Complex Numbers

Candidates should:

a) understand the idea of a complex number, recall the meaning of the terms *real part*, *imaginary part*, *modulus*, *argument*, *conjugate*, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;

b) be able to carry out operations of *addition*, *subtraction*, *multiplication* and *division* of two complex numbers;

c) be able to use the result that, for a polynomial equation with real coefficients, any non-real roots occur in *conjugate pairs*;

d) be able to represent complex numbers geometrically by means of an *Argand diagram*, and understand the geometrical effects of conjugating a complex number and of adding and subtracting two complex numbers;

e) find the two square roots of a complex number;

f) illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $|z - a| < k$, $|z - a| = |z - b|$, $\arg(z - a) = \alpha$.

Section 1: Introduction to complex numbers

Suppose we wished to solve the quadratic equation $x^2 + 4x + 5 = 0$.
Using the quadratic formula, the solutions would be:

$$x = \frac{-4 \pm \sqrt{4^2 - 4\times 1\times 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2}.$$

We notice a problem however since $\sqrt{-4}$ is not a real number. So the equation $x^2 + 4x + 5 = 0$ does not have any real roots.

However, suppose we introduced the symbol i to represent $\sqrt{-1}$. We could then find expressions for the solutions of the quadratic:

$$x = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm \sqrt{4\times -1}}{2} = \frac{-4 \pm 2i}{2}.$$

So the equation has two solutions: $x = -2 + i$ or $x = -2 - i$.

These two solutions are called **complex numbers**.

1.1 Some definitions

Suppose that z is a complex number. Let $z = a + ib$.
The **real part** of z, written $\text{Re}(z)$, is a.
The **imaginary part** of z, written $\text{Im}(z)$, is b.
The **complex conjugate** of z, written z^* or $\bar{z}$, is $z^* = a - ib$.

Example 1: Let $z = 5 - 2\text{i}$, $w = -2 + \text{i}$ and $u = 7\text{i}$.
Then:

$\text{Re}(z) = 5$	$\text{Im}(z) = -2$	$z^* = 5 + 2\text{i}$
$\text{Re}(w) = -2$	$\text{Im}(w) = 1$	$w^* = -2 - \text{i}$.
$\text{Re}(u) = 0$	$\text{Im}(u) = 7$	$u^* = -7\text{i}$.

1.2 Solving quadratic equations

Example 2: Solve the quadratic equation $2x^2 + 4x + 5 = 0$, giving your answers as complex numbers in surd form.

Solution: Using the quadratic formula:

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 2 \times 5}}{4} = \frac{-4 \pm \sqrt{-24}}{4}$$
$$= \frac{-4 \pm i\sqrt{24}}{4} = \frac{-4 \pm 2\sqrt{6}\,i}{4}$$

So the solutions are $x = -1 + \frac{\sqrt{6}}{2}i$ or $x = -1 - \frac{\sqrt{6}}{2}i$.

Notice that the two solutions are **complex conjugates** of each other. The solutions form a **conjugate pair**. This leads to this very important result:

Consider the equation $ax^2 + bx + c = 0$ where *a, b* and *c* are **real numbers**. If the equation has complex roots, then the two roots are always **conjugates** of each other.
Note: If a quadratic equation has any complex coefficients then this result doesn't apply.

Example: If $z = 1 - i$ is one solution of the quadratic equation $z^2 - 2z + 2 = 0$, then the second solution must be the **complex conjugate** (as the quadratic has real coefficients). So the second solution is $z = 1 + i$.

Section 2: Calculating with complex numbers

2.1 Adding and subtracting

Two complex numbers are added or subtracted by collecting together their real and imaginary parts.

So

$$(x + iy) + (u + iv) = (x + u) + i(y + v)$$

and

$$(x + iy) - (u + iv) = (x - u) + i(y - v)$$

We can also easily multiply a complex number by a real number:

$$k(x + iy) = kx + iky$$

Example: If $z = 4 + 2\mathrm{i}$ and $w = 3 - \mathrm{i}$, then

$z + w = (4 + 2\mathrm{i}) + (3 - \mathrm{i}) = 7 + \mathrm{i}$

$z - w = (4 + 2\mathrm{i}) - (3 - \mathrm{i}) = 1 + 3\mathrm{i}$ (being careful with the negative signs!)

$3z + 2w = 3(4 + 2\mathrm{i}) + 2(3 - \mathrm{i}) = (12 + 6\mathrm{i}) + (6 - 2\mathrm{i}) = 18 + 4\mathrm{i}$

$2w - z^* = 2(3 - \mathrm{i}) - (4 - 2\mathrm{i}) = (6 - 2\mathrm{i}) - (4 - 2\mathrm{i}) = 2$

2.2 Multiplying

Complex numbers can be multiplied using the general method for expanding brackets.

