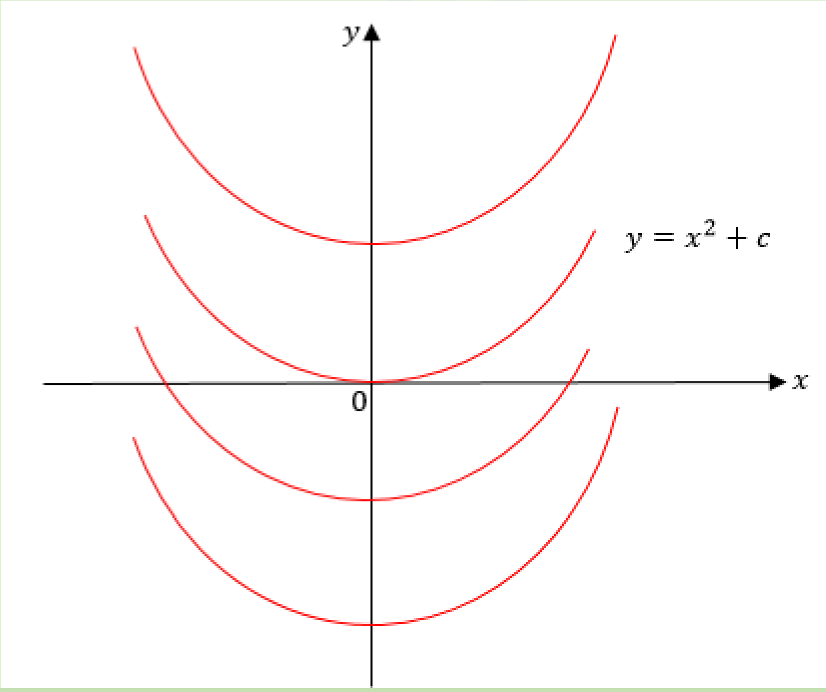
TOPIC 5: CALCULUS Prepared by G. Manuwere

5.3 First Order Differential equations

Learning Objectives

After completing this section you should be able to:

- solve differential equations by integration in the case where variables are separable
- formulate a statement involving a rate of change as a differential equation
- sketch typical examples of a family of curves representing a general solution of a differential equation
- find a particular solution of a differential equation given initial conditions
- solve problems involving 1st order differential equation



"I can do all things through Christ who strengthens me. See Philippians 4:13"

A Differential equation is an equation linking a quantity and its derivatives. If y is the quantity, then the equation might involve $\frac{dy}{dx}$ and y itself. Examples of first order differential equations are

$2x\frac{dy}{dx} = y$, $(x+1)\frac{dy}{dx} = x(y+3)$, $\frac{dy}{dx} = \frac{\cos x}{y}$, $\frac{dy}{dx} - e^x y^2 = 0$.

5.3.1 General Solution

Separation of variables.

To solve a differential equation, we may rely on separation of variables first, followed by integration.

Example 1

Find the general solution of the following differential equations

(a) $(x-3)\frac{dy}{dx} = y$

(b) $\frac{dy}{dx} = 2y(y+1)$

(c) $x\frac{dy}{dx} = \frac{1}{y} + y$.

Solution

(a) $(x-3)\frac{dy}{dx} = y$

Separating variables we have

$\frac{dy}{y} = \frac{dx}{x-3}$

Integrating both sides,

$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x-3} dx$

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$\Longrightarrow \ln y = \ln(x-3) + \ln c$
$\Longrightarrow \ln y = \ln[c(x-3)]$
$\Longrightarrow y = c(x-3)$
This is the general solution of the differential equation which is in terms of a constant c.

(b) $\frac{dy}{dx} = 2y(y+1)$

Separating variables and integrating we have,

$\int \frac{1}{y(y+1)} dy = \int 2\, dx$

Taking $\frac{1}{y(y+1)} \equiv \frac{A}{y} + \frac{B}{y+1}$

$A = 1$ and $B = -1$

$\therefore \int \frac{1}{y} - \frac{1}{y+1} dy = \int 2\, dx$

$\ln y - \ln(y+1) = 2x + c$

$\ln\left(\frac{y}{y+1}\right) = 2x + c$

$e^{\ln\left(\frac{y}{y+1}\right)} = e^{2x+c}$

$\frac{y}{y+1} = Ae^{2x}$ where $A = e^c$.

$y = (y+1).Ae^{2x}$

$y = yAe^{2x} + Ae^{2x}$

$y(1 - Ae^{2x}) = Ae^{2x}$

$y = \frac{Ae^{2x}}{1-Ae^{2x}}$.

(c) $x\frac{dy}{dx} = \frac{1}{y} + y$

$x\frac{dy}{dx} = \frac{1+y^2}{y}$

$\int \frac{y}{1+y^2} dy = \int \frac{1}{x} dx$

$\frac{1}{2}\int \frac{2y}{1+y^2} dy = \int \frac{1}{x}\, dx$

$\frac{1}{2}\ln(1+y^2) = \ln x + c$

$\frac{(1+y^2)^{\frac{1}{2}}}{x} = c$

$1 + y^2 = Ax^2.$

The graph of a general solution to a differential equation is a **family of curves** with similar characteristics, corresponding to the value of the constant.

Example 2

Solve the differential equation $\frac{dy}{dx} = 2x$ and sketch the family of curves.

