Complex Numbers

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SYLLABUS (6042) REQUIREMENTS

- Find the conjugates, moduli and arguments of complex numbers
- Carry out operations with complex numbers
- Represent complex numbers on an Argand diagram
- Solve polynomial equations with at least one pair of non- real roots
- Express complex numbers in polar form
- Carry out operations of complex numbers expressed in polar form
- Illustrate equations and inequalities involving complex numbers by means of loci in an Argand diagram
- Derive the DeMoivre's Theorem
- Prove the DeMoivre's Theorem
- Prove trigonometrical identities using DeMoivre's Theorem
- Solve equations using the DeMoivre's Theorem
- Solve problems involving complex numbers
- N^{th} roots of unity

The Complex Number System

- If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Now $b^2 - 4ac$ is called the discriminant.
  - (i) If $b^2 - 4ac = 0$, there is one repeated real root
  - (ii) If $b^2 - 4ac > 0$, there are two distinct and real roots
  - (iii) If $b^2 - 4ac < 0$, there are no real roots but we have imaginary roots represented by $\boldsymbol{i}$.

Example

Solve the equation $x^2 + 4x + 20 = 0$

Suggested solution

$x^2 + 4x + 20 = 0$

$x^2 + 4x = -20$

$x^2 + 4x + (+2)^2 = -20 + (+2)^2$

$(x + 2)^2 = -20 + (+2)^2$

$(x + 2)^2 = -16$

$x + 2 = \pm\sqrt{-16}$

$x + 2 = \pm\sqrt{16 \times -1}$

$x + 2 = \pm 4\sqrt{-1}$

$x + 2 = \pm 4i$

$\therefore x = -2 \pm 4i$

The symbol $\boldsymbol{i}$ is used to denote $\sqrt{-1}$

$\Rightarrow -1 = i^2$

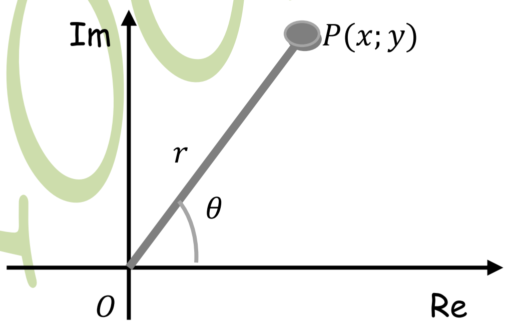
The General Complex Number

- A Complex number is represented in the form $x + iy$, where x and y are real numbers.
- x represents the real part and y represents the imaginary part.
- The set of real numbers ($\mathbb{R}$) is also a subset of the complex numbers ($\mathbb{C}$)

NB: Real numbers can be expressed in the form $x + 0i$

The modulus and argument of a Complex Number

- Complex numbers can be represented by points on a plane
- The diagram of points in Cartesian coordinates representing complex numbers is called an Argand diagram
- The y-axis represents the imaginary part and the x-axis represents the real part of a complex number $x + yi$.



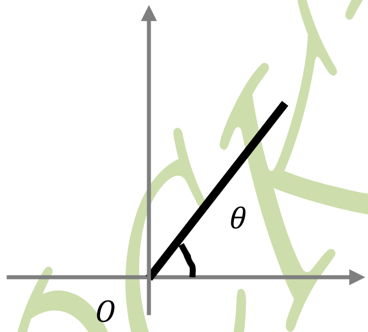
- If the complex number $x + yi$ is denoted by z, and hence $z = x + yi$, $|z|$ is defined as the distance frpm the origin O to the point P representing z.
- Thus $|z| = OP = r$.
- The modulus of a complex number z is given by: $z = \sqrt{x^2 + y^2}$
- The argument of z, $arg\,(z)$ is defined as the angle between the line OP and the positive x axis is usually in the range $(-\pi, \pi)$ or $(-180°, 180°)$

- $(\pi, -\pi)$ is sometimes referred to as the Principal argument.
- The argument of a complex number z is given by $\arg(z) = \theta$, where:

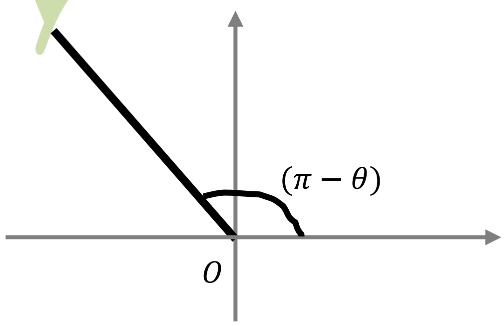
$$tan\theta = \frac{y}{x}$$

NB: One must be very careful when x or y, or both are negative. The quadrant in which it appears will determine whether its argument is negative or positive and whether it is acute or obtuse.

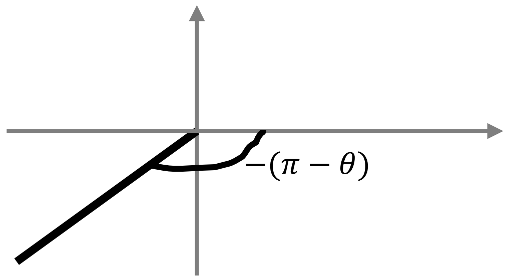
(i) Angles in first quadrant are measured anticlockwise from the positive real axis so θ is the required angle.



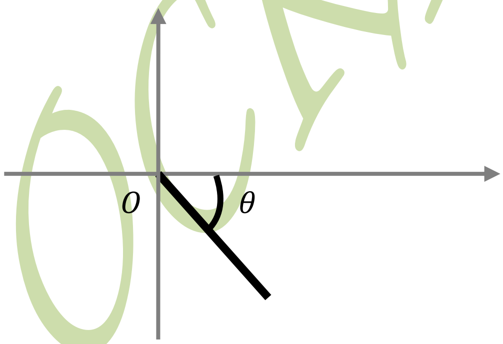
(ii) Angles in second quadrant are measured anticlockwise from the positive real axis so the required angle is $(\pi - \theta)$ or $(180° - \theta)$ or $\pi - \tan^{-1}\left(\frac{y}{x}\right)$ or $180° - \tan^{-1}\left(\frac{y}{x}\right)$



(iii) Angles in third quadrant are measured clockwise from the positive real axis and is negative so the required angle is $-(\pi-\theta)$ or $-\left[\pi-tan^{-1}\left(\frac{y}{x}\right)\right]$ or $-\pi+tan^{-1}\left(\frac{y}{x}\right)$ or $-\left[180°-\tan^{-1}\left(\frac{y}{x}\right)\right]$.



(iv) Angles in fourth quadrant are measured clockwise from the positive real axis and is negative so the required angle is $-\theta$ or $\left[2\pi-tan^{-1}\left(\frac{y}{x}\right)\right]$ or $\left[360°-tan^{-1}\left(\frac{y}{x}\right)\right]$.



NB: Degrees are also applicable

Solved Problems

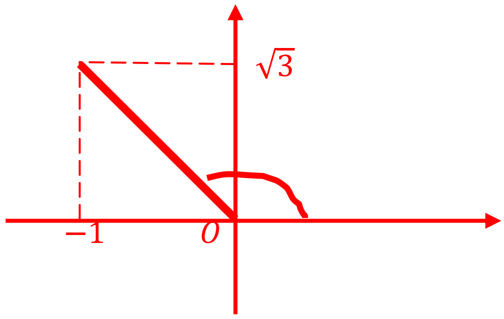
Example

Find the modulus and argument of the complex numbers:

a) $-1+\sqrt{3}i$ b) $-\sqrt{3}-i$ c) $\sqrt{3}-i$ d) $1+\sqrt{3}i$

Suggested solution

a) $-1+\sqrt{3}i$

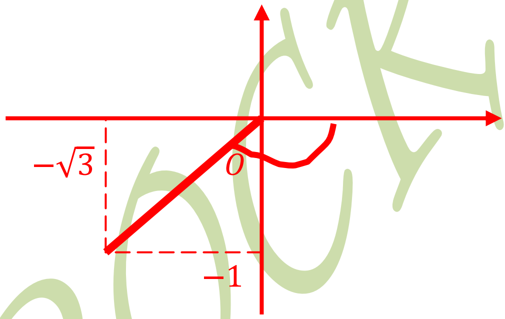


(i) $\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=2$

(ii) From the argand diagram, θ lies in the second quadrant hence

$$\theta=\pi-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

b) $-\sqrt{3}-i$

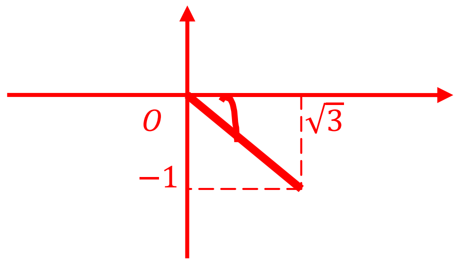


(i) $\sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}=\sqrt{4}=2$

(ii) From the argand diagram, θ lies in the third quadrant hence

$$\theta=-\left[\pi-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]=-\left(\pi-\frac{\pi}{6}\right)=-\frac{5\pi}{6}$$

c) $\sqrt{3}-i$

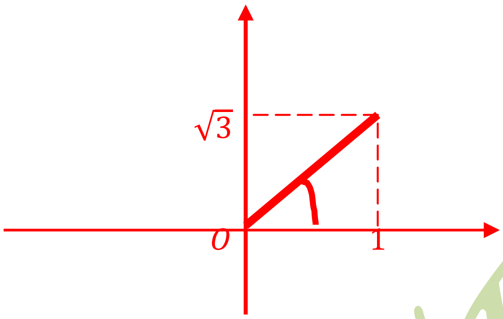


(i) $\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{4} = 2$

(ii) From the argand diagram, θ lies in the fourth quadrant hence

$$\theta = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

d) $1 + \sqrt{3}i$



(i) $\sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = \sqrt{4} = 2$

(ii) From the argand diagram, θ lies in the first quadrant hence

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

Addition, Subtraction and Multiplication of complex number of the form $x + iy$

- In general, if $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then:
  - (i) $z_1 + z_1 = (a_1 + a_2) + i(b_1 + b_2)$
  - (ii) $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
  - (iii) $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_2 b_1 + a_1 b_2)$

Example

Given that $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$, find

a) $z_1 + z_2$ b) $z_1 - z_2$ c) $z_1 z_2$

Suggested Solution

a) $z_1 + z_2 = (3 + 4i) + (1 - 2i)$

$= 3 + 4i + 1 - 2i$

$= 4 + 2i$ or

$z_1 + z_2 = (3 + 4i) + (1 - 2i)$

$= (3 + 1) + i(4 - 2)$

$= 4 + 2i$

b) $z_1 - z_2 = (3 + 4i) - (1 - 2i)$

$= 3 + 4i - 1 + 2i$

$= 2 + 6i$ or

$z_1 - z_2 = (3 + 4i) - (1 - 2i)$

$= (3 - 1) + i[4 - -2]$

$= 2 + 6i$

c) $z_1 z_2 = (3 + 4i)(1 - 2i)$

$= 3 - 6i + 4i - 8i^2$

$= 3 - 2i + 8 \ (since\ i^2 = -1)$

$= 11 - 2i$ or

$z_1 z_2 = (3 + 4i)(1 - 2i)$

$= (3 \times 1 - 4 \times -2) + i(1 \times 4 + 3 \times -2)$

$= 11 - 2i$

The conjugate of a complex number and the division of complex numbers of the form $x + iy$

- The conjugate of a complex number $Z = x + iy$, is denoted Z^* or $\bar{Z}$, is the complex number $Z^* = x - iy$ eg the conjugate of $-3 + 2i$ is $-3 - 2i$
- On an Argand diagram, the point representing the complex number Z^* is the reflection of the point representing Z on the $x\ axis$
- The important property of Z^* is that the product ZZ^* is real since:

$$ZZ^* = (x + iy)(x - iy)$$
$$= (x^2 + ixy - ixy - i^2y^2$$
$$= x^2 + y^2$$

NB: $ZZ^* = |z|^2$

- When dividing complex numbers we use the complex conjugate.

Example

Simplify $\frac{z_1}{z_2}$ where $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$

Suggested solution

$$\frac{z_1}{z_2} = \frac{(3 + 4i)}{(1 - 2i)}$$

[Multiply the numerator and denominator of $\frac{z_1}{z_2}$ by $Z_2{}^*$ ie $(1 + 2i)$]

$$= \frac{(3 + 4i)(1 + 2i)}{(1 - 2i)(1 + 2i)}$$

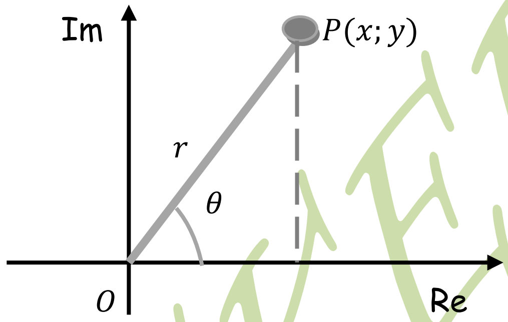
$$= \frac{(3 + 6i + 4i + i^2 8)}{(1^2 + 2^2)}$$

$$= \frac{(3 + 10i - 8)}{5}$$

$$= \frac{-5+10i}{5}$$

$$= -1+2i$$

The Polar form of a complex number



- In the diagram above $x = rCos\theta$ and $y = rSin\theta$
- If P is the point representing the complex number $z = x + iy$, it follows that z may be written in the form $rCos\theta + irSin\theta$
- This is called the polar form or modulus argument form of a complex number.
- A complex number may be written in the form $Z = r(Cos\theta + iSin\theta)$, where $|Z| = r$ and $\arg(Z) = \theta$
- For brevity, $r(Cos\theta + iSin\theta)$ can be written as (r, θ)

Example

1. Express $\frac{3}{1+i\sqrt{3}}$ in polar form, giving exact values of r and θ where possible, or value to two d.p.
2. Write in the form $(a + ib)$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.
   a) $3\sqrt{2}\left(Cos\,\frac{\pi}{4} + iSin\frac{\pi}{4}\right)$

b) $4\left(Cos\,\frac{-5\pi}{2} + iSin\frac{-5\pi}{2}\right)$

Suggested solution

1) $\frac{3}{1+i\sqrt{3}} = \frac{3(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})}$