Examples:

$(2 + 5i)(4 - 3i) = 8 - 6i + 20i - 15i^2$ Remember: $i^2 = -1$

$= 8 - 6i + 20i - 15(-1)$

$= 23 + 14i$

$(3 + 2i)^2 = (3 + 2i)(3 + 2i) = 9 + 6i + 6i + 4i^2$

$= 9 + 12i + 4(-1)$

$= 5 + 12i$

2.3 Dividing

To divide complex conjugates, you multiply through by the complex conjugate of the denominator:

Example: If $z = 3 - i$ and $w = 1 - 2i$, then

$$z \div w = \frac{3-i}{1-2i}$$

$$= \frac{3-i}{1-2i} \times \frac{1+2i}{1+2i}$$ (as the complex conjugate of w is $w^* = 1 + 2i$)

$$= \frac{3+6i-i-2i^2}{1+2i-2i-4i^2} = \frac{3+5i-2(-1)}{1-4(-1)}$$ (as $i^2 = -1$)

$$= \frac{5+5i}{5}$$

Therefore:

$$z \div w = 1 + i$$

Note: When a complex number is multiplied by its complex conjugate the answer is always purely a real number.
To show this, suppose $z = x + iy$.
Then $zz^* = (x + iy)(x - iy) = x^2 - 2ixy + 2ixy - i^2y^2$
$= x^2 + y^2$

Worked examination question:

a) Express in the form $a + ib$,

(i) $(3 + i)^2$

(ii) $(2 + 4i)(3 + i)$.

b) The quadratic equation

$$z^2 - (2+4i)z + 8i - 6 = 0$$

has roots z_1 and z_2.

i) Verify that $z_1 = 3 + i$ is a root of the equation.

ii) By considering the coefficients of the quadratic, write down the sum of the roots;

iii) Explain why z_1^*, the complex conjugate of z_1, is **not** a root of the quadratic;

iv) Find the other root, z_2, in the form $a + ib$.

Solution:

a) (i) $(3 + i)^2 = (3 + i)(3 + i) = 9 + 3i + 3i + i^2$

$= 9 + 6i - 1 = 8 + 6i$ Remember: $i^2 = -1$

(ii) $(2 + 4i)(3 + i) = 6 + 2i + 12i + 4i^2$

$= 6 + 14i - 4 = 2 + 14i$

b) (i) We substitute $z = 3 + i$ into the expression $z^2 - (2+4i)z + 8i - 6$ to check that it gives an answer of 0.

$(3 + i)^2 - (2 + 4i)(3 + i) + 8i - 6$

$= 8 + 6i - (2 + 14i) + 8i - 6$ (using the results from (a))

$= 8 + 6i - 2 - 14i + 8i - 6$

$= 0 + 0i = 0$ (as required)

(ii) In a quadratic equation, the sum of roots is given by the expression $-b/a$.
So here the sum of the roots is $(2 + 4i)$ (since $b = -(2 + 4i)$ and $a = 1$)

(iii) In a quadratic equation with real coefficients, any complex roots form a conjugate pair. However, this quadratic does not have real coefficients so the roots are not complex conjugates of each other.

(iv) From (ii), $z_1 + z_2 = 2 + 4i$

So $3 + i + z_2 = 2 + 4i$

i.e. $z_2 = 2 + 4i - (3 + i) = -1 + 3i$.

Examination Question 1:

Given that $z = \dfrac{1+i}{1-2i}$, find z in the form $a + ib$.

Examination Question 2:

$z_1 = -3 + 4i$ $\qquad z_2 = 1 + 2i$

Express $z_1 z_2$ and $\dfrac{z_1}{z_2}$ each in the form $a + ib$, where $a, b \in \mathbb{R}$.

Examination Question 3:

The complex numbers z and w are such that

$z = -2 + 5i$ $\qquad zw = 14 + 23i$

Find w in the form $p + qi$, where p and q are real.

Examination Question 4:

Given that $z = -2 + 2\sqrt{3}i$, show that $z^2 + 4z$ is real.

Examination Question 5

(a) Show that $(3 - i)^2 = 8 - 6i$.

(b) The quadratic equation

$$az^2 + bz + 10i = 0,$$

where a and b are real, has a root $3 - i$.

(i) Show that $a = 3$ and find the value of b.

(ii) Determine the other root of the equation, giving your answer in the form $p + iq$.

(Hint for (ii): Use the fact that the sum of the roots in any quadratic is $-b/a$).

2.4 Equivalence of two complex numbers

Two complex numbers are equal to each other if and only if their real parts are equal and their imaginary parts are equal, i.e

If $z = x + iy$ and $w = u + iv$, then $z = w$ if and only if $x = u$ and $y = v$.

2.5 Solving linear equations with complex coefficients

Linear equations can be solved by substituting $z = x + iy$.

Example 1: Solve $4z - 2 + 5i = 6 - 7i$

Solution: Let $z = x + iy$.

Then: $4(x + iy) - 2 + 5i = 6 - 7i$

$4x + 4iy - 2 + 5i = 6 - 7i$ (expanding bracket)

$(4x - 2) + (4y + 5)i = 6 - 7i$

Therefore, comparing real and imaginary parts:

$4x - 2 = 6$ i.e. $x = 2$

$4y + 5 = -7$ i.e. $y = -3$

So the solution to the original equation must be $z = 2 - 3i$.

Example 2: Find z when $2z - 5z^* = 9 + 14i$.

Solution: Let $z = x + iy$. Then $z^* = x - iy$.