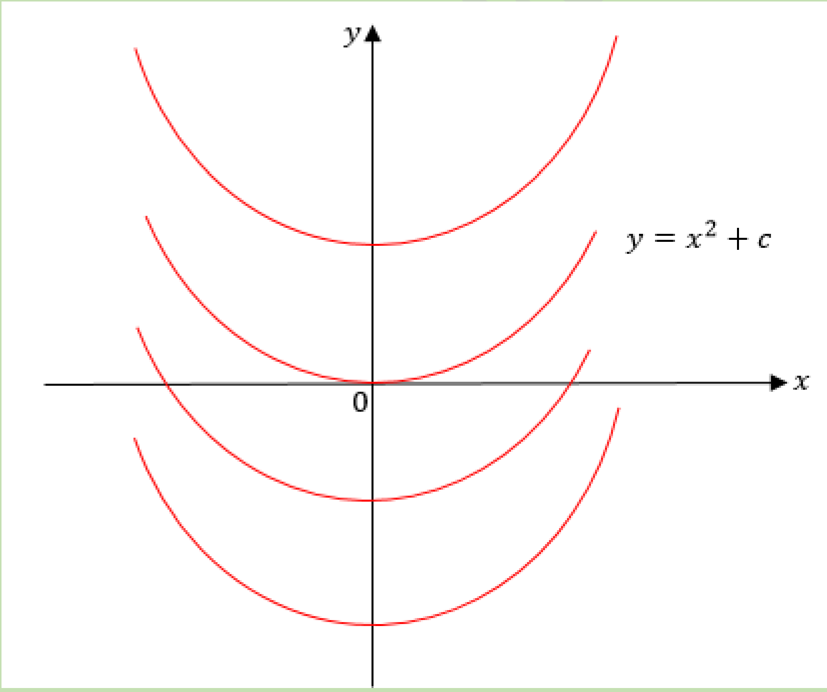
Solution

$\frac{dy}{dx} = 2x$

$\int dy = \int 2x\, dx$

General solution,

$y = x^2 + c$



Exercise 5.3.1

Find the general solutions of the following differential equations. Express y in terms of x where necessary

(a) $x\frac{dy}{dx} = y(1-x)$

(b) $cos^2x\ \frac{dy}{dx} = 1 - y^2,\ 0 < y < 1.$

(c) $\frac{dy}{dx} + y = x^2y$

(d) $\frac{dy}{dx} = \frac{x^2y+y}{x^2-1}$

(e) $\frac{dy}{dx} = \frac{x(y^2+1)}{y(x^2+1)}$

(f) $2\frac{dy}{dx} = 3x^2(y^2-1)$

(g) $\tan x\frac{dy}{dx} = \cot y$

(h) $\frac{dy}{dx} = (1+x^2)(1+y^2)$

(i) $y(x+1) = (x^2+2x)\frac{dy}{dx}$

(j) $\sec x\frac{dy}{dx} = e^y$

(k) $y - x\frac{dy}{dx} = xy.$

(l) $\frac{dy}{dx} = y^2 - y$

5.3.2 Particular solution

You have already seen that a differential equation has an infinite number of different solutions corresponding to different values of the constant of integration.

If you are given some more information, you can find out which of the possible solutions is the one that matches the situation in question. For example, you might be told that when $x = 0$, $y = 1$. This tells you that the correct solution is the one with the curve that passes through the point (0, 1). You can use this information to find out the value of c for this particular solution by substituting the values $x = 0$ and $y = 1$ into the general solution.

For the general solution $y = x^2 + c$, if we are given that $x = 0$ when $y = 1$ we get $c = 1$.

Thus $y = x^2 + 1$ is called the **particular solution** to the differential equation. This represents a specific curve.

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Example 1

(a) Find the particular solution to the following differential equations giving y in terms of x.

(i) $\frac{dy}{dx} = (x+1)(y+1)$, given that $(y > -1)$ and that $x = 0$ when $y = 0$.

(ii) $\frac{dy}{dx} = \frac{y^2-1}{2\tan x}$ given that when $y = 3$, $x = \frac{\pi}{2}$.

(b) A curve is defined by the differential equation $\frac{dy}{dx} = 5y$. Given that the curve passes through the point $(0; 5)$, find the equation of the curve and sketch it.

Solution

(a) (i) $\int \frac{1}{y+1} dy = \int (x+1) dx$

$\ln(y+1) = \frac{x^2}{2} + x + c$ General solution.

Substituting $x = 0$ and $y = 0$, we have,

$\ln 1 = c$

$\Longrightarrow c = 0$

$\therefore \ln(y+1) = \frac{x^2}{2} + x$

$y + 1 = e^{\frac{x^2}{2}+x}$

$y = e^{\frac{x^2}{2}+x} - 1$

(ii) $\frac{dy}{dx} = \frac{y^2-1}{2\tan x}$

$\int \frac{dy}{y^2-1} = \int \frac{1}{2\tan x} dx$

$\int \frac{1}{(y-1)(y+1)} dy = \frac{1}{2} \int \frac{\cos x}{\sin x} dx$

Taking $\frac{1}{(y-1)(y+1)} \equiv \frac{A}{y-1} + \frac{B}{y+1}$

$A = \frac{1}{2}$ and $B = -\frac{1}{2}$

$\therefore \int \frac{1}{2(y-1)} - \frac{1}{2(y+1)} dy = \frac{1}{2} \int \frac{\cos x}{\sin x} dx$

$\int \frac{1}{(y-1)} - \frac{1}{(y+1)} dy = \int \frac{\cos x}{\sin x} dx$

$[\ln(y-1) - \ln(y+1)] = \ln \sin x + c$

$\ln\left(\frac{y-1}{y+1}\right) = \ln \sin x + c$ General solution.