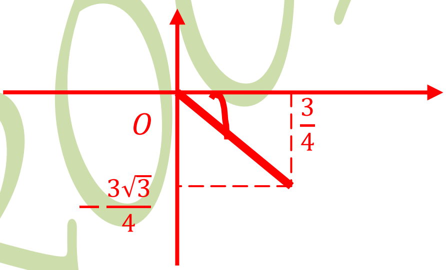
$$= \frac{3 - i3\sqrt{3}}{1^2 + \left(\sqrt{3}\right)^2}$$

$$= \frac{3 - i3\sqrt{3}}{4}$$

$$= \frac{3}{4} - i\,\frac{3\sqrt{3}}{4}$$

NB: Multiply the numerator and denominator of $\frac{3}{1+i\sqrt{3}}$ by the conjugate

i.e. $(1-i\sqrt{3})]$



(i) $\sqrt{\left(\frac{3}{4}\right)^2 + \left(-\frac{3\sqrt{3}}{4}\right)^2} = \sqrt{\frac{9}{16} + \frac{9\times 3}{16}} = \sqrt{\frac{36}{16}} = \frac{6}{4} = \frac{3}{2}$

(ii) From the argand diagram, θ lies in the second quadrant hence

$$\theta = -\tan^{-1}\left(\frac{\frac{3\sqrt{3}}{4}}{\frac{3}{4}}\right) = -\tan^{-1}(\sqrt{3}) = -\frac{\pi}{3}$$

Therefore the solution is $\frac{3}{2}[Cos\left(-\frac{\pi}{3}\right) + iSin\left(-\frac{\pi}{3}\right)$

2) (a) $3\sqrt{2}\left(Cos\ \frac{\pi}{4} + iSin\frac{\pi}{4}\right) = 3\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$$= \frac{3 \times 2}{2} + i\frac{3 \times 2}{2}$$

$$= 3 + 3i$$

(b) $4\left(Cos\ \frac{-5\pi}{2} + iSin\frac{-5\pi}{2}\right) = 4\left[-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right]$

$$= -2\sqrt{3} - 2i$$

$$= -2(\sqrt{3} + i)$$

Products and Quotients of complex number in their Polar form

- If $z_1 = r_1(Cos\theta_1 + iSin\theta_1)$ and $z_2 = r_2(Cos\theta_2 + iSin\theta_2)$ then:

  (a) $z_1z_2 = r_1r_2[Cos(\theta_1 + \theta_2) + iSin(\theta_1 + \theta_2)]$ and

  (b) $\frac{z_1}{z_2} = r_1r_2[Cos(\theta_1 - \theta_2) + iSin(\theta_1 - \theta_2)]$

Example

Simplify z_1z_2 where $z_1 = 2\left(Cos\frac{\pi}{3} + iSin\frac{\pi}{3}\right)$ and $z_2 = 3\left(Cos\frac{\pi}{6} - iSin\frac{\pi}{6}\right)$

Suggested Solution

$$z_1z_2 = 2\left(Cos\frac{\pi}{3} + iSin\frac{\pi}{3}\right)3\left(Cos\frac{\pi}{6} - iSin\frac{\pi}{6}\right)$$

$$= 6\left[Cos\frac{\pi}{3}Cos\frac{\pi}{6} - iCos\frac{\pi}{3}Sin\frac{\pi}{6} + iSin\frac{\pi}{3}Cos\frac{\pi}{6} - i^2\ Sin\frac{\pi}{3}Sin\frac{\pi}{6}\right]$$

$$= 6\left[\left(Cos\frac{\pi}{3}Cos\frac{\pi}{6} + i\ Sin\frac{\pi}{3}Sin\frac{\pi}{6}\right) + i\left(Sin\frac{\pi}{3}Cos\frac{\pi}{6} - Cos\frac{\pi}{3}Sin\frac{\pi}{6}\right)\right]$$

$$= 6\left[Cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\ Sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right]$$

NB Use the identities: $Cos(A - B) = CosACosB + i\ SinASinB$

$Sin(A - B) = SinACosB + i\ CosASinB$

$= 6\left[Cos\frac{\pi}{6} + i\,Sin\frac{\pi}{6}\right]$

Problems involving complex numbers

- You can solve problems by equating real parts and imaginary parts from each side of an equation involving complex numbers.
- This technique can be used to find the square roots of a complex number
- If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$

Worked Examples

Example 1

If $3 + 5i = (a + ib)(1 + i)$ where a and b are real, find the value of a and the value of b

Suggested Solution

$(a + ib)(1 + i) = a(1 + i) + ib(1 + i) = a + ai + bi - b = (a - b) + i(a + b)$

So $(a - b) + i(a + b) = 3 + 5i$

$\Rightarrow a - b = 3$ (i) (Equating real parts)

$a + b = 5$ (ii) (Equating imaginary parts)

Adding (i) and (ii): $2a = 8 \Rightarrow a = 4$

$a - b = 3$ (i)

$\Rightarrow 4 - b = 3$

$\therefore b = 1$

Example 2

Find the square root of $3 + 4i$.

Suggested Solution

Suppose the square root of $3 + 4i$ is $a + ib$ where a and b are real.

$\Rightarrow (a + ib)^2 = 3 + 4i$

$a^2 + 2abi + i^2b^2 = 3 + 4i$

$(a^2 - b^2) + 2abi = 3 + 4i$

Equating real parts and Imaginary parts together:

$a^2 - b^2 = 3$ (i)

$2ab = 4$ (ii)

From (ii):

$b = \frac{2}{a}$ (iii)

$\Rightarrow a^2 - \left(\frac{2}{a}\right)^2 = 3$

$\Rightarrow a^2 - \frac{4}{a^2} = 3$

$\Rightarrow a^4 - 3a^2 - 4 = 0$

$\Rightarrow a^4 - 4a^2 + a^2 - 4 = 0$

$\Rightarrow a^2(a^2 - 4) + 1(a^2 - 4) = 0$

$\Rightarrow (a^2 + 1)(a^2 - 4) = 0$

$\Rightarrow a^2 + 1 = 0$ or $a^2 - 4 = 0$

$\Rightarrow$ *No real solution* or $a^2 - 4 = 0$

$\Rightarrow\ a^2 - 4 = 0$

$\therefore a = \pm 2$

$b = \frac{2}{a}$ (iii)

$\Rightarrow b = \pm\frac{2}{2}$

$\therefore b = \pm 1$

$\Rightarrow$ The roots are $\pm(2 + i)$

Example 3

Simplify $\frac{(1+i)^4}{(2-2i)^3}$, giving your answer in the form $a + bi$

Suggested Solution

$\frac{(1+i)^4}{(2-2i)^3} \equiv \frac{(1+i)^4}{2^3(1-i)^3}$

$$= \frac{(1+i)^4}{8(1-i)^3}$$

Let $1 + i \equiv i(1-i)$ br

$$\Rightarrow \frac{(1+i)^4}{8(1-i)^3} = \frac{[i(1-i)]^4}{8(1-i)^3}$$

$$= \frac{i^4(1-i)^4}{8(1-i)^3}$$

$$= \frac{1(1-i)^{4-3}}{8}$$

$$= \frac{1-i}{8}$$

$$= \frac{1}{8} - \frac{i}{8}$$

Polynomials: Roots of Polynomial equations with real coefficients

- If the roots α and β of a quadratic equation are complex, α and β are always a complex conjugate pair
- Given any complex root of a quadratic equation you can find the equation
- Complex roots of a polynomial equation with real coefficients occur in conjugate pairs
- Suppose the equation $ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots + k$ has n roots α , β and $\gamma, \ldots$ then the

  (i) sum of the roots $= -\frac{b}{a}$

  (ii) sum of the products of all possible pairs of roots $= \frac{c}{a}$

  (iii)sum of products of all possible combinations of roots taken three at a time, and so on $= -\frac{d}{a}$

  (iv)product of n roots $= \frac{(-1)^n k}{a}$.

Worked problems

Example 1

Given that the root of $3z^3 - 10z^2 + 20z - 16 = 0$ is $1 - \sqrt{3}i$. Find the other roots.

Suggested Solution

The other root is $1 + \sqrt{3}i$ (conjugate).

Since sum of roots $=$ coefficient of $-\frac{z^2}{z^3}$:

Let the 3rd root $= x$.

Hence $x + (1 + \sqrt{3}i + (1 - \sqrt{3}i) = -\left(-\frac{10}{3}\right)$

$x + 1 + 1 = \frac{10}{3}$

$x = \frac{10}{3} - 2 = \frac{4}{3}$.

$\therefore$ The roots are $(1 + \sqrt{3}i)$ and $\frac{4}{3}$.

Example 2

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The equation $x^3 - 2x^2 + 4x - 8 = 0$ is $2i$ as one of its roots. Find the other roots. **[3]**

Suggested Solution

The other root is $-2i$ (conjugate).

Since sum of roots $=$ coefficient of $-\frac{x^2}{x^3}$:

Let the 3rd root $= x$.

Hence $x + 2i$ -$2i = -\left(-\frac{2}{1}\right)$

$x = 2$.

$\therefore$ The roots are 2 and $-$ 2i .

Example 3

$7 + 2i$ is one of the roots of a quadratic equation. Find its equation.

Suggested Solution

The other root is $7 - 2i$ (conjugate).

NB: The equation with roots α and β is $(x-\alpha)(x-\beta)=0$

$\Rightarrow [x-(7-2i)][x-(7+2i)]=0$

$\Rightarrow x^2 - x(7+2i) - x(7-2i) + (7-2i)(7+2i) = 0$

$\Rightarrow x^2 - 7x - 7xi - 7x + 7xi + (7^2 + 2^2) = 0$

$\Rightarrow x^2 - 14x + 53 = 0$

Example 4

Show that $x = 2$ is a solution of the cubic equation $x^3 - 6x^2 + 21x - 26 = 0$.

Hence solve the equation completely.

Suggested Solution

Let $f(x) = x^3 - 6x^2 + 21x - 26$

If $x = 2$ the $f(2) = 0$

$\Rightarrow f(2) = (2)^3 - 6(2)^2 + 21(2) - 26 = 8 - 24 + 42 - 26 = 0$

$\therefore (x-2)$ is a solution.

$$
\begin{array}{r|l}
 & x^2 - 4x + 13 \\
\hline
x - 2 & x^3 - 6x^2 + 21x - 26 \\
- & \underline{x^3 - 2x^2} \\
 & -4x^2 + 21x - 26 \\
- & \underline{-4x^2 + 8x} \\
 & 13x - 26 \\
- & \underline{13x - 26} \\
 & 0
\end{array}
$$

$f(x) = 0$

$\Rightarrow (x-2)(x^2 - 4x + 13) = 0$

$\Rightarrow x - 2 = 0$ or $x^2 - 4x + 13 = 0$

$x^2 - 4x = -13$

$x^2 - 4x + (-2)^2 = -13 + (-2)^2$

$(x-2)^2 = -9$

$x - 2 = \pm\sqrt{-9}$

$x - 2 = \pm 3i$

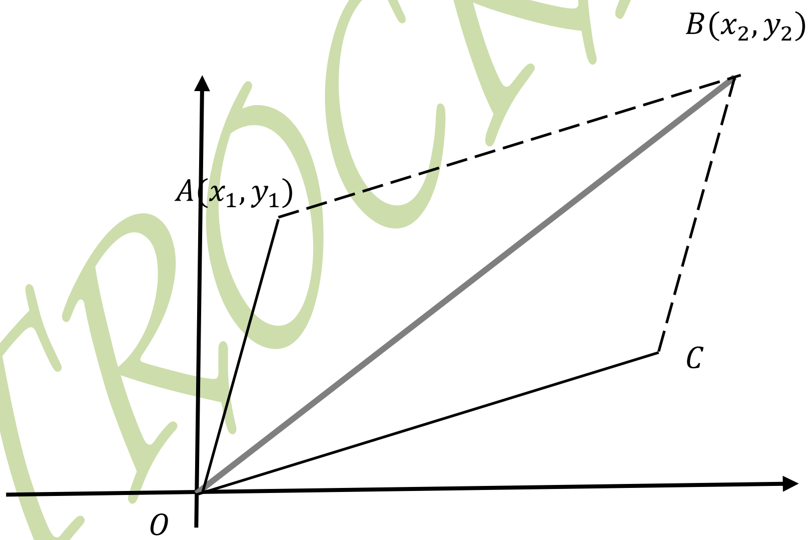
$x = 2 \pm 3i$

$\therefore x = 2; 2 + 3i$ or $2 - 3i$

NB: For a cubic equation either

- all the three roots are real or
- one of the roots is real and the other two roots form a complex conjugate pair.

Further consideration of $|Z_2 - Z_1|$ and $arg(Z_2 - Z_1)$



- Let $Z = Z_2 - Z_1$ where $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$.
- The points A and B represent Z_1 and Z_2 respectively, on Argand diagram.
- $Z = Z_2 - Z_1 = (x_2 - x_1) + i(y_2 - y_1)$. Hence $OABC$ becomes a parallelogram.
- $|Z_2 - Z_1| = |\overrightarrow{OC}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{1}{2}}$ i.e. $|Z_2 - Z_1|$ is the length of AB in the Argand diagram.

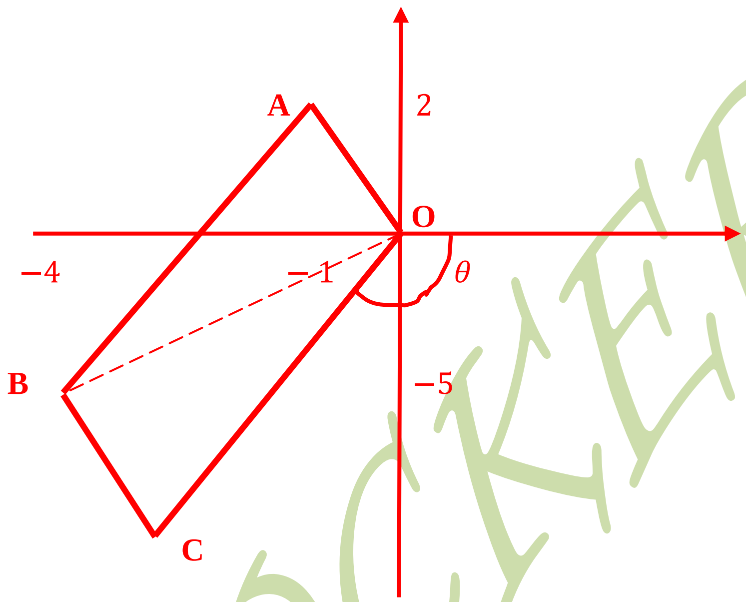
- arg $(Z_2 - Z_1)$ is the angle between OC and the positive direction of the x $axis$.