So:

$2(x + iy) - 5(x - iy) = 9 + 14i$

$2x + 2iy - 5x + 5iy = 9 + 14i$

i.e. $-3x + 7iy = 9 + 14i$.

Comparing real and imaginary parts, we see that $x = -3$ and $y = 2$.

So the solution is $z = -3 + 2i$.

Example 3: Solve $(4 + 2i)z + (3 - 2i) = 9 - 4i$

Solution: Write $z = x + iy$.

Then $(4+2i)(x+iy)+(3-2i)=9-4i$

So $(4x+4iy+2ix+2i^2y)+(3-2i)=9-4i$

i.e. $(4x-2y+4iy+2ix)+(3-2i)=9-4i$ (using $i^2 = -1$)

Collecting the real and imaginary terms together on the left hand side:

$(4x-2y+3)+i(4y+2x-2)=9-4i$

Comparing real and imaginary parts on both sides, we get the equations:

	$4x - 2y + 3 = 9$	i.e.	$4x - 2y = 6$	or	$2x - y = 3$	(1)
and	$4y + 2x - 2 = -4$	i.e.	$2x + 4y = -2$	or	$x + 2y = -1$	(2)

Equations (1) and (2) can be solved simultaneously:

$5x = 5$ $(2\times(1) + (2))$

i.e. $x = 1$

From equation (1), $y = 2x - 3 = 2 - 3 = -1$.

Therefore the solution to the equation is $z = 1 - i$.

Alternative solution:

The equation to be solved is $(4 + 2i)z + (3 - 2i) = 9 - 4i$

So … $(4 + 2i)z = (9 - 4i) - (3 - 2i)$

$= 6 - 2i$

Therefore

$$z = \frac{6-2i}{4+2i} = \frac{6-2i}{4+2i} \times \frac{4-2i}{4-2i}$$

So $$z = \frac{24-12i-8i-4}{16+4} = \frac{20-20i}{20}$$

This gives the solution $z = 1 - i$

Worked examination style question

It is given that $z = x + iy$ and that z^* is the complex conjugate of z.

a) Express $2z - 3z^*$ in the form $p + qi$.
b) Find the value of z for which $2z - 3z^* = -5 + 15i$

Solution:

If $z = x + iy$, then $z^* = x - iy$.

a) So $2z - 3z^* = 2(x + iy) - 3(x - iy) = 2x + 2iy - 3x + 3iy = -x + 5iy$.

b) If $2z - 3z^* = -5 + 15i$
then …. $-x + 5iy = -5 + 15i$
So … $x = 5$ and $y = 3$.

Examination style question:

Given that $(3 - 2i)(x + 5i) = y + 11i$

where x and y are real numbers:

a) find two equations for x and y;
b) find the values of x and y.

Examination Question 1

It is given that $z = x + iy$ and that z^* is the complex conjugate of z.

a) Express $z + 2z^*$ in the form $p + qi$.
b) Find the value of z for which $z + 2z^* = 9 + 2i$.

Examination Question 2:

It is given that $z = x + iy$, where x and y are real numbers.

a) Write down, in terms of x and y, an expression for z^*, the complex conjugate of z.
b) Find, in terms of x and y, the real and imaginary parts of
$$2z - iz^*$$
c) Find the complex number z such that
$$2z - iz^* = 3i.$$

Examination question 3:

(a) (i) Calculate $(2+i\sqrt{5})(\sqrt{5}-i)$.

(ii) Hence verify that $\sqrt{5}-i$ is a root of the equation

$$(2+i\sqrt{5})z = 3z^*$$

where z^* is the complex conjugate of z.

(b) The quadratic equation

$$x^2 + px + q = 0$$

in which the coefficients p and q are real, has a complex root $\sqrt{5}-i$.

(i) Write down the other root of the equation.

(ii) Find the sum and the product of the two roots of the equation.

Examination question 4:

Solve the simultaneous equations

$$iz + 2x = 1$$
$$z - (1 + i)w = i$$

giving your answers for z and w in the form $a + ib$.

2.6 Finding the square root of a complex number

One particular more difficult problem that occasionally occurs in FP1 examinations is to find the square root of a complex number. You should be prepared in case it comes up in your exam!!

Example: Find the square roots of 9 – 12i.

Solution: Let $a + bi$ be a square root of $9 - 12i$, where a and b are real numbers.
Then $(a + bi)^2 = 9 - 12i$.
So:

$$(a + bi)(a + bi) = 9 - 12i$$

i.e. $a^2 + abi + abi + b^2i^2 = 9 - 12i$

i.e. $a^2 - b^2 + 2abi = 9 - 12i$.

Comparing real and imaginary parts, we get two equations:

$a^2 - b^2 = 9$ ①

and $ab = -6$ ②

From equation ②, we get $b = \dfrac{-6}{a}$.

Substituting this into ① gives: $a^2 - \left(\dfrac{-6}{a}\right)^2 = 9$

i.e. $a^2 - \dfrac{36}{a^2} = 9$

i.e. $a^4 - 36 = 9a^2$

or $a^4 - 9a^2 - 36 = 0$

This is a quadratic equation in a^2. It can be factorised:

$$(a^2 - 12)(a^2 + 3) = 0$$

So $a^2 = 12$ or $a^2 = -3$.