Putting $y = 3$ and $x = \frac{\pi}{2}$ we have,

$\ln\left(\frac{1}{2}\right) = \ln \sin \frac{\pi}{2} + c$

$c = \ln \frac{1}{2}$

$\therefore \ln\left(\frac{y-1}{y+1}\right) = \ln \sin x + \ln \frac{1}{2}$

$\ln\left(\frac{y-1}{y+1}\right) = \ln\left(\frac{\sin x}{2}\right)$

$\frac{y-1}{y+1} = \frac{\sin x}{2}$

$2y - 2 = y \sin x + \sin x$

$(2 - \sin x)y = 2 + \sin x$

$y = \frac{2+\sin x}{2-\sin x}.$

(b) $\frac{dy}{dx} = 5y$

$\int \frac{dy}{y} = 5dx$

$\ln y = 5x + c$ General solution

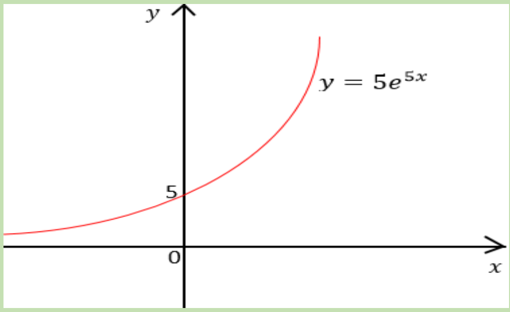
At $(0; 5)$

$\ln 5 = c$

Hence $\ln y = 5x + \ln 5$

$\Longrightarrow \ln \frac{y}{5} = 5x$

$\Longrightarrow y = 5e^{5x}$ Particular solution.



Exercise 5.3.2

1 Solve the following differential equations using the given initial conditions.

(a) $\frac{dy}{dx} = 4x\sqrt{y}$ $\quad x = 3, y = 81.$

(b) $\frac{y}{x}\frac{dy}{dx} = \frac{y^2-1}{x^2-1}$ $\quad x = 2, y = 3$

(c) $(\cos y)\frac{dy}{dx} = x^2 cosec^2 y$ $\quad x = \frac{1}{2}, y = \frac{\pi}{2}.$

(d) $\frac{dy}{dx} = \sqrt{xe^y}$ $\quad (x \geq 0)$ $\quad x = 0, y = 0.$

(e) $(x^2 - 1)\frac{dy}{dx} = -(x^2 + 1)y$ $\quad (x > 1),$ $\quad x = 3, y = 1.$

(f) $e^y\frac{dy}{dx} + \sin x = 0$ $\quad x = \frac{\pi}{2}, y = \frac{1}{2}$

(g) $(1 + x^2)\frac{dy}{dx} - y(y + 1)x = 0$ $\quad x = 0, y = 1.$

(h) $\frac{dy}{dx} = \sqrt{1 - y^2}$ $\quad x = \frac{\pi}{2}, y = 0.$

2 A curve is defined by the differential equation $\frac{dy}{dx} = \frac{\ln x}{\cot y}$. Given that the curve passes through the point where $y = \frac{\pi}{6}$ and $x = e$, find the equation of the curve and hence find the value of y when $x = 1$, giving your answer correct to 3 significant figures.

3 The gradient function of a curve is given by $\frac{y+1}{x^2-1}$. Find the equation of the curve given that it passes through the point $(-3; 1)$.

4 If $y = 2$ when $x = 1$, find the coordinates of the point where the curve represented by $\frac{2y}{3}\frac{dy}{dx} = e^{-3x}$ crosses the y axis.

5 For all positive values of x the gradient of a curve at a point $(x; y)$ is $\frac{y}{x^2+x}$. The point $A(3, 6)$lies on this curve.

(a) Calculate the equation of the normal to the curve at A.

(b) Find the equation of the curve in the form $y = f(x)$.

6 Find the equation of the curve which passes through the point $\left(\frac{1}{2}, 1\right)$ and is defined by the differential equation $ye^{y^2}\frac{dy}{dx} = e^{2x}$. Show that the curve also passes through the point $(2, 2)$ and sketch the curve.

Modelling a situation by a differential equation

Physical situations can be interpreted mathematically using differential equations. Consider the following situations:

1. The birth rate of a population, p at a time t years is proportional to the size of the population.

   Using p for population size and t for time, this situation can be modelled as $\frac{dp}{dt} = kp$ where k is a constant.

2. The rate of decay of a radioactive material is proportional to the mass, m, of the material.

   In this case, the rate of decay can be modelled by the differential equation $-\frac{dm}{dt} = km$ where k is a constant.

Example 1

A cyclist travelling on a level road stops pedaling and freewheels for 5 seconds. The distance travelled by the cyclist in t seconds is x metres. The relationship between x and t while the cyclist is freewheeling can be modelled by the differential equation

$\frac{dx}{dt} = \frac{250}{(5+t)^2}$.

(a) Find the general solution of this differential equation.

(b) (i) State the appropriate initial condition to be satisfied by the differential equation.
(ii) Find the particular solution satisfying this condition.
(c) Deduce that the cyclist travels 25 metres while freewheeling.

Solution

(a) $\frac{dx}{dt} = \frac{250}{(5+t)^2}$. Separating variables and integrating we have,

$$\int dx = \int \frac{250}{(5+t)^2}\,dt$$

$$\int dx = 250 \int (5+t)^{-2}\,dt$$

$$x = \frac{-250}{5+t} + c$$ ← General solution

(b) (i) initially $x = 0$ when $t = 0$

(ii) Substituting the initial conditions into the general solution we have,

$$0 = \frac{-250}{5} + c$$

$$\Longrightarrow c = 50$$

$$\therefore x = \frac{-250}{5+t} + 50$$ Particular solution

(c) The cyclist freewheeled for 5 seconds

$$\therefore x = \frac{-250}{5+5} + 50$$

$$= \frac{-250}{10} + 5$$

$$= 25 \text{ m}$$

Example 2

In a model to estimate the depreciation of the value of a car, it is assumed that the value, $\$V$, at age t months, decreases at a rate proportional to V. Using this model, write down a differential eqaution relating V and t. Given that the car has an initial value of \$6 000, solve the differential equation and show that $V = 6\,000e^{-kt}$ where k is a positive constant.

The value of the car is expected to decrease to \$3 000 after 36 months.

Calculate

(a) the value, to the nearest dollar, of the car when it is 15 months old,

(b) the age of the car, to the nearest month, when its value is \$2 000.