NB: $arg(u^*) - arg\ (u) \equiv arg\left(\frac{u^*}{u}\right)$

Example

Find $|Z_2 - Z_1|$ and $arg\ (Z_2 - Z_1)$ if $Z_1 = -1 + 2i$ and $Z_2 = -4 - 5i$.

Solution



$$Z = Z_2 - Z_1 = (-4 - 5i) + i(-1 + 2i)$$
$$= (-4 + 1) - i(5 + 2)$$
$$= -3 - 7i$$

Now:

$$|Z_2 - Z_1| = \sqrt{(-3)^2 + (-7)^2}$$
$$= \sqrt{9 + 49} = \sqrt{58} \text{ and}$$

$$\arg\ (Z_2 - Z_1) = \theta = -\left[\pi - \tan^{-1}\left(\frac{7}{3}\right)\right]$$
$$= -1.975688113 \text{ rad}$$
$$= -1.98 \text{ rad}$$

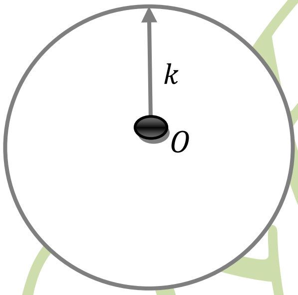
LOCI ON ARGAND DIAGRAM

- A locus is a path traced out by a plant subjected to certain restrictions.
- Paths can be traced out by points representing variable complex numbers on an Argand diagram just as they can in any other coordinate system.

Types of LOCI

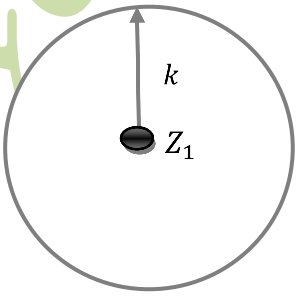
1) $|Z| = k$ represents a circle with centre O and radius k.

   If the point P represents the complex number Z: $|Z| = k$, then the distance of P from the origin O is a constant and so P will trace out a circle.



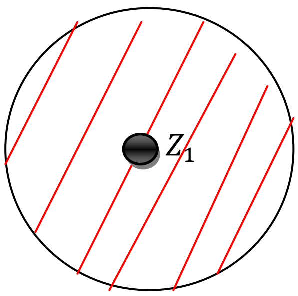
2) $|Z - Z_1| = k$ represents a circle with centre Z_1 and radius k.

   If $|Z - Z_1| = k$, where Z_1 is a fixed complex number represented by point A on an argand diagram then $|Z - Z_1|$ represents the distance AP and is constant. It follows that P must lie on a circle with centre A and radius k.

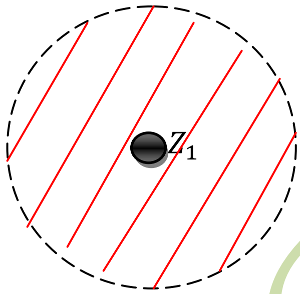


3) $|Z - Z_1| \leq k$ and $|Z - Z_1| < k$

   If $|Z - Z_1| \leq k$ or $|Z - Z_1| < k$ then the point representing P cannot lie only on the circumference (NB: for $|Z - Z_1| \leq k$), but also anywhere inside the circle. The locus P is therefore the region on (NB: for $|Z - Z_1| \leq k$) and within the circle with centre Z_1 and radius k.



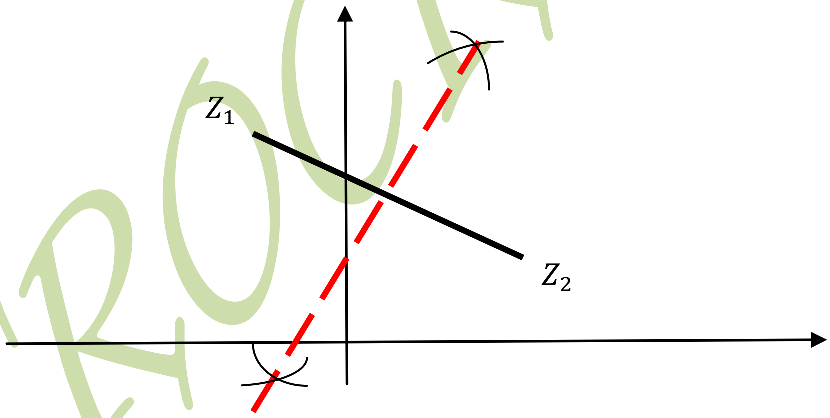
a) $|Z - Z_1| \leq k$



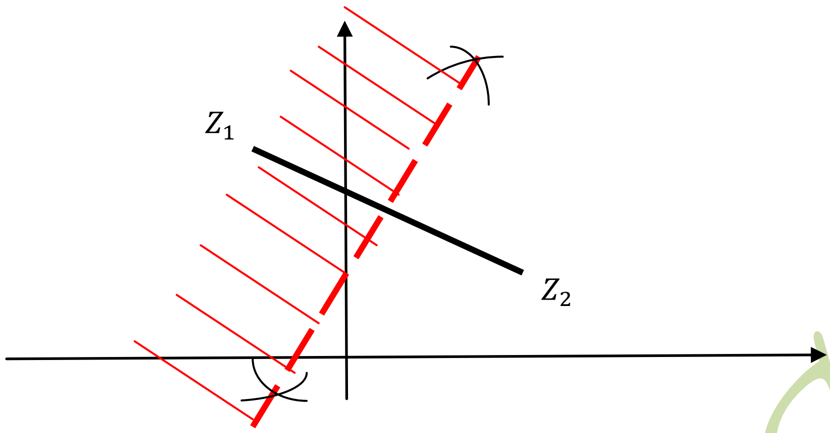
b) $|Z - Z_1| < k$

NB: $|Z - Z_1| = k|Z - Z_2|$ also represents a circle

4) $|Z - Z_1| = |Z - Z_2|$ represents a straight line. It is the perpendicular bisector of the line joining Z_1 and Z_2. NB: $\frac{|Z-Z_1|}{|Z-Z_2|} = k \Longrightarrow |Z - Z_1| = |Z - Z_2|$



5) $|Z - Z_1| \leq |Z - Z_2|$. The locus Z is not only the perpendicular bisector of AB, but also the whole half plane, in which A lies, bounded by this bisector.

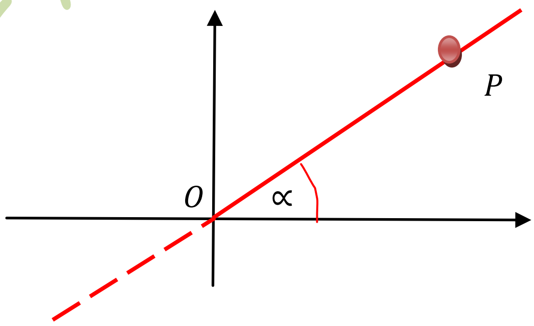


NB: All the loci considered so far have been related to distances - there are also simple Loci in Argand diagrams involving angles.

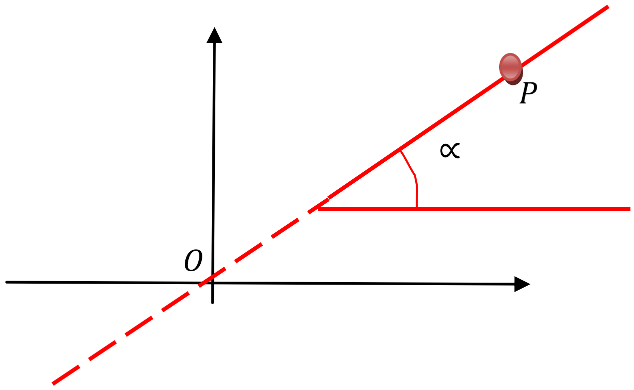
The simplest case is the locus of P subject to the conditions that arg $(z) = \propto$ where $\propto$ is a fixed angle.

6) arg $(z) = \propto$ represents the half line through O inclined at an angle $\propto$ to the positive direction of Ox.

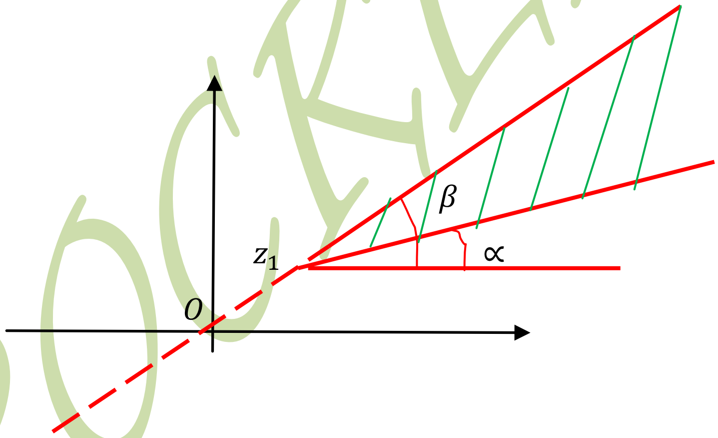
NB: The locus of P is only a half line - the other half, shown dotted in the diagram below, would have the equation $\arg(z) = \pi + \propto$ possibly $\pm 2\pi$ if $\pi + \propto$ falls outside the specified range for $\arg(z)$



7) arg $(z - z_1) = \propto$ represents the half line through the point z_1 inclined at an angle $\propto$ to the positive direction of Ox.



8) $\propto \leq \arg(z - z_1) \leq \beta$ indicates that the angle between AP and the positive $x - axis$ lies between $\propto$ and β, so that P can be on or within the two half line as sown in the diagram below.



9) $arg\left(\frac{z-a}{z-b}\right) = \theta$ describes an arc with end points A and B making an angle θ. Draw an arc starting from A to B.

NB: If θ is positive, then draw the arc going anticlockwise ( ) and if θ is negative then draw the arc going clockwise ( )

NB: $arg\left(\frac{z-a}{z-b}\right) = \theta \quad \equiv \quad arg(z-a) - arg(z-b) = \theta$

Solved Examples

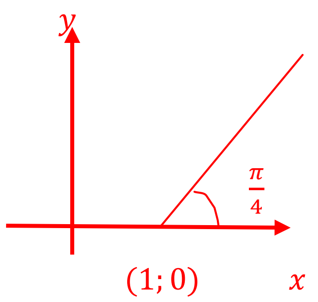
Question 1

Sketch on argand diagram the locus of points satisfying:

a) $\arg(z-1)=\frac{\pi}{4}$ b) $|z-2-i|=5$ c) $|z|=3$

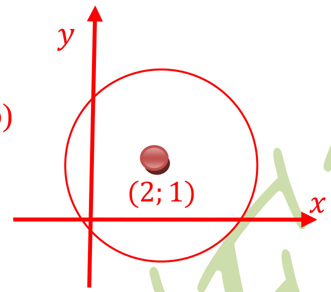
Suggested Solution

a)



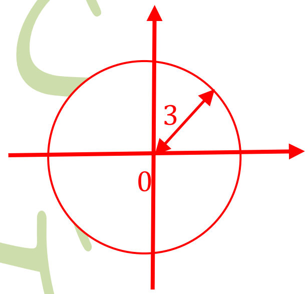
$\arg(z-1)=\frac{\pi}{4}$

b)



$|z-(2+i)|=5$

c)



$|z|=3$

Question 2

Sketch on argand diagram the locus of points satisfying:

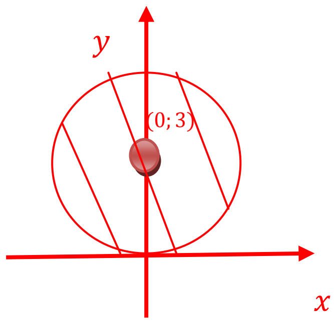
a) $|z-3i|\le 3$

b) $\frac{\pi}{2}\le \arg(Z-4-2i)\le\frac{5\pi}{6}$

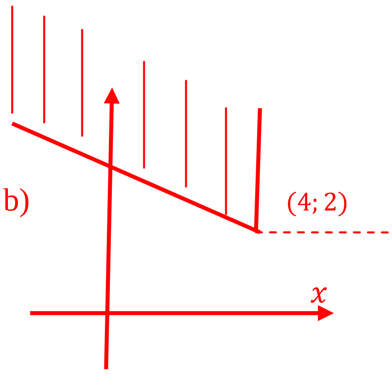
c) $arg\left(\frac{z-3i}{z+4}\right)\le\frac{\pi}{3}$

Suggested Solution

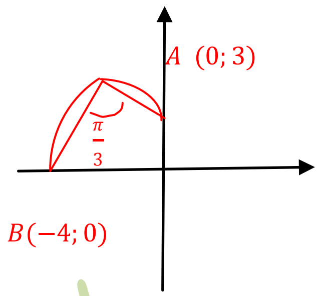
a)



b)



c)



$|z - 3i| \leq 3$ $\quad$ $\frac{\pi}{2} \leq arg\ (Z - 4 - 2i) \leq \frac{5\pi}{6}$ $\quad$ $arg\left(\frac{z-3i}{z+4}\right) \leq \frac{\pi}{3}$

Question 3

The point P represents a complex number z on an Argand diagram, where

$$|z - 6 + 3i| = 3|z + 2 - i|$$

Show that the locus of P is a circle, giving the coordinates of the centre and the radius of this circle.