As a is real, we must have $a = \pm\sqrt{12} = \pm 2\sqrt{3}$.

If $a = 2\sqrt{3}$, then $b = \frac{-6}{a} = \frac{-6}{2\sqrt{3}} = -\sqrt{3}$.

If $a = -2\sqrt{3}$, then $b = \frac{-6}{a} = \frac{-6}{-2\sqrt{3}} = \sqrt{3}$.

Therefore the square roots of $9 - 12i$ are $2\sqrt{3} - \sqrt{3}i$ and $-2\sqrt{3} + \sqrt{3}i$.

Note: All questions about finding the square root of a complex number can be solved using the same method.

Examination question:
Find the roots of the equation $z^2 = 21 - 20i$.

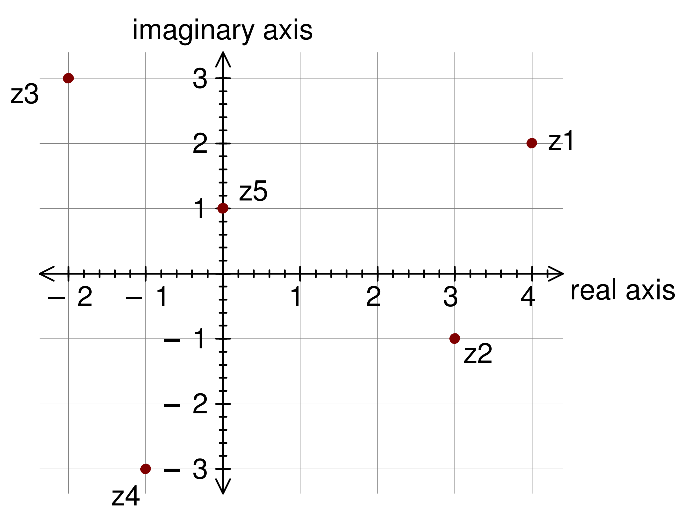
Hint: Remember to put $z = a + bi$.

Section 3: Argand Diagrams

3.1 Representing complex numbers on Argand diagrams

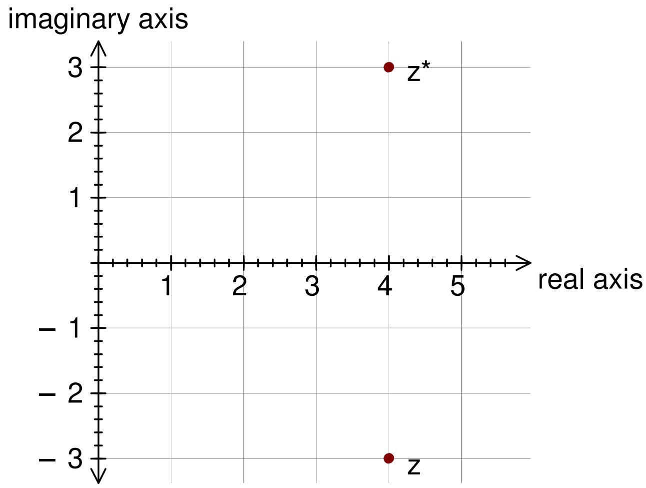
Complex numbers can be shown on an Argand diagram. The horizontal axis represents the real part of the complex number whilst the vertical axis represents the imaginary part.

Example:
Plot the complex numbers $z_1 = 4 + 2i$, $z_2 = 3 - i$, $z_3 = -2 + 3i$, $z_4 = -1 - 3i$ and $z_5 = i$ on an Argand diagram.



Example 2: Let $z = 3 - 4i$. Show z and z^* on an Argand diagram.

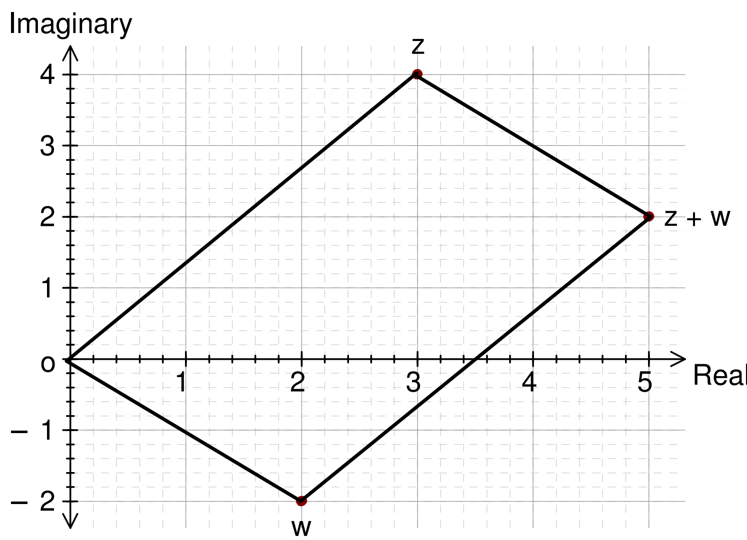
Solution: $z^* = 3 + 4i$.