Solution

$$-\frac{dV}{dt} \propto V$$

$$\Longrightarrow \frac{dV}{dt} = -kV$$

Separating variables and integrating we have,

$$\int \frac{1}{V} dV = \int -k dt$$

$$\ln V = -kt + c$$

$e^{\ln V} = e^{-kt+c}$

$V = Ae^{-kt}$ where $A = e^{c}$

Initially, $V = 6\,000$ when $t = 0$

Thus $6000 = A$

$\Longrightarrow V = 6000e^{-kt}$

(a) After 36 months the value is \$3 000, hence

$3000 = 6000e^{-36k}$

$\frac{1}{2} = e^{-36k}$

$-\frac{1}{36}\ln\frac{1}{2} = k$

$k = \frac{1}{36}\ln 2$

Thus $V = 6000e^{-\frac{t}{36}\ln 2}$

$\Longrightarrow V = 6000e^{\ln(2)^{-\frac{t}{36}}}$

$V = 6000(2)^{-\frac{t}{36}}$

When $t = 15$,

$V = 6000(2)^{-\frac{15}{36}} =$ \$4 495 to the nearest dollar

(b) When $V = 2000$,

$2000 = 6000(2)^{-\frac{t}{36}}$

$\frac{1}{3} = (2)^{-\frac{t}{36}}$

$\ln\frac{1}{3} = -\frac{t}{36}\ln 2$

$t = -36\left(\frac{\ln\frac{1}{3}}{\ln 2}\right) = 57$ months, to the nearest month.

Example 3

A certain type of tree grows in such a way that its height h metres, t years after the tree has been planted can be modelled by a differential equation. It is assumed that the rate of increase of the height is directly proportional to the cube root of $(9-h)$.

(a) Given that $h = 1$, and $\frac{dh}{dt} = \frac{1}{5}$ when $t = 0$,

(i) form a differential equation in terms of h.

(ii) solve the differential equation and obtain an expression for t in terms of h.

(b) Hence find the time taken for the tree to reach a height of 4m. (Zimsec N2011 p2)

Solution

(i) $\frac{dh}{dt} \propto (9-h)^{\frac{1}{3}}$

$\frac{dh}{dt} = k(9-h)^{\frac{1}{3}}$

(ii) putting $h = 1$, and $\frac{dh}{dt} = \frac{1}{5}$,

$\frac{1}{5} = k(8)^{\frac{1}{3}}$

$k = \frac{1}{10}$

$\therefore \frac{dh}{dt} = \frac{1}{10}(9-h)^{\frac{1}{3}}$

Separating variable and integrating

$\int \frac{1}{(9-h)^{\frac{1}{3}}} dh = \int \frac{1}{10} dt$

$-\frac{3}{2}(9-h)^{\frac{2}{3}} = \frac{1}{10}t + c$

When $t = 0, h = 1$, hence

$-\frac{3}{2}(8)^{\frac{2}{3}} = c$

$c = -6$

$-\frac{3}{2}(9-h)^{\frac{2}{3}} = \frac{1}{10}t - 6$

$\frac{1}{10}t = 6 - \frac{3}{2}(9-h)^{\frac{2}{3}}$

$t = 60 - 15(9-h)^{\frac{2}{3}}$

(b) When $h = 4$

$t = 60 - 15(9-4)^{\frac{2}{3}}$

$t = 16.139733926807$

$= 16$ years.

Example 4

A tree is planted as a seedling of negligible height. The rate of increase in its height, in metres per year, is given by the formula $0.2\sqrt{25-h}$, where h is the height of the tree, in metres, t years after it is planted.

(a) Explain why the height of the tree can never exceed 25 metres.

(b) Write down a differential equation connecting h and t, and solve it to find an expression for t as a function of h .

(c) How long does it take for the tree to put on

(i) its first metre of growth,

(ii) its last metre of growth?

(d) Find an expression for the height of the tree after t years. Over what interval of values of t is this model valid?

Solution

(a) when $h = 25$, $\frac{dh}{dt} = 0$ so the tree stops growing.

(b) $\frac{dh}{dt} = 0.2\sqrt{25-h}$

$\int \frac{dh}{\sqrt{25-h}} = \int 0.2 dt$

$\int (25-h)^{-\frac{1}{2}} dh = 0.2 \int dt$

$-2\sqrt{25-h} = 0.2t + c$

Negligible height at planting means $h = 0$ when $t = 0$

$\therefore -2\sqrt{25-0} = 0.2(0) + c$

$\Longrightarrow c = -10$

Thus $-2\sqrt{25-h} = 0.2t - 10$

$\Longrightarrow \sqrt{25-h} = 5 - 0.1t$

$t = 50 - 10\sqrt{25-h}$

(c) (i) When $h = 1$,

$t = 50 - 10\sqrt{25-1}$

$= 1.0$ years

(ii) When $h = 25$

$t = 50 - 10\sqrt{25-25}$

$= 50$ years

(d) $\sqrt{25-h} = 5 - 0.1t$

Making h subject of the formula gives $h = 25 - (5 - 0.1t)^2$

$\Longrightarrow h = 25 - (25 - t + 0.01t^2)$

$\Longrightarrow h = t - 0.01t^2$ Valid for $0 \le t \le 50$

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Example 5

A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant 20°C. As the liquid warms up, the rate of increase of its temperature θ°C after time t minutes is proportional to the temperature difference $(20-\theta)$°C. Initially the temperature of the liquid is 10°C and the rate of increase of the temperature is 1°C per minute.

(a) By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-\frac{1}{10}t}$.

(b) Find the time it takes the liquid to reach a temperature of 15°C, and state what happens to θ for large values of t.

(c) Sketch a graph of θ against t.

