find the coordinates of the foot of the perpendicular from the point to the plane.

(a) $2x - y + 3z = 4$ $\text{A}(1,2,3)$

(a) $3x + 3y - 5z = -9$ $\text{B}(-4, -5,3)$

(b) $4x - 2y - 7z = 21$ $\text{C}(1, 4, -9)$

(c) $2x - y + 5z + 4 = 0$ $\text{D}(5, -3, 2)$

(d) $x + 2y + 6z - 10 = 0$ $\text{E}(-5, 0, 1)$

(e) $2x - 5y + 3z + 6 = 0$ $\text{F}(-2, 3, -3)$

(f) $4x - y + 8z = 2$ $\text{G}(3, -2, 0)$

(g) $3x - 5y + 6z = 9$ $\text{H}(2, 5, -7)$

(h) $2x + 4y - z = 2$ $\text{J}(3, -2, 5)$

3 The plane π has vector equation $r = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

(a) Verify that π does not pass through the origin.

(b) Find the distance from the origin to π.

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4 Determine, in surd form, the perpendicular distance of the point (−5,−2, 8) from the plane whose Cartesian equation is $2x - y + 3z = 17$.

5 Find shortest distance of the origin O from the plane with equation
(a) $x + 2y + 2z = 5$ (b) $4x + 3y - 5z = 20$

6 Find shortest distance from the point $A(1,3,-2)$ to the plane with Cartesian equation

$x + 3y - 5z = 5$.

7 Find shortest distance from the point $P(3,1,3)$ to the plane with Cartesian equation $x - y + 2z = 2$.

8 Find shortest distance of the point $P(1,2,9)$ from the plane with Cartesian equation $-x + 4y + 8z = 16$.

Miscellaneous Exercises 2.2

1 Three points have coordinates $A(1, -8, -2)$, $B(9, 2, 4)$ and $C(-3, 2, 10)$.

(a) Find the vector product $\overrightarrow{AB}$ x $\overrightarrow{AC}$

(b) Deduce the area of the triangle ABC.

The plane P contains the point B and is perpendicular to AB. The plane Q contains the point C and is perpendicular to AC.

(c) Find the equation of the line of intersection of P and Q. OCR

2 Plane π_1, has equation $r.(i + j - 2k) = 6$, and plane π_2, passes through the points $A(2, 3, -2)$, $B(7, -6, 2)$ and $C(-7, 5, 5)$.

(a) (i) Find the Cartesian equation of the plane π_2.

(ii) Show that the planes π_1 and π_2 are perpendicular to each other. [7]

(b) The point P has coordinates $(4, 1, -1)$. Find the distances of P from π_1 and π_2. [4]

(c) Given that the point $Q(10, -6, -1)$ lies in both planes π_1 and π_2 find the vector equation of their line of intersection. [4]

3 (a) The line L_1 has equation $r = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and the plne π_1 passes through the points AB and C with coordinates $(2, -1, 3)$, $(4, 2, -5)$ and $(-1, 3, -2)$ respectively.

Find the

(i) Cartesian equation of π_1. [5]

(ii) angle between π_1 and L_1. [3]

(b) The plane π_2 has equation $\boldsymbol{r}.\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 6$.

The line L_2 lies in the plane π_2 and is perpendicular to L_1. The line L_2 passes through the point $(4, 2, 1)$.

Find the

(i) vector equation of L_2, [4]

(ii) vector equation of the line of intersection of the planes π_1 and π_2. [4]

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4 The planes π_1 and π_2 have respective equations

$$\left[r - \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}\right].\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 \text{ and } \left[r - \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}\right].\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0.$$

(i) Find the acute angle between π_1 and π_2.

(ii) Find the vector equation of the line L_1, the intersection of π_1 and π_2.

(i) Show that the length of the perpendicular from the point P$(0, -6, -9)$ to the line L_1 is 5.

(ii) The line L_2 has equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 10 \\ 5 \end{pmatrix}$,

(iii) Find the coordinates of the point of intersection of L_1 and L_2.

(iv) Find the Cartesian equation of the plane containing the two lines.

[20]

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5 The points A, B and C have position vectors $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{7i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ relative to the origin O.

(a) Find

(i) the Cartesian equation of the line AB,

(ii) the equation of the plane ABC,

(iii) the area of the triangle ABC. [9]

(b) A point D has position vector $\boldsymbol{2i} - \boldsymbol{j} - 6\boldsymbol{k}$.

(i) Find the distance of the point D from the plane ABC. [3]