Note: In an Argand diagram, z^* is a reflection of z in the real axis.

Example 3: Let $z = 3 + 4i$ and $w = 2 – 2i$. Show z, w and $z + w$ on an Argand diagram.

Solution: $z + w = 5 + 2i$.

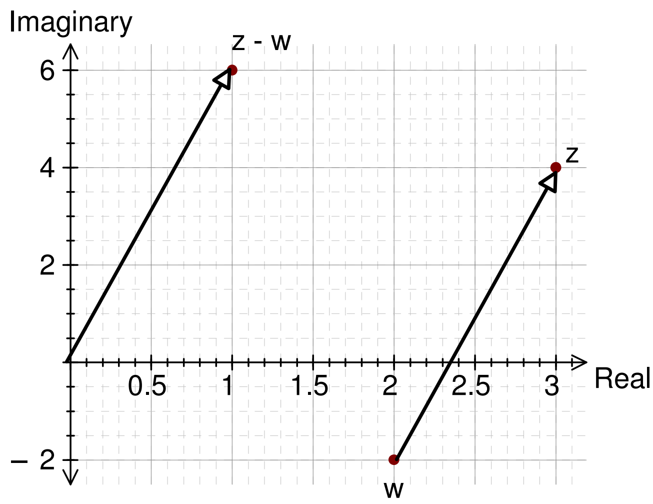


Notice that the points representing the origin, z, w and $z + w$ form a parallelogram.

General result: If complex numbers z, w and $z + w$ are represented by points Z, W and S in an Argand diagram, then OZSW is a parallelogram.

Example 4:
Let $z = 3 + 4i$ and $w = 2 – 2i$. Show z, w and $z - w$ on an Argand diagram.

Solution: $z – w = 1 + 6i$.



Notice this general result: If the complex numbers z and w are represented in an Argand diagram by points Z and W, then the translation which takes W to Z represents the complex number z – w.

Examination question:

Given that $z_1 = 1 + 2i$ and $z_2 = \frac{3}{5} + \frac{4}{5}i$, write $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $p + iq$, where $p, q \in \mathbb{R}$.

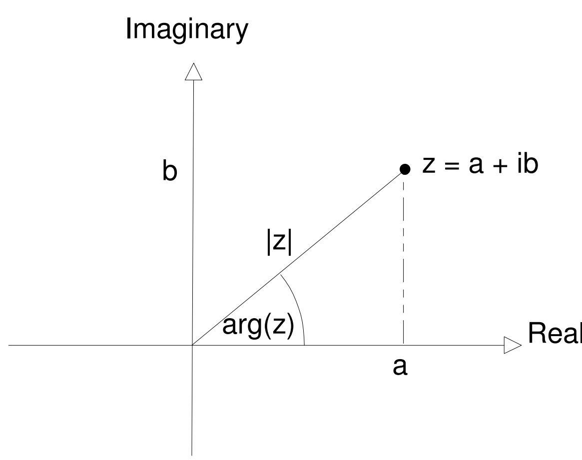
Plot the points representing $z_1, z_2, z_1 z_2$ and $\frac{z_1}{z_2}$ on an Argand diagram.

In the diagram, the origin and the points representing $z_1 z_2, \frac{z_1}{z_2}, z_3$ are the vertices of a rhombus.

Find z_3 and sketch the rhombus on this Argand diagram.

3.2 Modulus and argument of a complex number

Consider the complex number $z = a + ib$:



The **modulus** of z, written |z|, is the distance of z from the origin.

Therefore: $|z| = \sqrt{a^2 + b^2}$.

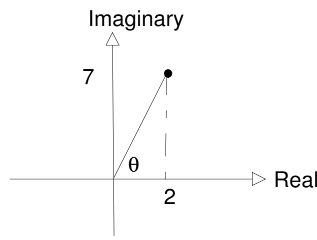
The **argument** of a complex number z is the angle that the line joining O to z makes with the positive real axis. Anticlockwise rotation is positive and clockwise rotation is negative. The argument is usually measured in radians and is chosen so that $-\pi < \arg(z) \leq \pi$.

Example: Find the modulus and argument of these complex numbers:

a) $2 + 7i$
b) $5 - 2i$
c) $-4 + 3i$
d) $-2 - 3i$
e) -5
f) $3i$

Solution: It helps to sketch an Argand diagram in each case:

a) $|2 + 7i| = \sqrt{2^2 + 7^2} = \sqrt{53}$

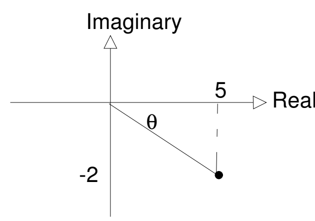


arg(2 + 7i) is angle θ in the diagram:

$$\tan\theta = \frac{7}{2}$$

i.e. $\theta = 1.29$ radians

b) $|5 - 2i| = \sqrt{5^2 + (-2)^2} = \sqrt{29}$

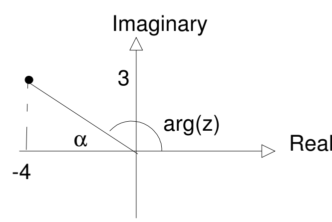


arg(5 - 2i) is shown by angle θ in the diagram.