Solution

(a) $\frac{d\theta}{dt} \propto (20-\theta)$

$\frac{d\theta}{dt} = k(20-\theta)$

Separating variables and integrating gives

$\int \frac{1}{20-\theta}\,d\theta = k\int dt$

$-\ln(20-\theta) = kt + c$

$\theta = 10°\text{C}$ when $t = 0$

$c = -\ln 10$

$-\ln(20-\theta) = kt - \ln 10$

$\ln\left(\frac{10}{20-\theta}\right) = kt$

$\frac{10}{20-\theta} = e^{kt}$

Putting the initial conditions $\frac{d\theta}{dt} = 1$ and $\theta = 10°\text{C}$ on $\frac{d\theta}{dt} = k(20-\theta)$ gives

$\frac{1}{10} = k$

Thus $\frac{10}{20-\theta} = e^{\frac{1}{10}t}$

$$10 = 20e^{\frac{1}{10}t} - \theta e^{\frac{1}{10}t}$$

$$\theta e^{\frac{1}{10}t} = 20e^{\frac{1}{10}t} - 10$$

$$\theta = \frac{20e^{\frac{1}{10}t} - 10}{e^{\frac{1}{10}t}}$$

$$\theta = 20 - 10e^{-\frac{1}{10}t}$$

(b) When $\theta = 15°\text{C}$

$$15 = 20 - 10e^{-\frac{1}{10}t}$$

$$\Longrightarrow e^{-\frac{1}{10}t} = 0.5$$

$$\Longrightarrow t = -10\ln 0.5$$

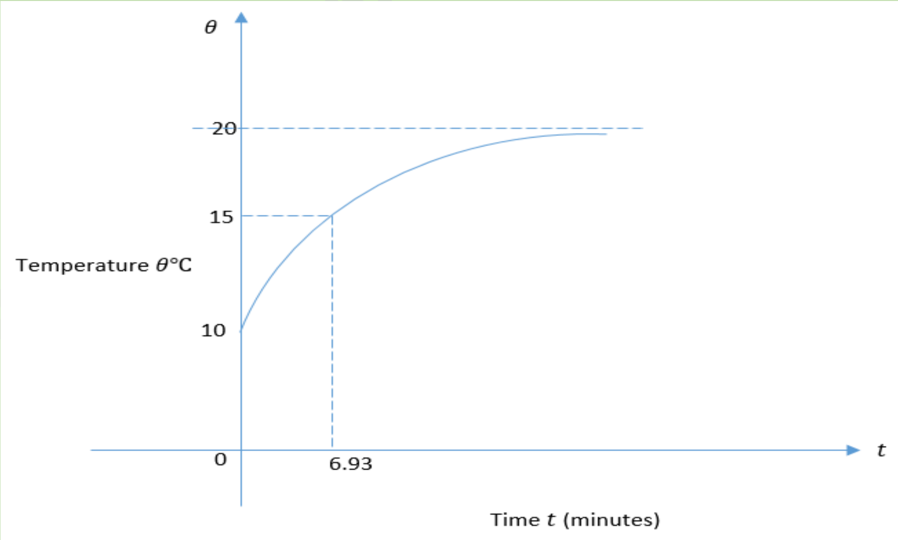
$$= 10\ln 2$$

$$\Longrightarrow t = 6.93 \text{ minutes.}$$

$$\lim_{t\to\infty}\left(20 - 10e^{-\frac{1}{10}t}\right) = 20°\text{C}$$

The temperature approaches 20°C.

(d) When $t = 0, \theta = 10$, When $t = 6.93, \theta = 15, \theta = 20$ is an asymptote



Example 4

In a certain chemical reaction, a substance is transformed into a compound. Tha mass of the substance after any time t is m and the substance is being transformed at a rate that is proportional to the mass of the substance at that time. Given that the original mass is 50g and that 20g is transformed after 200 seconds,

(a) form and solve a differential equation relating m and t,

(b) find the mass of the substance transformed in 300 seconds.

Solution

(a) $\frac{dm}{dt} \propto m$

$\therefore \frac{dm}{dt} = km$, where k is a constant.

(b) Seperating variables and integrating we have,

$\int \frac{1}{m} dm = \int k dt$

$\ln m = kt + c$

Initally $m = 50$ when $t = 0$

$\therefore \ln 50 = c$

Hence $\ln m = kt + \ln 50$

$\Longrightarrow \ln \frac{m}{50} = kt$

Also when $t = 200$ $m = 30$.

Thus $\ln \frac{30}{50} = 200k$

$\Longrightarrow k = \frac{1}{200} \ln \frac{3}{5}$

Hence $\ln \frac{m}{50} = \frac{t}{200} \ln \frac{3}{5}$

$\Longrightarrow \frac{m}{50} = \left(\frac{3}{5}\right)^{\frac{t}{200}}$

$m = 50 \left(\frac{3}{5}\right)^{\frac{t}{200}}$

When $t = 300$, $m = 50 \left(\frac{3}{5}\right)^{\frac{3}{2}}$

Hence transformed mass is $50-50\left(\frac{3}{5}\right)^{\frac{3}{2}}=26.76g$

Example 5

A species of tree is growing in height and the typical maximum height it can reach in its lifetime is 12 m.

The rate of growth of its height, H m, is proportional to the difference between its height and the maximum height it can reach.

When a tree of this species was planted, it was 1m in height and at that instant the tree was growing at the rate of 0.1 m per month.

(a) Show clearly that $110\frac{dH}{dt}=12-H$ where t is the time, measured in months, since the tree was planted.

(b) Determine a simplified solution for the above differential equation, giving the answer in the form $H=f(t)$.

(c) Find, correct to 2 decimal places, the height of the tree after 5 years.

(d) Calculate, correct to the nearest year, the number of years it will take for the tree to reach a height of 11m.