Solution

Let $z = x + iy$

$|z - 6 + 3i| = 3|z + 2 - i|$

$|x + iy - 6 + 3i| = 3|x + iy + 2 - i|$

$|(x - 6) + i(y + 3)| = 3|(x + 2) + i(y - 1)|$

$(x - 6)^2 + (y + 3)^2 = 9[(x + 2)^2 + (y - 1)^2]$

$x^2 - 12x + 36 + y^2 + 6y + 9 = 9[x^2 + 4x + 4 + y^2 - 2y + 1]$

$8x^2 + 8y^2 + 48x - 24y = 0$

$x^2 + 6x + y^2 - 3y = 0$

$$(x+3)^2 - 9 + \left(y - \frac{3}{2}\right)^2 - \frac{9}{4} = 0$$

$$(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{45}{4}$$

center: $\left(-3, \frac{3}{2}\right)$ and radius: $\frac{3}{2}\sqrt{5}$

Question 3

Sketch on argand diagram the locus of points satisfying:

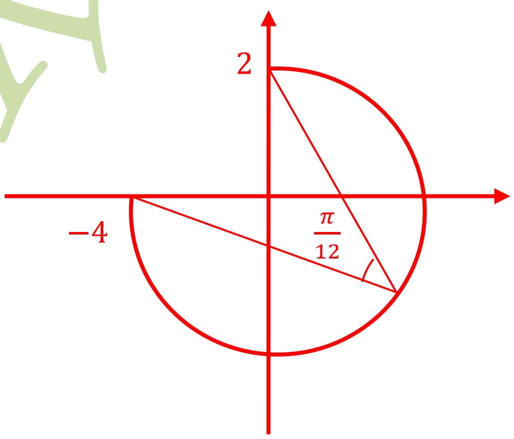
$$arg(z-2i) - arg(z+4) = -\frac{\pi}{12}$$

Solution

$$arg(z-2i) - arg(z+4) = -\frac{\pi}{12}$$

$$arg\left(\frac{z-2i}{z+4}\right) = -\frac{\pi}{12}$$

$$arg\left[\frac{z-(0+2i)}{z-(-4+0i)}\right] = -\frac{\pi}{12}$$



Solved Past Examination Questions

Question 1

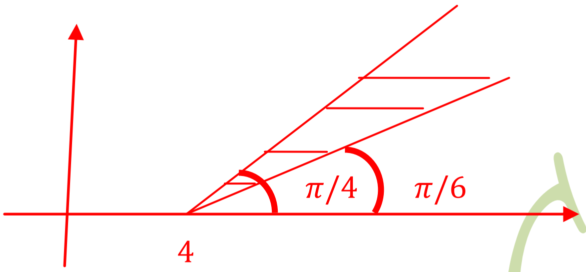
ZIMSEC JUNE 2019 PAPER 2

On a single diagram shade the region defined by the inequalities

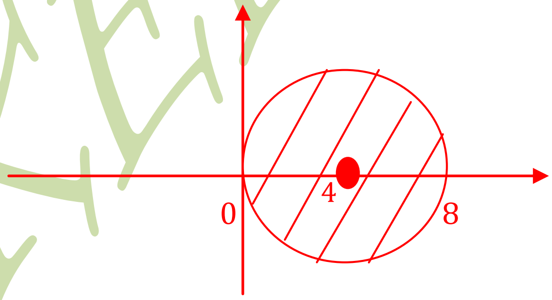
$\frac{\pi}{6} \leq \arg(Z-4) \leq \frac{\pi}{4}$ and $|z-4| \leq 4$ **[3]**

Solution

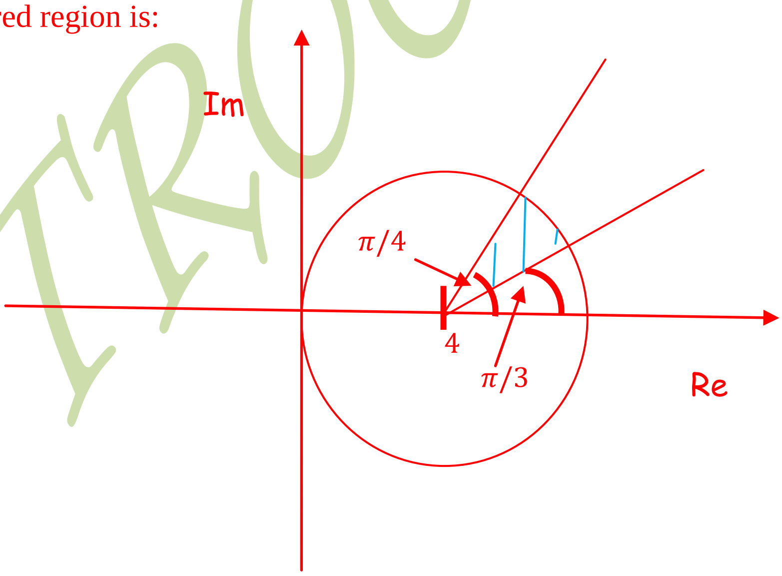
$\frac{\pi}{6} \leq \arg(Z-4) \leq \frac{\pi}{4}$



$|z-4| \leq 4$



The required region is:



Question 2

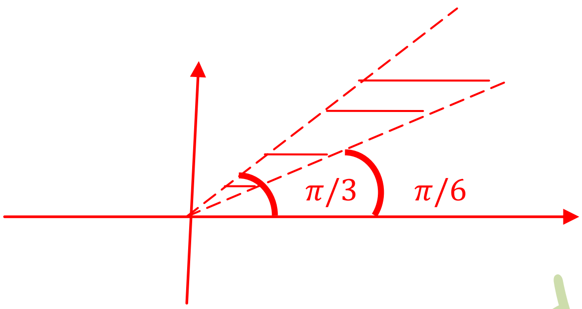
ZIMSEC NOVEMBER 2019 PAPER 2

The complex number $\boldsymbol{z}$ satisfies the inequalities $2 < |\mathbf{z}| < 3$ and $\frac{\pi}{6} < \arg\ \mathbf{z} < \frac{\pi}{3}$.

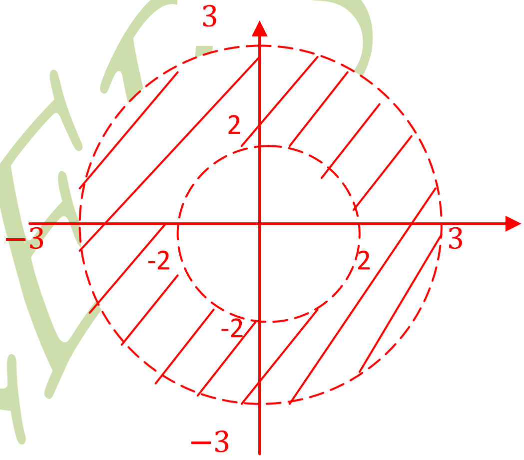
Sketch and shade on an Argand diagram the region represented by the inequalities. **[4]**

Solution

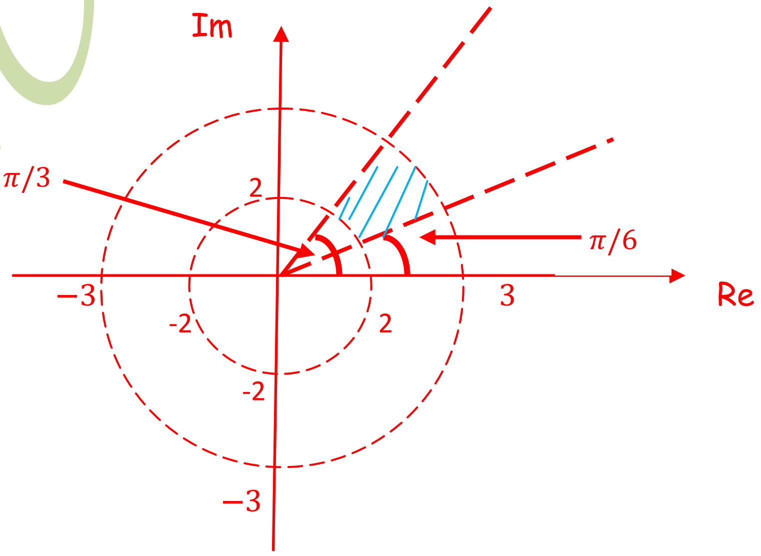
$\frac{\pi}{6} \leq \arg \mathbf{z} \leq \frac{\pi}{3}$



$2 < |\mathbf{z}| < 3$



The required region is:



DEMOIVRE'S THEOREM

- Given that $Z = r(Cos\theta + iSin\theta)$ is a complex number and n is a positive integer, then

$$Z^n = [r(Cos\theta + iSin\theta)]^n = r^n(Cosn\theta + iSinn\theta)$$

NB: DeMoivre's theorem holds not only when n is a positive integer, but also when it is negative and even when it is fractional

- The DeMoivre's theorem can also be written as

$$Z^n = re^{in\theta}$$

- $Z = r(Cos\theta + iSin\theta)$ can also be written as $Z = re^{i\theta}$

NB: One very important application of DeMoivre's theorem is in condition of complex numbers of the form $(a + ib)^n$

Solved Problems

Example 1

Simplify $\left(Cos\frac{\pi}{6} + iSin\frac{\pi}{6}\right)^3$

Suggested solution

$$\left(Cos\frac{\pi}{6} + iSin\frac{\pi}{6}\right)^3 = Cos\frac{3\pi}{6} + iSin\frac{3\pi}{6}$$

$$= Cos\frac{\pi}{2} + iSin\frac{\pi}{2}$$

$$= 0 + i$$

$$= i$$

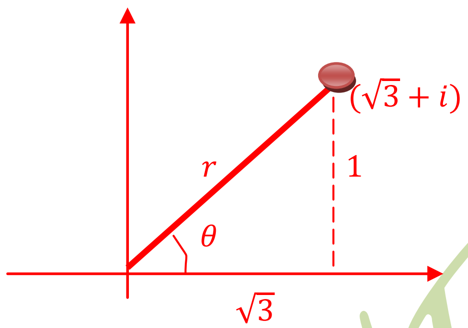
Example 2

Find $\left(\sqrt{3}+i\right)^{10}$ in the form $a+ib$.

Suggested solution

NB: (i) Clearly it would not be practical to multiply $(\sqrt{3}+i)$ by itself ten times.

(ii) Express it in polar form.



$$r=\sqrt{\left(\sqrt{3}\right)^2+(1)^2}=\sqrt{4}=2$$

$$tan\theta=\frac{1}{\sqrt{3}}\Rightarrow\theta=\frac{\pi}{6}$$

Thus $\left(\sqrt{3}+i\right)=2\left(Cos\frac{\pi}{6}+iSin\frac{\pi}{6}\right)$ and

$$\left(\sqrt{3}+i\right)^{10}=2^{10}\left(Cos\frac{\pi}{6}+iSin\frac{\pi}{6}\right)^{10}=2^{10}\left(Cos\frac{10\pi}{6}+iSin\frac{10\pi}{6}\right)$$

$$=1024\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$=512-i512\sqrt{3}$$

Example 3

Simplify $\left(Cos\frac{\pi}{6}-iSin\frac{\pi}{6}\right)^3$

Suggested solution

NB: DeMoivre's theorem applies only to expression in the form $(Cos\theta + iSin\theta)$ and not $(Cos\theta - iSin\theta)$, so the expression to be simplified must be written in the form $[Cos(-\theta) + iSin(-\theta)]$

$$\Rightarrow \left(Cos\frac{\pi}{6} - iSin\frac{\pi}{6}\right) = Cos\left(-\frac{\pi}{6}\right) + iSin\left(-\frac{\pi}{6}\right)$$

Hence

$$\left(Cos\frac{\pi}{6} - iSin\frac{\pi}{6}\right)^3 = \left[Cos\left(-\frac{\pi}{6}\right) + iSin\left(-\frac{\pi}{6}\right)\right]^3$$

$$= Cos\left(-\frac{3\pi}{6}\right) + iSin\left(-\frac{3\pi}{6}\right)$$

$$= Cos\left(-\frac{\pi}{2}\right) + iSin\left(-\frac{\pi}{2}\right)$$

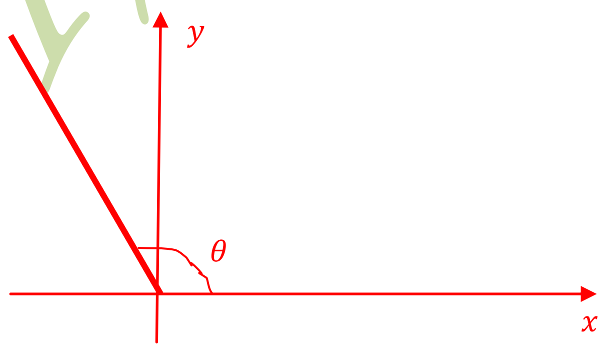
$$= Cos\left(\frac{\pi}{2}\right) - iSin\left(\frac{\pi}{2}\right)$$

$$= -i$$

Example 4

Find $\frac{1}{\left(-2+2\sqrt{3}i\right)^3}$ in the form $a + ib$.

Suggested solution



$$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$$

$$\theta = \pi - tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) \Rightarrow \theta = \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

Now $\frac{1}{\left(-2+2\sqrt{3}i\right)^3} = \left(-2 + 2\sqrt{3}i\right)^{-3}$

$$= \left[4\left\{Cos\left(\frac{2\pi}{3}\right) + iSin\left(\frac{2\pi}{3}\right)\right\}\right]^{-3}$$

$$= 4^{-3}\left[Cos\left(-3 \times \frac{2\pi}{3}\right) + iSin\left(-3 \times \frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{64}[Cos(-2\pi) + iSin(-2\pi)]$$

$$= \frac{1}{64}(1 + 0)$$

$$= \frac{1}{64}$$

Example 5

If $z = cos\theta + isin\theta$, show that

$$\frac{1}{z} = cos\theta - isin\theta.$$

Hence use the DeMoivre's theorem to show that

$$cos\theta - isin\theta \equiv Cos(-\theta) + iSin(-\theta).$$

Suggested solution

$$\frac{1}{z} = \frac{1}{cos\theta + isin\theta}$$

$$= \frac{1(cos\theta - sin\theta)}{(cos\theta + isin\theta)(cos\theta - isin\theta)}$$

$$= \frac{cos\theta - sin\theta}{cos^2\theta + sin^2\theta}$$

$$= \frac{cos\theta - sin\theta}{1}$$

$$= cos\theta - sin\theta \ (as\ required)$$

Now:

$\frac{1}{z} = z^{-1} = (cos\theta + isin\theta)^{-1}$

$= \cos(-\theta) + isin(-\theta)$ Using DeMoivre's theorem

$cos\theta - sin\theta \equiv \cos(-\theta) + isin(-\theta) \quad (as\ required)$

APPLICATION OF DEMOIVRE'S THEOREM IN ESTABLISHING TRIGONOMETRIC IDENTITIES

Example 1

Show that $Cos3\theta = 4Cos^3\theta - 3Cos\theta$

Suggested Solution

$Cos3\theta + iSin3\theta = (Cos\theta + iSin\theta)^3$ (*Using DeMoivre's Theorem*)

$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}}{2!}b^2 + \cdots$

Now:

$Cos3\theta + iSin3\theta = Cos^3\theta + 3Cos^2\theta(iSin\theta) + 3Cos\theta(iSin\theta)^2 + (iSin\theta)^3$

$= Cos^3\theta + 3iCos^2\theta Sin\theta - 3Cos\theta Sin^2\theta - iSin^3\theta$ (Since $i^2 = -1$)

Now $Cos3\theta$ is the real part of the LHS of the equation, and the real parts of both sides can be equated

$$Cos3\theta = Cos^3\theta - 3Cos\theta Sin^2\theta$$

$$= Cos^3\theta - 3Cos\theta(1 - Cos^2\theta) \qquad \text{(Since } Cos^2\theta + Sin^2\theta = 1)$$

$$= 4Cos^3\theta - 3Cos\theta$$

Example 2

Express $Tan3\theta$ in terms of $Tan\theta$.