$$\tan\theta = \frac{2}{5} \Rightarrow \theta = 0.381$$

i.e. $\theta = -0.381$ radians (it is negative as the number is below the axis).

c) $|-4 + 3i| = \sqrt{(-4)^2 + 3^2} = 5$

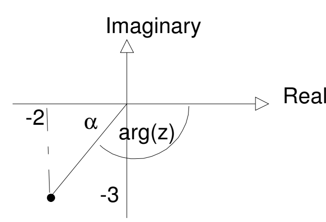


arg(-4 + 3i) is the angle shown on the diagram.
To find it, it is easiest to first find angle α:

$$\tan(\alpha) = \tfrac{3}{4} \qquad \Rightarrow \qquad \alpha = 0.644$$

Therefore, arg(-4 + 3i) = $\pi - 0.644 = 2.50$ radians.

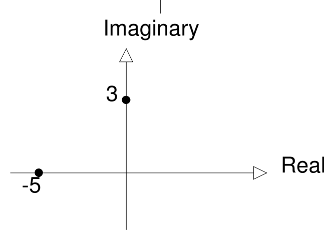
d) $|-2 - 3i| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$



arg(-2 – 3i) is the angle shown on the diagram. As -2-3i is below the real axis, the argument will be negative.
It is simplest to initially find angle α:

$$\tan(\alpha) = \tfrac{3}{2} \qquad \Rightarrow \qquad \alpha = 0.983$$

$\pi - 0.983 = 2.16$ radians.
Therefore arg(-2 – 3i) = -2.16 radians.

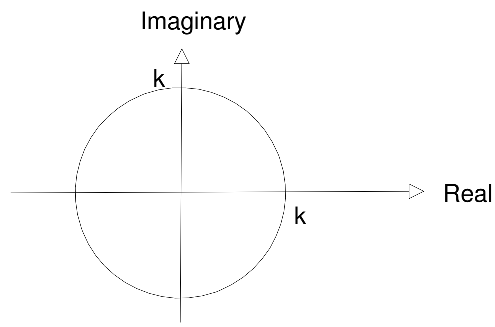


e) $|-5| = 5$
arg(-5) = π rads

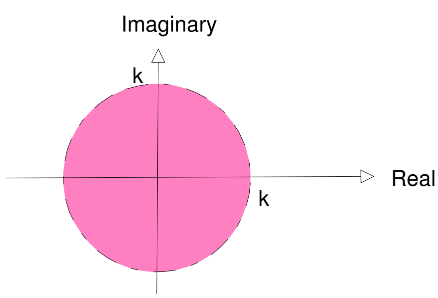
f) $|3i| = 3$
$\arg(3i) = \frac{1}{2}\pi$ rads

3.3 Loci

Situation 1: $|z| = k$
Since $|z|$ denotes the distance in an Argand diagram that a complex number is from the origin, the loci of points satisfying the relationship $|z| = k$ is a circle centre the origin, radius k.

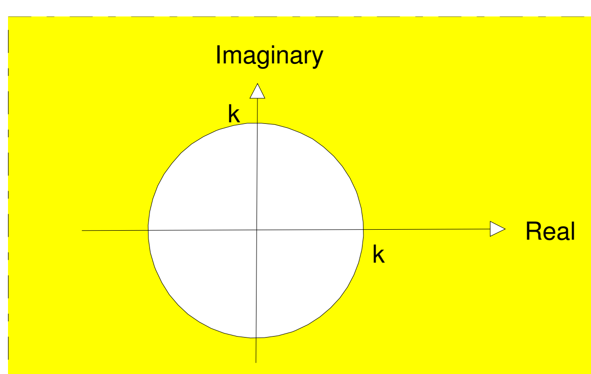


Related loci are:
* **$|z| < k$** the inside of a circle radius k, centre O (not including the circle circumference);

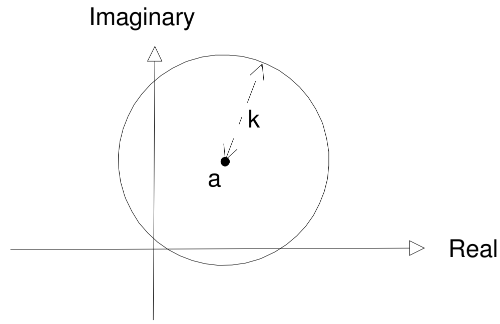


(the boundary is shown with a dotted line as it does not form part of the locus.)

* **$|z| \geq k$** the outside of a circle radius k, centre O (including the circle circumference).



Situation 2: $|z - a| = k$
Since $|z - w|$ represents the distance between the two complex numbers z and w on an Argand diagram, the loci of points satisfying the relation $|z - a| = k$ is a circle centre the complex number a and radius k.