Solution

(a) $\frac{dH}{dt}\propto 12-H$

$\frac{dH}{dt}=k(12-H)$

Initially $H=1$ and $\frac{dH}{dt}=0.1$

$\therefore 0.1=11k$

$k=\frac{1}{110}$

Hence $\frac{dH}{dt}=\frac{1}{110}(12-H)$

$110\frac{dH}{dt}=(12-H)$

(b) Separating variables and integrating

$\int\frac{dH}{12-H}=\int\frac{1}{110}dt$

$-\ln(12 - H) = \frac{1}{110}t + c$

$\ln\left(\frac{1}{12-H}\right) = \frac{1}{110}t + c$

$H = 1$ when $t = 0$

$\ln\frac{1}{11} = c$

Hence $\ln\left(\frac{1}{12-H}\right) = \frac{1}{110}t + \ln\frac{1}{11}$

$\ln\left(\frac{11}{12-H}\right) = \frac{1}{110}t$

$e^{\ln\left(\frac{11}{12-H}\right)} = e^{\frac{1}{110}t}$

$\frac{11}{12-H} = e^{\frac{1}{110}t}$

$12e^{\frac{1}{110}t} - He^{\frac{1}{110}t} = 11$

$H = 12 - 11e^{-\frac{1}{110}t}$

(c) 5 years = 60 months. When $t = 60$

$H = 12 - 11e^{-\frac{60}{110}}$

$= 5.624638933$

=5.62 m

(d) When $H = 11$

$11 = 12 - 11e^{-\frac{1}{110}t}$

$e^{-\frac{1}{110}t} = \frac{1}{11}$

$-\frac{1}{110}t = -\ln 11$

$t = 110\ln 11$

$= 263.76848$ months

=21.98070667 years ≈ 22 years.

Exercise 5.3.3

1 The rate at which students of a school enter their Assembly Hall is modelled as being inversely proportional to the square root of the number of students already in the Hall.
Taking x to be the number of students in the Hall t minutes after they start to enter, show that this information gives rise to the differential equation

$$\frac{dx}{dt} = \frac{k}{\sqrt{x}}$$

and state one assumption about x which is necessary for the model to be valid. Solve this equation, given that initially there are 9 students in the Hall and that after one minute there are 100 students, giving an equation for t in terms of x.

Find the predicted number of students in the Hall after 3 minutes. (Zimsec J2007p2)

2 In 2005 a community of 10 000 people planned an extension to its water facilities.

Assume that the birth rate and death rate are 55 persons per 1 000 and 14 persons per 1 000 respectively, and that there are no other changes in population. The population is P persons at time t years after 2005.

The rate of increase of the population is given by the product of the net-rate of increase and the population at any time.

Show that the differential equation which models this situation can be written as

$$\frac{dP}{dt} = \frac{41}{1000}P.$$

Solve this differential equation and determine the predicted population for the year 2020.

3 It is assumed that the length, l, in cm of a certain snake at time, t, months after birth increases at a rate proportional to $(10-l)^{\frac{1}{2}}$.

When $t = 0$ $\quad l = 1$ and $\frac{dl}{dt} = 0{,}3$.

(i) Show that l and t satisfy the differential equation

$$\frac{dl}{dt} = 0.1(10-l)^{\frac{1}{2}}.$$

(ii) Solve the differential equation and obtain an expression for l in terms of t.

(iii) Find the maximum length of the snake.

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4 A race called the Matrices lives on an isolated island called Geometry. Demographical studies have shown that the number of births per unit time is proportional to the population, x, at any time, t. The number of deaths per unit time is proportional to the square of the population.

(i) Show that the above information can be modeled by the differential equation

$\frac{dx}{dt} = kx - hx^2$, where k and h are positive constants.

(ii) Solve the differential equation for x in terms of t, given that $x = \frac{k}{3h}$ when $t = 0$.

(iii) Show that the limit to the size of the population is $\frac{k}{h}$ as t approaches infinity.

Differential equations involving Rates of change

Example 1

Water is flowing into a cylindrical water tank of base diameter 6m at a constant rate of 0.48π m^3 per minute. The depth of the water at time t minutes is h meters. There is a tap at the bottom of the tank which when opened water leaves the tank at a rate of $0.6\pi h$ m^3 per minute.

(a) Show that t minutes after the tap has been opened,

$$75\frac{dh}{dt} = 4 - 5h$$

(b) Find the value of t when $h = 0.5$ m given that initially the depth of the water in the tank was 0.2 m.

Solution

(a) $\frac{dV}{dt} = 0.48\pi - 0.6\pi h$

Volume of a cylinder (V) = cross sectional area x height

$$V = 9\pi h$$

$$\frac{dV}{dh} = 9\pi$$

Using parametric equations $\frac{dh}{dt} = \frac{dh}{dv}$ x $\frac{dv}{dt}$

$$\Longrightarrow \frac{dh}{dt} = \frac{1}{9\pi} \text{ x } (0.48\pi - 0.6\pi h)$$

$$\Longrightarrow \frac{dh}{dt} = \frac{4}{75} - \frac{1}{15}h$$

Multiplying both sides by 75

$$\Longrightarrow 75\frac{dh}{dt} = 4 - 5h$$

(b) Separating variables and integrating we have,

$$\int \frac{1}{4-5h} dh = \int \frac{1}{75} dt$$

$$-\frac{1}{5}\ln(4-5h) = \frac{1}{75}t + c$$

$$\Longrightarrow \frac{1}{5}\ln\frac{1}{4-5h} = \frac{1}{75}t + c$$ General solution

When $t = 0$, $h = 0.2$

Hence $\frac{1}{5}\ln\frac{1}{3} = c$

Thus $\frac{1}{5}\ln\frac{1}{4-5h} = \frac{1}{75}t + \frac{1}{5}\ln\frac{1}{3}$

$$\Longrightarrow \frac{1}{5}\ln\frac{3}{4-5h} = \frac{1}{75}t$$

$$t = 15\ln\frac{3}{4-5h}$$ Particular solution

When $h = 0.5$

$$t = 15\ln\frac{3}{4-5(0.5)} = 10.39720771$$

$$= 10.4 \text{ minutes}$$

Example 2

A water tank has the shape of a cuboid with base area $4m^2$ and height $3m$ and is initially empty. Water is poured into the tank at a constant rate of $0.05m^3$ per minute. There is a small hole at the bottom of the tank through which water leaks out. The depth of the water in the tank is h metres when water has been poured for t minutes.

(i) In a simple model it is assumed that water leaks out of the tank at a constant rate of $0.025m^3$ per minute.