Suggested Solution

$$Tan3\theta = \frac{Sin3\theta}{Cos\ 3\theta}$$

NB: $Sin3\theta$ and $Cos3\theta$ are obtained from the expansion of $(Cos\theta + iSin\theta)^3$.

Now

$$Tan3\theta = \frac{Sin3\theta}{Cos\ 3\theta} = \frac{3Cos^2\theta Sin\theta - Sin^3\theta}{Cos^3\theta - 3Cos\theta Sin^2\theta}$$

Dividing every term by $Cos^3\theta$

$$Tan3\theta = \frac{\left(\frac{3Cos^2\theta Sin\theta}{Cos^3\theta} - \frac{Sin^3\theta}{Cos^3\theta}\right)}{\left(\frac{Cos^3\theta}{Cos^3\theta} - \frac{3Cos\theta Sin^2\theta}{Cos^3\theta}\right)}$$

$$= \frac{\left(\frac{3Sin\theta}{Cos\theta} - \frac{Sin^3\theta}{Cos^3\theta}\right)}{\left(\frac{Cos^3\theta}{Cos^3\theta} - \frac{3Sin^2\theta}{Cos^2\theta}\right)}$$

$$= \frac{3Tan\theta - Tan^3\theta}{1 - 3Tan^2\theta}$$

Example 3

Express $Cot3\theta$ in terms of $Cot\theta$.

Suggested Solution

$$Cot3\theta = \frac{Cos3\theta}{Sin\ 3\theta}$$

NB: $Sin3\theta$ and $Cos3\theta$ are obtained from the expansion of $(Cos\theta + iSin\theta)^3$.

Now

$$Cot3\theta = \frac{Cos3\theta}{Sin\ 3\theta} = \frac{Cos^3\theta - 3Cos\theta Sin^2\theta}{3Cos^2\theta Sin\theta - Sin^3\theta}$$

Dividing every term by $Sin^3\theta$

$$Cot3\theta = \frac{\left(\frac{Cos^3\theta}{Sin^3\theta} - \frac{3Cos\theta Sin^2\theta}{Sin^3\theta}\right)}{\left(\frac{3Cos^2\theta Sin\theta}{Sin^3\theta} - \frac{Sin^3\theta}{Sin^3\theta}\right)}$$

$$= \frac{\left(\frac{Cos^3\theta}{Sin^3\theta} - \frac{3Cos\theta}{Sin\theta}\right)}{\left(\frac{3Cos^2\theta}{Sin^2\theta} - \frac{Sin^3\theta}{Sin^3\theta}\right)}$$

$$= \frac{Cot^3\theta - 3Cot\theta}{3Cot^2\theta - 1}$$

EXPRESSIONS FOR POWERS OF $Sin\theta$ AND $Cos\theta$ IN TERMS OF SINES AND COSINES OF MULTIPLES

- Expressions for powers of $Sin\theta$ and $Cos\theta$ in terms of sines and cosines of multiples of θ canbe derived using the following results:

  Suppose $z = Cos\theta + iSin\theta$, then

$$z^{-1} = \frac{1}{z} = (Cos\theta + iSin\theta)^{-1}$$

$$= Cos(-\theta) + iSin(-\theta)$$

$$= Cos\theta - iSin\theta$$

- Therefore if $z = Cos\theta + iSin\theta$ then $\frac{1}{z} = Cos\theta - iSin\theta$

  (i) Adding $z + \frac{1}{z} = 2Cos\theta$ and

  (ii) Subtracting $z - \frac{1}{z} = 2iSin\theta$

NB: If $z = Cos\theta + iSin\theta$: $z + \frac{1}{z} = 2Cos\theta$ and $z - \frac{1}{z} = 2iSin\theta$

- Also $z^n = (Cos\theta + iSin\theta)^n = Cos(n\theta) + iSin(n\theta)$,
- Then $z^{-n} = \frac{1}{z^n} = (Cos\theta + iSin\theta)^{-n}$

  $= Cos(-n\theta) + iSin(-n\theta)$

  $= Cos(n\theta) - iSin(n\theta)$
- Combining z^n and $\frac{1}{z^n}$ as before:

  (i) Adding $z^n + \frac{1}{z^n} = 2Cos(n\theta)$ and

  (ii) Subtracting $z^n - \frac{1}{z^n} = 2iSin(n\theta)$

NB: If $z = Cos\theta + iSin\theta$: $z^n + \frac{1}{z^n} = 2Cos(n\theta)$ and $z^n - \frac{1}{z^n} = 2iSin(n\theta)$

NB: A common mistake is to omit the i in $2iSin(n\theta)$, so make a point of remembering this result carefully.

Solved Examples

Example 1

Use DeMoivre's Theorem to show that $Cos^5\theta = \frac{1}{16}(Cos5\theta + 5Cos3\theta + 10Cos\theta)$.

Suggested Solution

Suppose $z = Cos\theta + iSin\theta$ then $z + \frac{1}{z} = 2Cos\theta$

Now

$$(2Cos\theta)^5 = \left(z+\frac{1}{z}\right)^5$$

$$\therefore \left(z+\frac{1}{z}\right)^5 = z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5$$

$$= z^5 + 5z^3 + 10z + 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^5$$

$$\Rightarrow 32\ Cos^5\theta = z^5 + \left(\frac{1}{z}\right)^5 + 5z^3 + 5\left(\frac{1}{z}\right)^3 + 10z + 10\left(\frac{1}{z}\right)$$

$$= \left(z^5 + \frac{1}{z^5}\right) + 5\left[z^3 + \left(\frac{1}{z}\right)^3\right] + 10\left[z + \left(\frac{1}{z}\right)\right]$$

Using the results established earlier: $z^n + \frac{1}{z^n} = 2Cos(n\theta)$

$z^5 + \frac{1}{z^5} = 2Cos(5\theta)$

$z^3 + \frac{1}{z^3} = 2Cos(3\theta)$

and $z + \frac{1}{z} = 2Cos\theta$

Hence $32\ Cos^5\theta\ = 2Cos(5\theta) + 5[2Cos(3\theta)] + 10(2Cos\theta)$

$$Cos^5\theta = \frac{2Cos(5\theta)}{32} + \frac{5[2Cos(3\theta)]}{32} + \frac{10(2Cos\theta)}{32}$$

$\therefore\ Cos^5\theta = \frac{1}{16}(Cos5\theta + 5Cos3\theta + 10Cos\theta)\ \{as\ required\}.$

NB: One very succesful application of the example above would be integrating $\boldsymbol{Cos^5\theta}$

$$\int Cos^5\theta = \int \frac{1}{16}(Cos5\theta + 5Cos3\theta + 10Cos\theta)$$

$$= \frac{1}{16}\left[\frac{Sin(5\theta)}{5} + \frac{5[Sin(3\theta)]}{3} + 10Sin\theta\right] + c$$

Example 2

a) Show that $Cos^3\theta\ Sin^3\theta = \frac{1}{32}(3Sin2\theta - Sin6\theta)$

b) Evaluate

$$\int_0^{\frac{\pi}{2}} Cos^3\theta\ Sin^3\theta.$$

Suggested Solution

$(2Cos\theta)^3 = \left(z + \frac{1}{z}\right)^3$ (i)

$(2iSin\theta)^3 = \left(z - \frac{1}{z}\right)^3$ (ii)

Multiplying (i) and (ii)

$$8Cos^3\theta \times 8i^3Sin^3\theta = \left(z + \frac{1}{z}\right)^3\left(z - \frac{1}{z}\right)^3$$

$$-64iCos^3\theta Sin^3\theta = \left[\left(z - \frac{1}{z}\right)\left(z + \frac{1}{z}\right)\right]^3 = \left(z^2 - \frac{1}{z^2}\right)^3$$

$$= (z^2)^3 - 3(z^2)^2\left(\frac{1}{z^2}\right) + 3(z^2)\left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3$$

$$= z^6 - 3z^2 + 3\left(\frac{1}{z^2}\right) - \frac{1}{z^6}$$

$$= \left(z^6 - \frac{1}{z^6}\right) - 3\left(z^2 - \frac{1}{z^2}\right)$$

Now $z^6 - \frac{1}{z^6} = 2iSin6\theta$ and $z^2 - \frac{1}{z^2} = 2iSin2\theta$

$\Longrightarrow -64iCos^3\theta Sin^3\theta = 2iSin6\theta - 3(2iSin2\theta)$

Dividing by $(-64i)$

$$Cos^3\theta Sin^3\theta = -\frac{1}{32}(Sin6\theta) + \frac{3}{32}(\ Sin2\theta) = \frac{1}{32}(3Sin2\theta - Sin6\theta)\ \{as\ required\}$$

b)

$$\int_0^{\frac{\pi}{2}} Cos^3\theta \; Sin^3\theta \, d\theta = \frac{1}{32}\int_0^{\frac{\pi}{2}}(3Sin2\theta - Sin6\theta)d\theta$$

$$= \frac{1}{32}\left[\frac{-3cos2\theta}{2} + \frac{cos6\theta}{6}\right]_0^{\pi/2}$$

$$= \frac{1}{32}\left[\frac{3}{2} - \frac{1}{6} - \frac{3}{2} + \frac{1}{6}\right)$$

$$= \frac{1}{32} \times \frac{8}{3}$$

$$= \frac{1}{12}$$

Exponential Form of a Complex Number

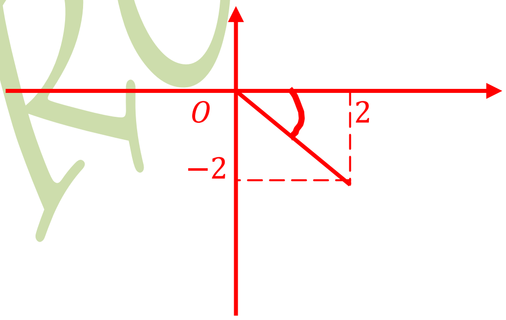
If $Z = r(Cos\theta + iSin\theta)$ then $Z = re^{i\theta}$ and $Z^n = re^{ni\theta}$

Example

Express $2 - 2i$ in the form $re^{i\theta}$.

Suggested Solution

$2 - 2i$



(i) $\sqrt{(2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$

(ii) From the argand diagram, θ lies in the fourth quadrant hence

$$\theta = -\tan^{-1}\left(\frac{2}{2}\right) = -\frac{\pi}{4}$$

$\therefore 2 - 2i = 2\sqrt{2}e^{-\frac{\pi i}{4}}$

The Cube Roots of Unity

- The cube roots of 1 are numbers: when they are cubed their value is 1.
- They satisfy the equation $z^3 - 1 = 0$.
- Clearly, one of the roots of $z^3 - 1$ is $= 1$

$$\Rightarrow (z - 1) \text{ must be a factor of } z^3 - 1.$$

- $\therefore$ Factorising (after performing long division) we get $(z - 1)(z^2 + z + 1)$
- Now the other roots come from the quadratic equation $z^2 + z + 1 = 0$.
- If one of these roots is denoted by w, then w satisfies the equation $z^2 + z + 1 = 0$ so that $w^2 + w + 1 = 0$.
- It can also be shown that if w is a roots of $z^3 = 1$ then w^2 is also a root, in fact, the other root.
- i.e. Substituting w^2 into the left hand side of $z^3 = 1$ gives

  $(w^2)^3 = w^6 = (w^3)^2 = 1^2 = 1$, as $w^3 = 1$ since w is a solution of $z^3 = 1$.
- Thus the cube roots are 1, w and w^2, where w and w^2 are non-real.
- w can be expressed in the form $a + ib$.

  i.e.

$$w^2 + w + 1 = 0$$

$$\Rightarrow \left(w + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + 1 = 0$$

$$\Rightarrow \left(w + \frac{1}{2}\right)^2 = -\frac{3}{4}$$

$$\Rightarrow w + \frac{1}{2} = \mp\sqrt{-\frac{3}{4}}$$

$$\Rightarrow w + \frac{1}{2} = \mp i\frac{\sqrt{3}}{2}$$

$$\Rightarrow w = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore w = \frac{-1 \pm i\sqrt{3}}{2}$$

NB: It doesn't matter whether w is labelled as $\frac{-1+i\sqrt{3}}{2}$ or as $\frac{-1-i\sqrt{3}}{2}$ because each is the square of the other.

In other words of $w = \frac{-1+i\sqrt{3}}{2}$ then:

$$w^2 = \left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{1-2i\sqrt{3}+i^2(3)}{4}$$

$$= \frac{1-3-2i\sqrt{3})}{4}$$

$$= \frac{-2-2i\sqrt{3})}{4}$$

$$= \frac{-1-i\sqrt{3}}{2}, \text{(which is the other root − conjugate)}$$

If $w = \frac{-1+i\sqrt{3}}{2}$, then $w^2 = \frac{-1-i\sqrt{3}}{2}$.

- Now the cube roots of unity are 1, w and w^2, where:
  - (i) $w^3 = 1$
  - (ii) $1 + w + w^2 = 0$
  - (iii)the non-real roots are $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$

Solved Examples

Example 1

Simplify $w^7 + w^8$ where w is a complex cube root of 1.