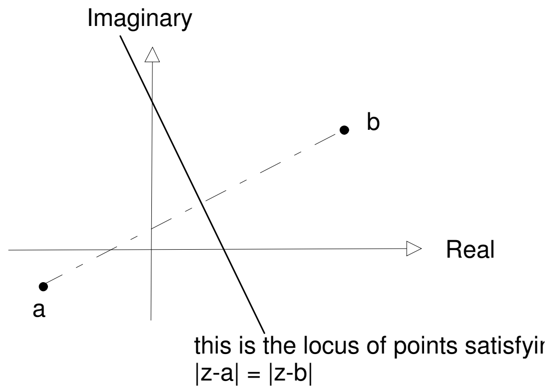


Related loci are:

* $|z - a| < k$ the inside of a circle radius k, centre a (not including the circle circumference);
* $|z - a| \geq k$ the outside of a circle radius k, centre a (including the circumference).

Situation 3: $|z - a| = |z - b|$

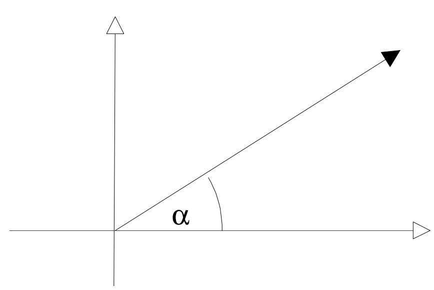
The points satisfying the relation $|z - a| = |z - b|$ will be the complex numbers that are equidistant from the complex numbers a and b. So the locus consists of the perpendicular bisector of the line segment joining complex numbers a and b.



Likewise the locus of points satisfying the relationship $|z - a| < |z - b|$ consists of points closer to a than to b.

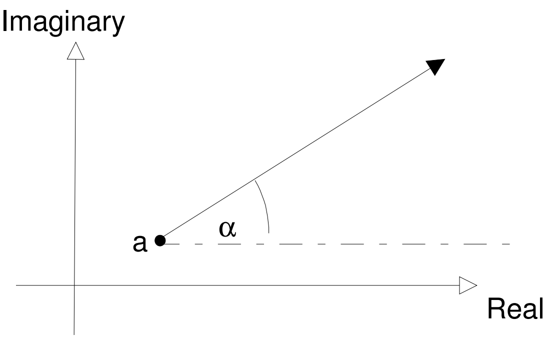
Situation 4: arg(z) = α

The points satisfying this relationship consist of a half-line emanating from the origin making an angle α with the positive real axis.



Situation 5: arg(z – a) = α

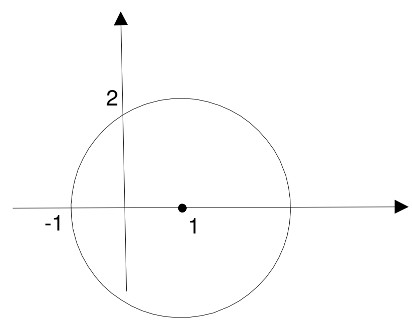
The required locus here is a half-line this time emanating from a, making an angle α with the (positive) horizontal direction:



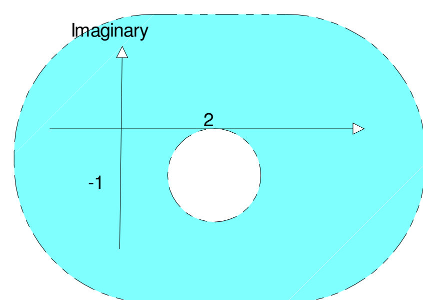
Example:
Sketch the loci satisfying these equations:

a) $|z - 1| = 2$
b) $|z - 2 + i| > 1$
c) $|z - 3| = |z - 1 - i|$
d) $|z| = |z + 2i|$
e) $\arg(z - 2) = \pi/3$
f) $\arg(z - 2 - i) = -\pi/4$.

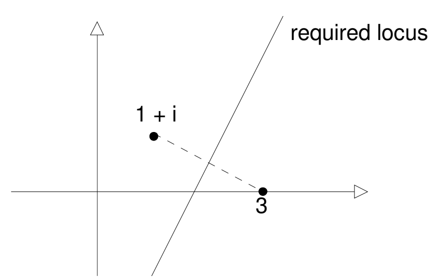
Solution:
a) The required locus is a circle centre 1, radius 2.



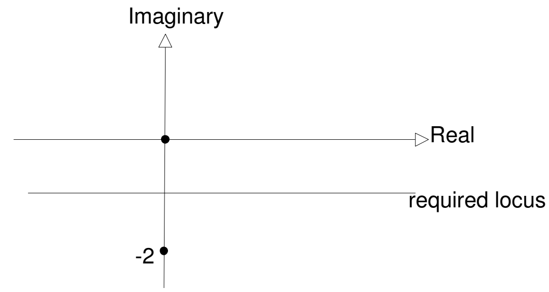
b) First we re-write the locus into the form $|z - a| > 1$:
We get: $|z - (2 - i)| > 1$.
This is the locus of points outside a circle centre $2 - i$, radius 1.