1. Show that the variable h satisfies the differential equation $\frac{dh}{dt} = \frac{1}{160}$.
2. Hence or otherwise, find the time when the tank starts to overflow.

(ii) In a more refined model, the variable h satisfies the differential equation

$$160\frac{dh}{dt} = 2 - h.$$

1. Solve the differential equation, expressing h in terms of t.
2. Hence sketch the graph of h against t.

Solution

(i) 1. $\frac{dv}{dt} = 0.05 - 0.025$

$\frac{dv}{dt} = 0.025$

$V = 4h$

$\frac{dV}{dh} = 4$

Using parametric equations $\frac{dh}{dt} = \frac{dh}{dv}x\frac{dv}{dt}$

$\frac{dh}{dt} = \frac{1}{4}$ x 0.025

$= \frac{1}{160}$

2. To find the time when $h = 3$

$\frac{dh}{dt} = \frac{1}{160}$

"I can do all things through Christ who strengthens me. See Philippians 4:13"

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Separating variables and integrating we have

$\int dh = \frac{1}{160}dt$

$h = \frac{t}{160} + c$

Initially, when $t = 0$, $h = 0$, hence

$0 = c$

Thus the particular solution of the differential equation is
$h = \frac{t}{160}$

When $h = 3$,

$t = 160$ x $3 = 480$ minutes.

(ii) 1. $160\frac{dh}{dt} = 2 - h$

Separating variables and integrating we have

$\int \frac{1}{2-h}dh = \int \frac{1}{160}dt$

$-\ln(2-h) = \frac{t}{160} + c$

$\ln\left(\frac{1}{2-h}\right) = \frac{t}{160} + c$

Initially $h = 0$, when $t = 0$.

$\ln\frac{1}{2} = c$

Thus the particular solution of the differential equation is

$\ln\left(\frac{1}{2-h}\right) = \frac{t}{160} + \ln\frac{1}{2}$

$\ln\left(\frac{1}{2-h}\right) - \ln\frac{1}{2} = \frac{t}{160}$

$\ln\left(\frac{2}{2-h}\right) = \frac{t}{160}$

$e^{\ln\left(\frac{2}{2-h}\right)} = e^{\frac{t}{160}}$

$\frac{2}{2-h} = e^{\frac{t}{160}}$

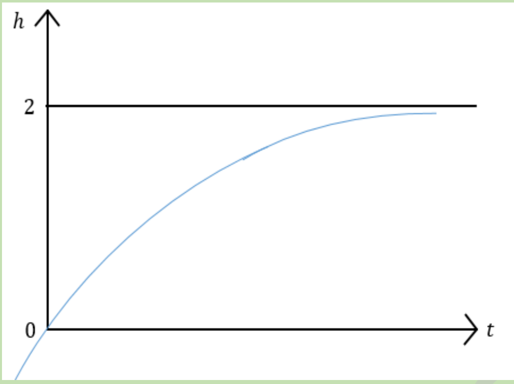
$2 = 2e^{\frac{t}{160}} - he^{\frac{t}{160}}$

$$he^{\frac{t}{160}} = 2e^{\frac{t}{160}} - 2$$

$$h = 2 - 2e^{-\frac{t}{160}}$$

2. $h = 0$, when $t = 0$

The equation is undefined when $h = 2$, OR $\lim_{t \to \infty} h = 2$, hence the line $h = 2$ is an asymptote



Example 3

The depth of an aquarium at the time t seconds is h metres. At time $t = 0$, the aquarium is empty and water begins to flow into it at a constant rate of $2m^3s^{-1}$. At the same time water begins to flow out at a rate proportional to $h^{\frac{1}{2}}$. When $h = 1, \frac{dh}{dt} = 0.002$.

(i) Show that h satisfies the differential equation $\frac{dh}{dt} = \frac{1}{500}(2 - \sqrt{h})$.

(ii) By making the substitution $x = 2 - h^{\frac{1}{2}}$, show that the equation in (i) becomes

$(2 - x)\left(\frac{dx}{dt}\right) = -0.001x$. (Zimsec N2012 p2)

Solution

(i) $\frac{dV}{dt} = 2 - \sqrt{h}$

$\frac{dh}{dt} = \frac{dh}{dV} \text{ x } \frac{dV}{dt}$

Substituting $\frac{dh}{dt} = 0.002$ and $\frac{dV}{dt} = 2 - \sqrt{h}$ and $h = 1$

$$0.002 = \frac{dh}{dV} \text{ x } \left(2 - \sqrt{1}\right)$$

$$\Rightarrow \frac{dh}{dV} = \frac{1}{500}$$

Taking $\frac{dh}{dt} = \frac{dh}{dV} \text{ x } \frac{dV}{dt}$

$$\Longrightarrow \frac{dh}{dt} = \frac{1}{500} \text{ x } \left(2 - \sqrt{h}\right)$$

(ii) Let $x = 2 - \sqrt{h}$

$$\frac{dx}{dh} = -\frac{1}{2} h^{-\frac{1}{2}} = -\frac{1}{2\sqrt{h}}$$

Now, $\frac{dx}{dt} = \frac{dx}{dh} \text{ x } \frac{dh}{dt}$

$$\frac{dx}{dt} = -\frac{1}{2\sqrt{h}} \text{x} \frac{1}{500} \text{ x } \left(2 - \sqrt{h}\right)$$

But $x = 2 - \sqrt{h}$ $\Longrightarrow h = (2 - x)^2$

Thus $\frac{dx}{dt} = -\frac{1}{2\sqrt{(2-x)^2}} \text{x} \frac{1}{500} \text{ x } \left(2 - \sqrt{(2 - x)^2}\right)$

$$\frac{dx}{dt} = -\frac{1}{(2-x)} \text{ x } \frac{1}{1000} \text{ x } (x)$$

$$\Longrightarrow (2 - x)\frac{dx}{dt} = -0.001x$$

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Exercise 5.3.4

1 A spherical balloon is allowed to deflate. The rate at which air is leaving the balloon is proportional to the volume V of air left in the balloon. When the radius of the balloon is 15 cm, air is leaving at a rate of $8cm^3s^{-1}$.