Suggested Solution

$w^7 = w^6 \times w = (w^3)^2 \times w = 1^2 \times w = w$ {because $w^3 = 1$}

$w^8 = w^6 \times w^2 = (w^3)^2 \times w^2 = 1^2 \times w^2 = w^2$ {because $w^3 = 1$ }

$\therefore w^7 + w^8 = w + w^2 = -1$ {because $1 + w + w^2 = 0$}

Example 2

Show that

$$\frac{1}{1+w}+\frac{1}{1+w^2}+\frac{1}{w+w^2}=0$$

Suggested Solution

$1+w+w^2=0 \Longrightarrow$ (i) $1+w=-w^2$

(ii) $1+w^2=-w$

(iii) $w+w^2=-1$

Now the equation simplifies to

$$\frac{1}{-w^2}+\frac{1}{-w}+\frac{1}{-1}$$

Multiply the first term by w and the second term by w^2 (**NB:** Multiply both on the numerator and the denominator)

$$\left(\frac{w}{w}\right)\frac{1}{-w^2}+\left(\frac{w^2}{w^2}\right)\frac{1}{-w}-1 \Longrightarrow \frac{w}{-w^3}+\frac{w^2}{-w^3}-1$$

But

$$w^3=1 \Longrightarrow \frac{w}{-1}+\frac{w^2}{-1}-1=-w-w^2-1=-1(w+w^2+1)$$

$=-1(0)=0$ $\{\text{Since } 1+w+w^2=0\}$

The Nth Roots of Unity

- The equation$z^n = 1$ clearly has at least one root, namely $z = 1$, but actually has many more , most of which (If not all) are complex.
- To find the remaining roots, the right hand side of the equation $z^n = 1$ should be expressed in exponential form,

$$\Rightarrow z^n = e^{2k\pi i}$$

- Taking the nth root of both sides gives

$$z = e^{\frac{2k\pi i}{n}}$$

- Different integer values of k will give rise to different roots
- Thus the equation $z^n = 1$ has roots:

$$z = e^{\frac{2k\pi i}{n}}, \quad k = 0,1,2,3, \dots, (n-1)$$

Worked Examples

Example 1

Find in the form $a + ib$, the roots of the equation $z^6 = 1$ and illustrate these roots on an argand diagram.

Suggested Solution

$z^6 = 1 = e^{\frac{2k\pi i}{6}} = e^{\frac{k\pi i}{3}} \quad k = 0,1,2,3,4,5.$

Thus the roots are:

$k = 0; \quad z = 1$

$k = 1; \quad z = e^{\frac{\pi i}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$$= \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$k = 2; \quad z = e^{\frac{2\pi i}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$k = 3; \quad z = e^{\pi i} = \cos(\pi) + isin(\pi)$

$$= -1$$

$k = 4; \quad z = e^{\frac{4\pi i}{3}} = \cos\left(\frac{4\pi}{3}\right) + isin\left(\frac{4\pi}{3}\right)$

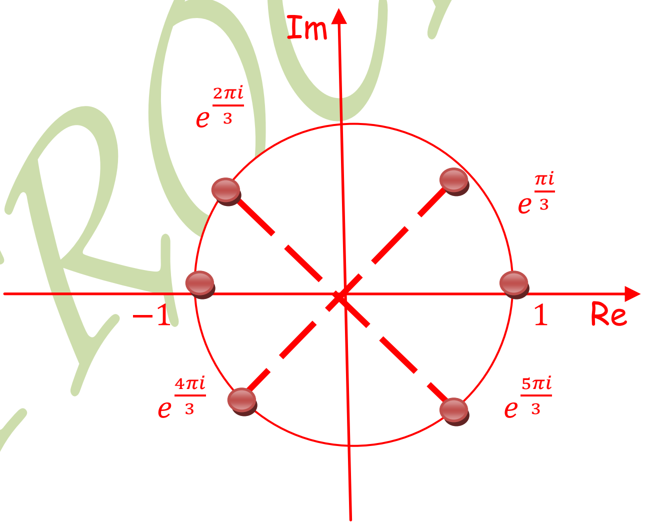
$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$k = 5; \quad z = e^{\frac{5\pi i}{3}} = \cos\left(\frac{5\pi}{3}\right) + isin\left(\frac{5\pi}{3}\right)$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

To summarise the sixth roots:

$$z = \pm 1 \text{ and } z = \pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$



NB: (i) The arguments of the roots should be between $-\pi$ and $+\pi$ instead of 0 and 2π. In the example above the roots would be given as $z = e^{\frac{k\pi i}{3}}$ for $k = 0, \pm 1, \pm 2, 3$.

(ii) Some equation may not involve unity so they are treated as the example below:

Example 2

Solve $z^6 = 64$

Suggested solution

$z^6 = 64$

$z^6 = 2^6\, e^{2k\pi i} \Longrightarrow 2e^{\frac{2k\pi i}{6}} \quad k = 0,1,2,3,4,5.$

The only difference would be the modulus of each root would be 2 instead of 1, with the consequence that the six roots of $z^6 = 64$ would lie on the circle $|z| = 2$ instead $|z| = 1$.

Solutions of the Binomial Equations

Case 1

$Z = A^N$ where A is a real positive number and N is a fraction.

$$Z = \sqrt[n]{A}\left[\cos\left(\frac{2k\pi}{n}\right) + isin\left(\frac{2k\pi}{n}\right)\right]$$

where $k = 0,1,2,3, \dots, (n-1)$

Case 2

$Z = A^N$ where A is a real negative number and N is a fraction.

$$Z = \sqrt[n]{|A|}\left[\cos\left(\frac{\pi + 2k\pi}{n}\right) + isin\left(\frac{\pi + 2k\pi}{n}\right)\right]$$

where $k = 0,1,2,3, \dots, (n-1)$

Example

Solve $z^3 = -8$.

Suggested Solution

$$Z_k = \sqrt[3]{|-8|}\left[\cos\left(\frac{\pi + 2k\pi}{3}\right) + isin\left(\frac{\pi + 2k\pi}{3}\right)\right]$$

where $k = 0,1,2$.

$Z_0 = -1 + i\sqrt{3}$

$Z_1 = -2$

$Z_3 = 1 - i\sqrt{3}$.

The roots of $z^n = \alpha$ where α is a non-real number

- Every complex number of the form $a + ib$ can be written in the form $re^{i\theta}$, where r is real and θ lies in an interval of 2π (Ussually from 0 to 2π or from $-\pi$ to π)
- Suppose that $\alpha = re^{i\theta}$
- Now $e^{i\theta+2\pi i} = e^{i\theta} \times e^{2\pi i} = e^{i\theta}$ (because $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$)
- Similarly, $e^{i\theta+2k\pi i} = e^{i\theta} \times e^{2k\pi i} = e^{i\theta}$
- So $z^n = \alpha = re^{i\theta+2k\pi i}$
- Taking the n^th root of both sides

$$Z = r^{\frac{1}{n}}e^{i\left(\frac{\theta+2k\pi}{n}\right)} \qquad k = 0,1,2,3,\dots,(n-1).$$

- ∴The equation$z^n = \alpha$, where $\alpha = re^{i\theta}$ has roots:

$$Z = \sqrt[n]{r}e^{i\left(\frac{\theta+2k\pi}{n}\right)} \qquad k = 0,1,2,3,\dots,(n-1).$$

$Z = \sqrt[n]{r}e^{i\left(\frac{\theta+2k\pi}{n}\right)} \qquad k = 0,1,2,3,\dots,(n-1)$ or

$Z = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + isin\left(\frac{\theta + 2k\pi}{n}\right)\right]$ where $k = 0,1,2,3,\dots,(n-1)$.

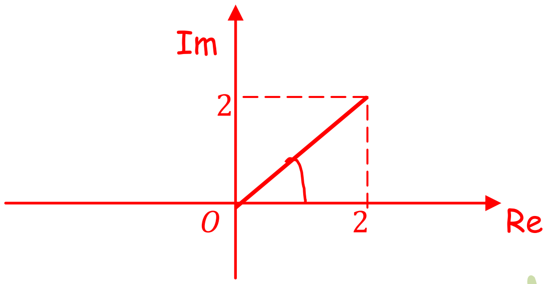
Worked Example

Example 1

Find the three roots of the equation $z^3 = 2 + 2i$.

Suggested Solution

Express $2 + 2i$ in exponential form.



(i) $\sqrt{(2)^2 + (2)^2} = \sqrt{8}$

(ii) From the argand diagram, θ lies in the first quadrant hence

$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\therefore 2 + 2i = \sqrt{8}e^{i\frac{\pi}{4}}$

$\Rightarrow z^n = \sqrt{8}e^{i\left(\frac{\pi}{4}+2k\pi\right)}$

$\Rightarrow Z = (\sqrt{8})^{\frac{1}{3}}e^{i\left(\frac{\frac{\pi}{4}+2k\pi}{3}\right)} = \sqrt{2}e^{i\left[\frac{(1+8k)\pi}{12}\right]}$ where $k = 0,1,2$

The roots are

$k = 0;\quad z = \sqrt{2}e^{i\frac{\pi}{12}}$

$k = 1;\quad z = \sqrt{2}e^{i\frac{9\pi}{12}}$

$k = 2;\quad z = \sqrt{2}e^{i\frac{17\pi}{12}}$ or $\left(\sqrt{2}e^{i\frac{-7\pi}{12}}\right)$

NB: These roots can be written in the form $r(cos\theta + isin\theta)$ i.e.

$\sqrt{2}\left[\cos\left(\frac{(1+8k)\pi}{12}\right) + isin\left(\frac{(1+8k)\pi}{12}\right)\right]$ for $k = 0,1,2(or - 1)$.

NB: You can also express them in the form $a + ib$.

SOLVED PAST EXAMINATION QUESTIONS

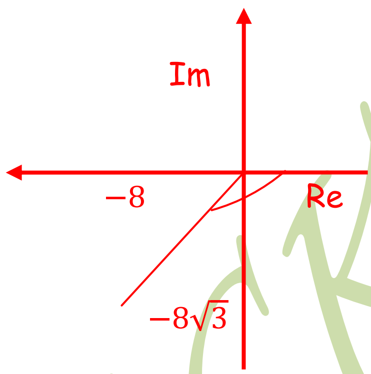
Question 1

ZIMSEC JUNE 2010 PAPER 2

Express $-8 - i8\sqrt{3}$ in the form $r(cos\theta + isin\theta)$. Hence or otherwise find all the fourth roots of $-8 - i8\sqrt{3}$.

Suggested Solution

Let $z = -8 - i8\sqrt{3}$



$$\theta = -\left[\pi - tan^{-1}\left(\frac{8\sqrt{3}}{8}\right)\right]$$

$$= -[\pi - tan^{-1}(\sqrt{3})]$$

$$= -\left(\pi - \frac{\pi}{3}\right)$$

$$= -\frac{2\pi}{3}$$

$$r = \sqrt{(8)^2 + (8\sqrt{3})^2}$$

$$= \sqrt{64 + 192}$$

$$= \sqrt{256}$$

$$= 16$$

$$\therefore z = 16\left[\cos\left(-\frac{2\pi}{3}\right) + isin\left(-\frac{2\pi}{3}\right)\right]$$

$$Z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \text{where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_k = \sqrt[4]{16}\left[\cos\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2k\pi}{4}\right\} + i\sin\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2k\pi}{4}\right\}\right] \text{where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_k = 2\left[\cos\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2k\pi}{4}\right\} + i\sin\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2k\pi}{4}\right\}\right] \text{where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_0 = 2\left[\cos\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2(0)\pi}{4}\right\} + i\sin\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2(0)\pi}{4}\right\}\right]$$

$$= 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$= \sqrt{3} - i$$

$$\Rightarrow Z_1 = 2\left[\cos\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2(1)\pi}{4}\right\} + i\sin\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2(1)\pi}{4}\right\}\right]$$

$$= 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\sqrt{3}$$

$$\Rightarrow Z_2 = 2\left[\cos\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2(2)\pi}{4}\right\} + i\sin\left\{\frac{\left(-\frac{2\pi}{3}\right) + 2(2)\pi}{4}\right\}\right]$$

$$= 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

$$= 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$= -\sqrt{3} + i$$

$$\Rightarrow Z_3 = 2\left[\cos\left\{\frac{\left(-\frac{2\pi}{3}\right)+2(3)\pi}{4}\right\} + i\sin\left\{\frac{\left(-\frac{2\pi}{3}\right)+2(3)\pi}{4}\right\}\right]$$

$$= 2\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]$$

$$= 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -1 - i\sqrt{3}$$

<u>Question 2</u>

<u>ZIMSEC JUNE 2013</u>

Using the substitution $w = z^4$, solve the equation $z^8 - z^4 - 6 = 0$ where z is a complex number.

<u>Suggested Solution</u>

$z^8 - z^4 - 6 = 0$

Let $w = z^4$

$\Rightarrow w^2 - w = 6$

$$\Rightarrow w^2 - w + \left(-\frac{1}{2}\right)^2 = 6 + \left(-\frac{1}{2}\right)^2$$

$$\Rightarrow \left(w - \frac{1}{2}\right)^2 = 6 + \frac{1}{4}$$

$$\Rightarrow \left(w - \frac{1}{2}\right)^2 = \frac{25}{4}$$

$$\Rightarrow w - \frac{1}{2} = \pm\sqrt{\frac{25}{4}}$$

$$\Rightarrow w - \frac{1}{2} = \pm\frac{5}{2}$$

$$\Rightarrow w = \frac{1}{2} \pm \frac{5}{2}$$

$$\Rightarrow w = \frac{1}{2} + \frac{5}{2} \text{ or } \frac{1}{2} - \frac{5}{2}$$

$\therefore w = 3 \text{ or } -2$

But $z^4 = w$

$\Rightarrow z^4 = 3 \text{ or } z^4 = -2$

<u>NOW</u>

$z^4 = -2$

$$Z_k = \sqrt[n]{|r|}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \text{ where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_k = \sqrt[4]{|-2|}\left[\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right] \text{ where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_k = \sqrt[4]{2}\left[\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right] \text{ where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_0 = \sqrt[4]{2}\left[\cos\left\{\frac{\pi + 2(0)\pi}{4}\right\} + i\sin\left\{\frac{\pi + 2(0)\pi}{4}\right\}\right]$$

$$= \sqrt[4]{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

$$= \sqrt[4]{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= 0.840896415 + i0.840896415$$

$$= 0.84 + i0.84 \text{ (to } 2s.f.)$$

$$\Rightarrow Z_1 = \sqrt[4]{2}\left[\cos\left\{\frac{\pi + 2(1)\pi}{4}\right\} + i\sin\left\{\frac{\pi + 2(1)\pi}{4}\right\}\right]$$

$$= \sqrt[4]{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= \sqrt[4]{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= -0.840896415 + i0.840896415$$

$$= -0.84 + i0.84 \text{ (to } 2s.f.)$$

$$\Rightarrow Z_2 = \sqrt[4]{2}\left[\cos\left\{\frac{\pi + 2(2)\pi}{4}\right\} + i\sin\left\{\frac{\pi + 2(2)\pi}{4}\right\}\right]$$

$$= \sqrt[4]{2}\left[\cos\left(\frac{5\pi}{4}\right) + isin\left(\frac{5\pi}{4}\right)\right]$$

$$= \sqrt[4]{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= -0.840896415 - i0.840896415$$

$$= -0.84 - i0.84 \text{ (to } 2s.f.)$$

$$\Rightarrow Z_3 = \sqrt[4]{2}\left[\cos\left\{\frac{\pi + 2(3)\pi}{4}\right\} + isin\left\{\frac{\pi + 2(3)\pi}{4}\right\}\right]$$

$$= \sqrt[4]{2}\left[\cos\left(\frac{7\pi}{4}\right) + isin\left(\frac{7\pi}{4}\right)\right]$$

$$= \sqrt[4]{2}\left(\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}\right)$$

$$= 0.840896415 - i0.840896415$$

$$= 0.84 - i0.84 \text{ (to } 2s.f.)$$

ALSO:

$z^4 = 3$

$$Z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + isin\left(\frac{\theta + 2k\pi}{n}\right)\right] \text{ where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_k = \sqrt[4]{3}\left[\cos\left(\frac{2k\pi}{4}\right) + isin\left(\frac{2k\pi}{4}\right)\right] \text{ where } k = 0,1,2 \text{ and } 3$$

$$\Rightarrow Z_0 = \sqrt[4]{3}\left[\cos\left\{\frac{2(0)\pi}{4}\right\} + isin\left\{\frac{2(0)\pi}{4}\right\}\right]$$

$$= \sqrt[4]{3}[\cos(0) + isin(0)]$$

$$= \sqrt[4]{3}(1)$$

$$= 1.316074013$$

$$= 1.3 \text{ (to } 2s.f.)$$

$$\Rightarrow Z_1 = \sqrt[4]{3}\left[\cos\left\{\frac{2(1)\pi}{4}\right\} + isin\left\{\frac{2(1)\pi}{4}\right\}\right]$$

$$= \sqrt[4]{3}\left[\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right]$$

$= \sqrt[4]{3}(i)$

$= 1.316074013i$

$= 1.3i$ (to $2s.f.$)

$$\Rightarrow Z_2 = \sqrt[4]{3}\left[\cos\left\{\frac{2(2)\pi}{4}\right\} + isin\left\{\frac{2(2)\pi}{4}\right\}\right]$$

$= \sqrt[4]{3}[\cos(\pi) + isin(\pi)]$

$= \sqrt[4]{3}(-1)$

$= -1.316074013$

$= -1.3$ (to $2s.f.$)

$$\Rightarrow Z_3 = \sqrt[4]{3}\left[\cos\left\{\frac{2(3)\pi}{4}\right\} + isin\left\{\frac{2(3)\pi}{4}\right\}\right]$$

$$= \sqrt[4]{3}\left[\cos\left(\frac{3\pi}{2}\right) + isin\left(\frac{3\pi}{2}\right)\right]$$

$= \sqrt[4]{3}(-i)$

$= -1.316074013i$

$= -1.3i$ (to $2s.f.$)

PRACTICE QUESTIONS

Question 1

Solve the following equation $z^4 + 8 + i8\sqrt{3} = 0$, giving your answer in the form $r(cos\theta + isin\theta)$

Question 2

Solve the following equations and express them in the form $re^{i\theta}$. Answers are in red.

a) $z^3 = 1 - i$ $\left[\sqrt[6]{2}e^{i\frac{(8k-1)\pi}{12}}\ for\ k = 1,2,3\right]$

b) $z^8 = 1 - 3i$ $\left[\sqrt[8]{2}e^{i\frac{(6k-1)\pi}{24}}\ for\ k = 1,2,3, \dots ,8\right]$

c) $(z + 1)^3 = 8i$ $\left[2e^{i\frac{(4k-1)\pi}{6}}\ for\ k = 0,1,2\right]$

Question 3

a) Use DeMoivre's theorem to show that $cos5\theta = co\theta(16cos^4\theta - 20cos^2\theta = 5)$.

b) By solving the equation $cos5\theta = 0$, deduce that $cos^2\theta\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{2}$.

c) Hence, or otherwise, write down the exact values of $cos^2\theta\left(\frac{3\pi}{10}\right)$,$cos^2\theta\left(\frac{7\pi}{10}\right)$ and $cos^2\theta\left(\frac{9\pi}{10}\right)$.

Question 4

a) Express $4 - 4i$ in the form $r(cos\theta + isin\theta)$, where $r > 0, -\pi < \theta < \pi$, where r and θ are exact values.

b) Hence, or otherwise, solve the equation $z^5 = 4 - 4i$ leaving your answers in the form $z = Re^{-ik\pi}$,where R is the modulus of z and k is a rational number such that $-1 \leq k \leq 1$.

c) Show on an Argand diagram the points representing your solution.

Question 5

Express $\frac{(\cos 3x+isin\ 3x)^2}{cox-isinx}$ in the form $cos\ nx + isin\ nx$ where n is an integer to be found.

Question 6

Use DeMoivre's theorem to evaluate

a) $(1-i)^6$

b) $\frac{1}{\left(\frac{1}{2}-\frac{1}{2}\right)^{16}}$

Question 7

a) If $z = r(cos\theta + isin\theta)$, use DeMoivre's theorem to show that $z^n + \frac{1}{z^n} = 2cosn\theta$.

b) Express $\left(z^2 + \frac{1}{z^2}\right)^3$ in term of $cos6\theta$ and $cos2\theta$.

c) Hence , or otherwise, show that $cos^3 2\theta = acos6\theta + bcos2\theta$, where a and b are constants.

d) Hence, or otherwise

$$\int_0^{\frac{\pi}{6}} cos^3 2\theta\, d\theta = k\sqrt{3},$$

where k is a constant

e) Express $\frac{(\cos 3x+isin\ 3x)^2}{cox-isinx}$ in the form $cos\ nx + isin\ nx$ where n is an integer to be found.

Question 8

The region R in an argand diagram is satisfied by the inequalities $|z| \leq 5$ and $|z| \leq |z-6|$. Draw an argand diagram and shade in the region R.

Question 9

a) Sketch in on the same Argand diagram:

(i) the locus of points representing $|z-2| = |z-6-8i|$,

(ii) the locus of points representing $arg\ (z-4-2i) \leq 0$, (iii)the locus of points representing $arg\ (z - 4 - 2i) \leq \frac{\pi}{2}$.

The region R in an argand diagram is satisfied by the inequalities $|z - 2| = |z - 6 - 8i|$ and $arg\ (z - 4 - 2i) \leq \frac{\pi}{2}$.

b) On your sketch in part (a), identify, by shading the region R.

Question 10

a) Find the solutions of the equation $z^6 - 1 = 0$.
   Hence, plot the answers on an Argand diagram.

b) Sketch on an Argand diagram the locus of points satisfying both
   $|z - i| = |z + 1 + 2i|$ and $|z + 3i| \leq 4$.

Question 11

a) Express $sin3\theta$ in terms of powers of $sin\theta$.

b) Find the fifth roots of unity in trigonometric form.

c) Find the square roots of the complex number $15 + 8i$ in the form $a + bi$ where a and b are real numbers.

Question 12

a) Simplify $\frac{Z_1}{Z_2}$ where $Z_1 = 3 + 4i$ and $Z_2 = 1 - 2i$.

b) Find $Z_1 Z_2$ if $Z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $Z_2 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

c) Express $4(\sqrt{3} - i)$in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$.

Question 13

a) Express $\sin 5\theta$ in terms of powers of $\sin\theta$ and hence show that
   $sin\ 5\theta - 5\ sin\ \theta = 16sin^5\theta - 20sin^3\theta.$

b) Find

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (16sin^5\theta - 20sin^3\theta)d\theta,$$

giving your answer in exact form.

Question 14

a) (i) Express $\dfrac{e^{\frac{\pi}{2}i}}{e^{\frac{\pi}{3}i}}$ in the form $a + ib$

(ii) Hence find the sixth roots of $a + ib$, the complex number obtained above. Give your answer in the form $r(\cos\theta + i\sin\theta)$

b) (i) Sketch on an argand diagram the locus of points of z where

$|z - 1 - i| = |z + 2 + 3i|$

(ii) Hence or otherwise state the Cartesian equation of this locus.

Question 15

a) The polynomial $2x^4 + x^3 + 17x^2 + 9x - 9$ is denoted $p(x)$.

(i) Show that $3i$ is a root of the equation $p(x)$.

(ii) State the other complex root of the equation $p(x) = 0$.

(iii)Hence or otherwise find the other 2 roots of the equation $p(x) = 0$.

b) Simplify $\dfrac{(1+i)^4}{(2-2i)^3}$ giving your answer in the form $a + ib$

c) Use DeMoivre's theorem to show that

$$tan4\theta(1 - 6tan^2\theta + tan^4\theta) = 4tan\theta - 4tan^3\theta.$$

ZIMSEC PAST EXAMINATIONS QUESTIONS PAPER 1

ZIMSEC NOVEMBER 2003 SPECIMEN

a) Given that the imaginary part of Z is $-\frac{1}{2}$, where $Z = \frac{2-3i}{1-ai}$, find possible values of a. **[2]**

b) Given that $Z_1 = 1 + i\sqrt{3}$ and $Z_2 = \sqrt{3} + i$.

(i) Calculate the modulus and argument of Z_1 and Z_2.

(ii) Hence plot on an Argand diagram $Z_1 Z_2$ and $\frac{Z_1}{Z_2}$. **[4]**

c) Given that $(a + ib)^2 = 8 + 6i$, find the values of a and b. **[4]**

ZIMSEC NOVEMBER 2003

Given that $z_1 = 1 + 3i$ and $z_2 = 3 + 2i$, find

(i) $|z_1|$, **[1]**

(ii) $arg\ z_2$, **[1]**

(iii) $z_1 z_2$, **[2]**

(iv) $\frac{z_1}{z_2}$, **[2]**

Show the complex numbers z_1 and z_2 on the same Argand diagram, clearly labelling $|z_1|$ and $arg\ z_2$. **[2]**

ZIMSEC JUNE 2004

a) Express $Z = \frac{2+i}{3-i}$ in modulus argument form. Hence find their simplest form the moduli and arguments of numbers:

(i) Z^2,

(ii) $\frac{1}{Z}$ **[6]**

b) (i) Shade the area represented on an argand diagram by:

$|Z - 1 + 2i| < 3$ **[2]**

(ii) Sketch the locus of Z if

$$\arg(Z-1) - \arg(Z+1) = \frac{\pi}{6},$$ **[3]**

ZIMSEC NOVEMBER 2004

Given that $Z = 4 - 2i$, find

(i) $|Z|$ and $arg\ Z$, **[2]**

(ii) $\frac{Z}{\bar{Z}}$ in the form $a + bi$, where $\bar{Z}$ represents the conjugate Z and a and b are real numbers. **[2]**

ZIMSEC NOVEMBER 2005

The complex number $z = 2 + 3i$ has a modulus k and argument $\propto$.

a) Determine the value k and $\propto$. **[2]**

b) ω is the complex number $z + 3iz$. Find ω in the form $a + ib$ and hence represent ω on the Argand diagram. **[3]**

ZIMSEC JUNE 2006

a) Express the complex number $z = \frac{6+4i}{1+5i}$ in the form $a + ib$. Hence find $|z|$ and $arg(z)$. **[4]**

b) Show by substitution that $w = 2 - 3i$ is a root of the equation $w^2 - 4w + 13 = 0$. **[3]**

ZIMSEC NOVEMEBER 2006

The complex number $z = x + iy$ satisfies the equation $\frac{z}{z+2} = 2 - 1$.

Find the value of x and the value of y. **[4]**

ZIMSEC JUNE 2008

Given the complex number $W = 2 - 3i$,

evaluate

(i) iW,

(ii) $W + iW$. **[3]**

Plot the points P, Q and R representing the complex numbers W, iW, $W + iW$ respectively on an Argand diagram. **[2]**

Hence name the quadrilateral $OPRQ$, where O is the origin. **[1]**

ZIMSEC NOVEMBER 2008

A complex number z has modulus 8 and argument $\frac{3\pi}{4}$.

State the modulus and argument of z^2. **[2]**

Using these values show the number z^2 on an Argand diagram, and hence express z^2 in the form $a + bi$. **[2]**

ZIMSEC JUNE 2009

The complex number $p = 3 - 5i$ and it is given that $q = 4ip$

a) State the relationship between

(i) $|p|$ and $|q|$,

(ii) $arg(p)$ and $arg(q)$, **[2]**

b) Given that $r = p + q$, find r in the form $a + bi$ where a and b are real numbers. **[2]**

c) The points P, Q and R in an Argand diagram represent the complex numbers p, q and r respectively.

(i) State the kind of quadrilateral that $OPRQ$ is, where O is the origin.

(ii) Find the area of $OPRQ$. **[3]**

ZIMSEC NOVEMBER 2009

The complex numbers z and w are given by $-3 + 2i$ and $w = 5 + 4i$.

Find

(i) $|z|$, **[1]**

(ii) $arg(z)$, **[2]**

(iii) $\frac{z}{w}$ in the form $a + ib$ where a and b are exact.

Hence represent $\frac{z}{w}$ in an Argand diagram. **[3]**

ZIMSEC NOVEMBER 2010

Express $z = \frac{1+i}{3+4i}$ in the form $a + bi$, where a and b are real. **[3]**

Hence or otherwise find $|z|$ in the form $c\sqrt{d}$ where d is a prime number. **[2]**

ZIMSEC JUNE 2011

It is given that $z_1 = 2 - 4i$ and $z_2 = 6 - 2i$.

a) Find $z_1 - z_2$ and $z_1 z_2$ in the form $a + ib$. **[3]**

b) if $w = \frac{1}{z_1}$, obtain the exact values of the modulus and argument of w. **[4]**

ZIMSEC NOVEMBER 2011

a) The complex number u is such that $(-1 + 3i)u = 5 - 3i$.

Find

(i) the modulus of u,

(ii) the argument of u. **[4]**

b) Given that complex number w is $2i$.

Find in the form $a + ib$

(i) $\frac{u}{w}$,

(ii) uw. **[4]**

ZIMSEC JUNE 2012

The complex number $w = \frac{4+3i}{3-2i}$.

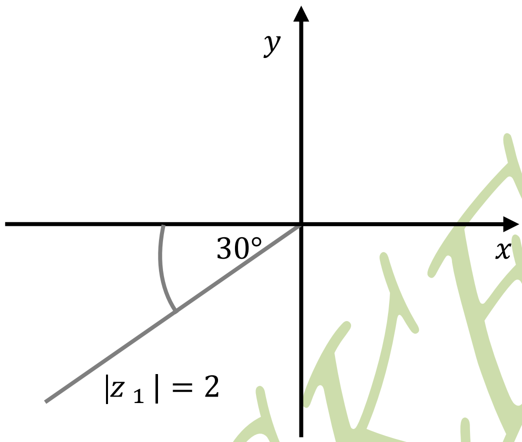
a) Express w in the form $x + iy$ where x and y are real. **[2]**

b) Find

(i) modulus of w,

(ii) argument of w. **[5]**

ZIMSEC NOVEMBER 2012



A complex number z_1 has modulus 2 and is positioned as shown in the Argand diagram above.