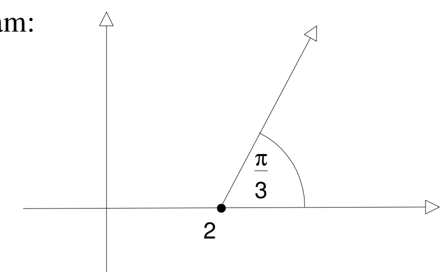
c) The equation $|z - 3| = |z - 1 - i|$ can be rewritten as
$|z - 3| = |z - (1 + i)|$.
So the required locus is the perpendicular bisector of the line segment joining the complex numbers 3 and $1 + i$:



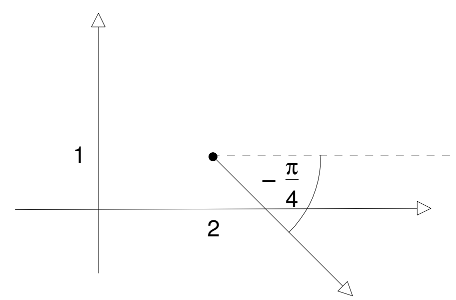
d) The equation $|z| = |z + 2i|$ represents the locus of points that are equidistant from the origin O and the complex number $-2i$:



e) $\arg(z - 2) = \pi/3$ represents the locus shown in the diagram:



f) The equation $\arg(z - 2 - i) = -\pi/4$ can be rewritten as $\arg(z - (2 + i)) = -\pi/4$.
This locus is shown on the diagram:



Examination question 1:
Two loci, L_1 and L_2, in the Argand diagram are defined by the following equations:

$$L_1: \quad |z + 2 - 3i| = 1;$$
$$L_2: \quad \arg(z - 4) = \tfrac{1}{2}\pi .$$

a) Sketch the two loci on one Argand diagram;
b) Find the smallest possible value of $|z_1 - z_2|$, where the points z_1 and z_2 lie on the loci L_1 and L_2 respectively.

Examination Question 2:
a) On the same Argand diagram, sketch the loci of points satisfying:
(i) $|z + 3 + i| = 5$;
(ii) $\arg(z + 3) = -\tfrac{3}{4}\pi$.

b) (i) From your sketch, explain why there is only one complex number satisfying both equations.
(ii) Verify that this complex number is $-7 - 4i$.

[Hint: To verify the last part, you need to check that the number $-7 - 4i$ satisfies both equations].

Examination question 3:

a) Sketch on one Argand diagram:

(i) the locus of points satisfying $|z - i| = |z - 2|$;

(ii) the locus of points satisfying $\arg(z - i) = \frac{1}{4}\pi$.

(b) Shade on your diagram the region in which

$$|z - i| \le |z - 2| \text{ and } -\tfrac{\pi}{2} \le \arg(z - i) \le \tfrac{\pi}{4}$$

Examination question 4:

Shade on an Argand diagram the region in which

$$|z - 2i| \le 1.$$

Examination question 5:

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \sqrt{3}i \text{ and } z_2 = iz.$$

a) (i) Express z_2 in the form $a + ib$.

(ii) Find the modulus and argument of z_2.

b) Label the points representing z_1 and z_2 on an Argand diagram.

c) On the same Argand diagram sketch the locus of points z satisfying

(i) $|z - z_1| = |z - z_2|$;

(ii) $\arg(z - z_1) = \arg(z_2)$.

Section 4: Mixed complex numbers questions

Worked examination style question:

The complex number 2 + 5i is one root of the equation $x^2 + px + q = 0$, where p and q are real.

a) Write down the second root of this equation.

b) Find the values of p and q.

Solution:

a) As the quadratic equation has real coefficients, the second root must be the complex conjugate of the first, i.e. 2 – 5i.

b) We know that the sum of the roots is given by $-b/a = -p$.
But the sum of the roots is (2 + 5i) + (2 – 5i) = 4.
Therefore $-p = 4$ i.e. $p = -4$.

The product of the roots is $c/a = q$.
But the product of the roots is (2 + 5i)(2 – 5i) = 4 – 10i + 10i + 25 = 29.
So $q = 29$.

Examination question (OCR January 2005):

(i) The complex number z is such that $z^2 = 1 + i\sqrt{3}$. Find the two possible values of z in the form $a + ib$, where a and b are exact real numbers.

(ii) With the value of z from part (i) such that the real part of z is positive, show on an Argand diagram the points A and B representing z and z^2 respectively.

Examination question (OCR May 2005):

The complex numbers z_1 and z_2 are such that $z_1 = 1 - i$ and $z_2 = -\sqrt{3} + i$.

(i) Find $\frac{z_1}{z_2}$ in the form $x + iy$, where x and y are exact real numbers.

(i) Find the exact modulus and argument (in terms of π) of z_1 and z_2.

Self-review:

Syllabus	Self-review ☺ 😐 ☹
Understand the idea of a complex number, recall the meaning of the terms *real part*, *imaginary part*, *modulus*, *argument*, *conjugate*, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;	
Carry out operations of *addition*, *subtraction*, *multiplication* and *division* of two complex numbers;	
Be able to use the result that, for a polynomial equation with real coefficients, any non-real roots occur in *conjugate pairs*;	
Represent complex numbers geometrically by means of an *Argand diagram*, and understand the geometrical effects of conjugating a complex number and of adding and subtracting two complex numbers;	
Find the two square roots of a complex number;	
Illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $\lvert z - a\rvert < k$, $\lvert z - a\rvert = \lvert z - b\rvert$, $\arg(z - a) = \alpha$.	