Show that $\frac{dV}{dt} = -\frac{2V}{1125\pi}$

2 A tank is shaped as a cuboid with a square base of side 10 cm. Water runs out through a hole in the base at a rate proportional to the square root of the height, h cm, of water in the tank. At the same time, water is pumped into the tank at a constant rate of 2 cm^3s^{-1}.

Show that $\frac{dh}{dt} = \frac{2-k\sqrt{h}}{100}$

3 During a flood, water is entering a lake at a constant rate of $12m^3$ per second. The volume of the water in the lake at any given time is kh^3 cubic metres, where h is the depth in metres of the water at the dam wall and k is a constant.

Show that the rate of change of depth is given by the equation $\frac{dh}{dt} = \frac{4}{kh^2}$

Obtain the general solution of this equation

Given that when the flood starts, the depth of the water at the dam wall is 24m, find in terms of k the time that it will take to increase to depth of 30m.

4 A right circular metallic cylinder has radius r cm and height h cm. Given that the volume of the cylinder is 200 cm^3, write down a formula for h in terms of r. The cylinder is melted and then rolled in a machine to form another cylinder. Given that at the time the radius is 4 cm the height h is increasing at a rate of 0.4 cm/minute, find the rate of change of the radius giving your answer to 4 decimal places.

The total surface area of the cylinder, S, is given by $S = 2\pi r^2 + \frac{400}{r}$.

Find

(i) $\frac{ds}{dr}$,

(ii) the rate at which the total surface area of the cylinder is changing when $r = 4$ cm and $\frac{dh}{dt} = 0.4$ cm. Comment on the result.

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Miscellaneous Exercise 5.3

1 A population of bacteria grows from an initial size of 1000. After t hours the size of the population is P. After 10 hours the size of the population is 4000.

At first the rate of growth is modelled as being proportional to the size of the population.

(a) Write down a differential equation modelling the population growth and solve it for P in terms of t.

To allow for constrains on the population growth, the model is revised to give

$\frac{dP}{dt} = kP(5000 - P)$, where k is a constant.

(b) Solve this differential equation to find t in terms of P, subject to the given conditions.

(c) Find the time it takes for the population to reach 4900, giving your answer in hours, correct to 2 decimal places.

2 Solve the differential equation $\frac{dy}{dx} = x - xy$ when $y = 0$ and $x = 0$.

3 A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant $20°C$. As the liquid warms up, the rate of increase of its temperature $\theta°C$ after time t minutes is proportional to the temperature difference $(20 - \theta)°C$. Initially the temperature of the liquid is 10℃ and the rate of increase of the temperature is 1℃ per minute.

By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-\frac{1}{10}t}$.

Find the time it takes the liquid to reach a temperature of 15℃, and state what happens to θ for large values of t. Sketch the value of θ against t.

4 Find a general solution of the differential equation $e^{2x}\frac{dy}{dx} = cosec^2 y$

5 A water tank with uniform cross section has a tap at its base. When the tap is opened water flows out at a rate proportional to the square root of the depth of the water in the tank. Given that the cross sectional area of the tank is 5m^2 and the depth of water, t minutes after opening is h metres, show that $\frac{dh}{dt} = -\frac{k}{5}\sqrt{h}$.

Given that the tap is opened when the depth of water is 2 metres, find an expression in terms of k for the time taken for the depth to reach 1 metre. (zimsec N2003 p1)

6 Show that a general solution of the differential equation $5\frac{dy}{dx} = 2y^2 - 7y + 3$ is given by

$y = \frac{Ae^x - 3}{2Ae^x - 1}$ where A is an arbitrary constant.

7 Show that a general solution of the differential equation $e^{x+2y}\frac{dy}{dx} + (1-x)^2 = 0$ is given by $y = \frac{1}{2}\ln[2e^{-x}(x^2+1) + k]$ where K is an arbitrary constant.

8 A disease is spreading through a population. The rate of increase is proportional to the product of the proportion of the people infected and the proportion of the people not affected. Initially the proportion infected is 0.1%, and 1 year later it is 10%.

Letting P be the proportion infected, find a differential equation in P. Solve this differential equation and use the solution to show that the time when 50% of the population will be infected is 1.47 years.

9 The rate at which a body looses temperature at any given instant is proportional to the amount by which the temperature of the body, at that instant, exceeds the temperature of its surroundings. A container of hot liquid is placed in a room of temperature 18℃ and in 6 minutes the liquid cools from 82℃ to 50℃. How long does it take for the liquid to cool from 26℃ to 20℃?

10 A biologist studying fluctuations in the size of a particular population decides to investigate a model for which $\frac{dP}{dt} = kP\cos kt$, where P is the size of the population at time t days and k is a positive constant.

(a) Given that $P = P_0$ when $t = 0$, express P in terms of k, t and P_0.

(b) Find the ratio of the maximum size of the population to the minimum size.

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11 Water is pouring into a cylindrical container at a constant rate of 1600 cm^3 per second and is leaking out of a hole at the base of the cylinder at a rate proportional to the square root of the height of water already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(a) Show that at time t seconds, the height h cm of the water in the container satisfies the differential equation

$\frac{dh}{dt} = 0.4 - k\sqrt{h}$, where k is a positive constant.

When $h = 25$ cm, water is leaking out of the hole at 400 cm^3 per second.

(b) Show that $k = 0.02$

(c) Show that the time taken to fill the cylinder from empty to a height of 100 cm is given by $\int_0^{100} \frac{50}{20-\sqrt{h}} dh$

Using the substitution $h = (20 - x)^2$, or otherwise,

(d) find the exact value of $\int_0^{100} \frac{50}{20-\sqrt{h}} dh$

(e) Hence find the time taken to fill the cylinder from empty to a height of 100 cm giving your answer in minutes and seconds to the nearest second.

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