(i) State the principal argument of z_1 and write z_1 in the form $a + ib$ where a and b are exact real numbers. **[3]**

(ii) Find exactly in the form $a + ib$, the complex number w, given that

$w = \dfrac{(-8\sqrt{3})i}{z_1}$. **[2]**

(iii)Show a sketch of w in an Argand diagram, labelling the modulus and argument values in your diagram. **[3]**

ZIMSEC JUNE 2013

Given that $p = 5 + i$ and $q = -2 + 3i$,

a) (i) show the complex numbers ip and $p + q$ on an argand diagram,

(ii) describe the geometrical transformation which maps ip ont p. **[3]**

b) Find

(i) the modulus and argument of p,

(ii) pq,

(iii) $\frac{p}{q}$. **[5]**

ZIMSEC NOVEMBER 2013

If $Z_1 = -1 + i$ and $Z_2 = -1 - \sqrt{3}i$,

Find

(i) the modulus and argument of Z_2. **[2]**

(ii) (a) $Z_1 Z_2$,

(b) $\frac{Z_1}{Z_2}$. **[4]**

ZIMSEC JUNE 2014

Given that $a = 2 + i$ and $b = 1 + 3i$,

(i) show on a single argand diagram the complex numbers

1. ab
2. $\frac{a}{b}$. **[6]**

(ii) find the modulus and argument of each case in (i)1 and (i)2. **[4]**

ZIMSEC NOVEMBER 2014

The complex number z satisfies the equation

$$z + 2\bar{z} = \frac{13}{-2 + 3i}$$

Find

(i) z in the form $x + iy$, **[3]**

(ii) the modulus and argument of $\frac{1}{z}$. **[4]**

ZIMSEC JUNE 2015

The complex number $w = 3 - 4i$ and u is such that $\frac{w}{u} = \frac{2}{13} + \frac{3}{13}\,i$

a) Find

(i) u in the form $x + iy$

(ii) 1. $|u|$

2. $\arg(u)$. **[7]**

b) Sketch u on an argand diagram showing clearly the $|u|$ and $arg(u)$. **[2]**

ZIMSEC NOVEMBER 2015

Two complex numbers $z = x + iy$ and $w = a + ib$ are such that

$z + iw = 2$ and $iz + w = 2 + 3i$.

Find

(i) 1. z,

2. w, **[4]**

(ii) the modulus of zw, **[2]**

(iii) the argument $\frac{z}{w}$. **[3]**

ZIMSEC JUNE 2016

(i) Express the complex number $w = 8 + \frac{4-1}{1+2i}$ in the form $x + iy$. **[4]**

(ii) Hence, or otherwise, find

1. $|w|$ in the form $a\sqrt{b}$.
2. argument of w. **[6]**

ZIMSEC JUNE 2017

The complex numbers z_1 and z_2 are such that $z_1 = 2 - 3i$ and $z_2 = 1 + 3i$.

a) Find

(i) $\frac{z_1}{z_2}$ in the form $x + iy$ **[3]**

(ii) $\left|\frac{z_1}{z_2}\right|$ **[2]**

(iii)$\arg\left(\frac{z_1}{z_2}\right)$ **[2]**

b) Hence represent $\frac{z_1}{z_2}$on a clearly labelled Argand diagram. **[2]**

ZIMSEC NOVEMEBR 2017

Given the complex numbers $w = 1 + 2i$ and $u = 3 - 1$, find

a) in the form $a + ib$, where a and b are real numbers

(i) $u + w$ **[1]**

(ii) uw **[1]**

b) the argument of uw. **[2]**

ZIMSEC JUNE 2018

The complex number $w = -2 + (2\sqrt{3})i$

Find

a) $|w|$ the modulus of w, **[1]**

b) the argument of the conjugate of w, **[2]**

c) $\frac{w+1}{w}$ in the form $x + iy$. **[3]**

ZIMSEC NOVEMBER 2019

A complex number is give by $= \frac{3+i}{2-i}$.

(a) Express u in the form $a + ib$ where a and b are real numbers. **[2]**

(b) Find the modulus and argument of u. **[2]**

(c) Show the complex number u on an Argand diagram. **[1]**

ZIMSEC PAST EXAMINATIONS QUESTIONS PAPER 2

ZIMSEC NOVEMBER 2019

a) The equation $x^4 - 4x^3 + 3x^2 + 2x - 6 = 0$ has a root $1 - i$.
   Find the other three roots. **[6]**

b) The complex number $\boldsymbol{z}$ satisfies the inequalities $2 < |\mathbf{z}| < 3$ and $\frac{\pi}{6} < \arg \mathbf{z} < \frac{\pi}{3}$.
   Sketch and shade on an Argand diagram the region represented by the inequalities. **[4]**

c) Solve the equation $z^4 - 8\sqrt{3} + 8i = 0$ giving your answers in the form $a + ib$, correct to 2 decimal places. **[6]**

ZIMSEC JUNE 2019

a) On a single diagram shade the region defined by the inequalities

$$\frac{\pi}{6} \leq arg(z - 4) \leq \frac{\pi}{6} \text{ and } |z - 4| \leq 4.$$ [3]

b) Solve the equation $z^3 = -5 + 12i$. **[6]**

c) Use DeMoivre's theorem to show that

$$sin\theta sin5\theta = 16sin^6\theta - 20sin^4\theta + 5sin^2\theta.$$ **[7]**

ZIMSEC JUNE 2018

a) It is given that $\left(x + 2\sqrt{2}\right)$ and $\left(x - 2\sqrt{2}\right)$ are factor of the polynomial $f(x) = x^4 - 6x^3 + ax^2 + bx - 104$
   (i) Find the value of a and b. **[5]**
   (ii) Hence, or otherwise, find the roots of the equation $f(x) = 0$. **[5]**

b) Find the real part of $\left(2 + \frac{1}{2}\right)^4$, giving your answer in exact form. **[6]**

ZIMSEC NOVEMBER 2017

d) Find the value of $\left(2+2\sqrt{3}i\right)^{4}$ using the De Moivre's Theorem. **[4]**

e) Express $\dfrac{sin6\theta}{4sin\theta}$ in terms of $cos\theta$. **[6]**

ZIMSEC JUNE 2017

a) Given that the complex numbers $W_1 = 1 + ix$ and $W_2 = x + iy$, where x and y are numbers, satisfy the equation $W_1 - W_2 = 3i$,
find the value of x and the value of y. **[4]**

b) Indicate by shading on a single Argand diagram the region in which both of the following inequalities are satisfied:

$$\frac{\pi}{4} \le \arg z \le \frac{\pi}{2}$$

$$|z - 3i| \le 3$$

[3]

c) Use De-Moivres theorem to

(i) find the value of $\left(cos\frac{1}{4}\pi + isin\frac{1}{4}\pi\right)^{12}$, **[2]**

(ii) Show that $\tan 4\theta = \dfrac{4tan\theta - 4tan^3\theta}{1-6tan^2\theta+tan^4\theta}$. **[2]**

ZIMSEC NOVEMBER 2016

a) Express in the form $r(cos\theta + isin\theta)$, the roots of the equation $z^7 - 8 - 8i = 0$. **[9]**

b) show $Arg(z+1) = \frac{\pi}{3}$ in an argand diagram. **[2]**

ZIMSEC NOVEMBER 2015

(a) Is $z_1 = 3 + i$, $z_2 = -3 - 4i$ and $z_3 = x + iy$, sketch the locus of points
$z_1 = 3 + P(x; y)$ on the Argand diagram for which $|z - z_1| = |z_2|$. **[3]**

(b) Hence, from (**a**) write down the number z corresponding to the point on the locus for which

(i) the imaginary part is i,

(ii) $\arg(z - z_1) = \frac{\pi}{2}$. **[3]**

(c) Given that $z = 3e^{-\frac{\pi i}{2}} + 4$,

find

(i) $|z|$,

(ii) $\arg(z)$. **[3]**

ZIMSEC JUNE 2015

a) Given that $= \frac{5+i}{2+3i}$, find the fifth roots of z in the form $re^{i\theta}$. **[8]**

b) Given that $1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where p and q are constants,

find

1. the other **two** roots.
2. the values of p and q. **[6]**

ZIMSEC JUNE 2013

(a) Using the substitution $w = z^4$, solve the equation $z^8 - z^4 - 6 = 0$ where z is a complex number. **[10]**

(b) The real part of the complex number $\frac{z+2}{z-2}$ is zero. Show that the locus of the point representing z in the Argand diagram plane is a circle centre $(0,0)$ and radius 2. **[4]**

(c) Sketch in an argand diagram the set of points representing all complex numbers z satsfying both the inequalities $|z - 3 - i| \leq 4$ and $\frac{\pi}{3} \leq \arg\left(z - 4 - 2i\right) \leq \frac{\pi}{2}$. **[3]**

ZIMSEC NOVEMBER 2012

(a) Simplify $\frac{(1+\mathrm{i})^4}{(2-2\mathrm{i})^3}$, giving your answer in the form $a + bi$. **[4]**

(b) (i) Simplify $\frac{cos3\theta + isin3\theta}{cos2\theta - isin2\theta}$, **[2]**

(ii) Use De Moivre's theorem to express $sin5\theta$ in terms of $sin\theta$. **[6]**

(c) (i) Sketch an argand diagram of the locus of z where $|z-1-i| = |z+2+3i|$

(ii) Hence or otherwise state the Cartesian equation of the locus. **[5]**

ZIMSEC NOVEMBER 2011

(a) Express in exponential form $\left(\frac{3}{5}+\frac{4i}{5}\right)^{20} - \left(\frac{3}{5}-\frac{4i}{5}\right)^{20}$. **[5]**

(b) (i) Prove that $\tan 4\theta = \frac{4tan\theta - 4tan^3\theta}{1-6tan^2\theta+tan^4\theta}$ based on DeMoivre's theorem.

(ii) Hence find the first four exact values of θ for which

$tan^4\theta - 4tan^3\theta - 6tan^2\theta - 4tan\theta + 1 = 0$. **[10]**

ZIMSEC NOVEMBER 2009

Given that $z = cos\theta + isin\theta$. Show that $z - \frac{1}{z} = 2isin\theta$. **[3]**

Hence express $sin^4\theta$ in terms of $cos^4\theta$ and $cos2\theta$ using De Moivre's theorem. **[4]**

a) Express $4(\sqrt{3}-i)$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$. **[3]**

b) Given that $x_1 = 1+2i$ is a root of the equation $x^4 - 4x^3 - 6x^2 + 20x - 75 = 0$, find the other three roots. **[5]**

ZIMSEC NOVEMBER 2008

a) Find the modulus and argument of $\frac{(1+i)^5}{(1-i)^7}$ for $-\pi < argz < \pi$. **[4]**

b) Sketch in an Argand diagram the set of points representing all complex numbers z satisfying both of the inequalities.

$|z-2i| < 2 \quad and \quad |z-2i| \leq |z|$ [3]

c) Use DeMoivre's theorem to express $sin5\theta$ in terms of $sin\theta$. **[5]**

ZIMSEC JUNE 2007

a) Illustrate on an Argand diagram the set of points representing the complex number z satisfying both

$|z-1-2i| \leq 3$ and $\arg(z-2-i) = \frac{3\pi}{4}$. **[3]**

b) Given that $z = 2\left(cos\frac{5\pi}{6} + isin\frac{5\pi}{6}\right)$ and $w = \sqrt{3}\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$, find the modulus and argument of

(i) zw, **[2]**

(ii) $\frac{z}{w}$. **[2]**

c) Given that $z = 1 + i\sqrt{3}$, prove that $z^{11} = 2^{10}(1 - i\sqrt{3})$. **[3]**

ZIMSEC NOVEMBER 2006

a) The equation $3z^3 - 10z^2 + 20z - 16 = 0$ has $1 - \sqrt{3}i$ as one of its roots.

(i) Find the other two roots. **[5]**

(ii) Sketch these roots in an Argand diagram. **[2]**

b) Express $3\sqrt{3} - 3i$ in the form $re^{i\theta}$. **[3]**

Hence find the 4th root of $3\sqrt{3} - 3i$, giving your answers correct to 2 decimal places. **[5]**

ZIMSEC NOVEMBER 2005

a) By using the substitution $z = x + iy$, show that the Cartesian equation of the circle representing the complex number z, where

$|z+1| = 2|z-1|$, can be expressed in the form $Ax^2 + Bx + Cy^2 + D = 0$, where A, B, C and D are integers. **[3]**

Sketch this circle on an Argand diagram. **[3]**

b) Using De Moivre's theorem to express $cos6\theta$ in terms of powers of $cos\theta$. **[6]**

c) Solve the equation $z^4 + 8 + i8\sqrt{3} = 0$ giving your answers in the form $r(cos\theta + isin\theta)$. **[8]**

ZIMSEC NOVEMBER 2004

A complex number Z has modulus 8 and argument $\frac{\pi}{4}$. Another complex number W has modulus $\frac{1}{2}$ and argument $\frac{\pi}{8}$.

a) Write each of the complex numbers in the form $a + ib$.

(i) ZW^4, **[6]**

(ii) $\frac{Z^2}{W^2}$. **[6]**

b) Find the smallest value n such that $|W^n| < 0.01$. **[3]**

ZIMSEC JUNE 2004

a) Use De Moivre's theorem to express $sin5\theta$ in terms of powers of $sin\theta$. **[5]**

b) Given that $Z^4 = 8 - i8\sqrt{3}$, find all possible values of Z giving your answers in the form $a + ib$ with a and b correct to 2 $decimal\ places$. **[7]**

c) Sketch on an Argand diagram the locus of Z, where

$$|Z + 4| = |Z - 4i|$$

[2]

Hence or otherwise state the Cartesian equation of the locus. **[1]**

ZIMSEC NOVEMBER 2003

a) Sketch the following locus on an Argand diagram

$$Arg\left(\frac{z-1}{z-4i}\right) = \frac{\pi}{3}$$

[4]

b) Express $cos^5\theta$ in terms of cosines of multiple angles. **[7]**

c) Show that $2 + 3i$ is a root of the equation $z^3 - 3z^2 + 9z + 13 = 0$.

Hence find the other two roots. **[6]**

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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