ALL SOULS HIGH SCHOOL

ADVANCED LEVEL PURE MATHEMATICS

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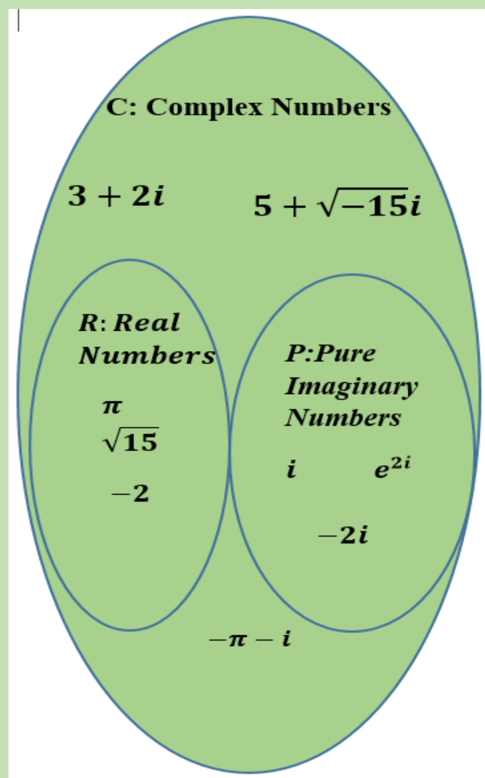
TOPIC 7: COMPLEX NUMBERS

Learning Objectives

After completing this chapter you should be able to:

- express complex numbers in polar and exponential form,
- carry out operations of complex numbers expressed in polar form,
- derive and prove the de Moivre's Theorem,
- prove trigonometrical identities using deMoivre's Theorem,
- solve equations using the deMoivre's Theorem,
- solve problems involving complex numbers.
- solve polynomial equations with at least one pair of non-real roots,
- illustrate equations and inequalities involving complex numbers by means of loci in an Argand diagram,

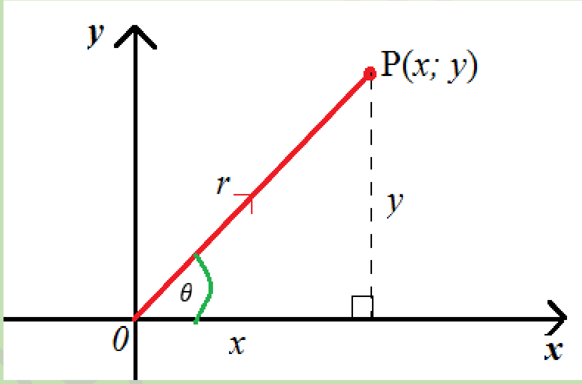


"I tell you, with complex numbers you can do anything" -John Derbyshire

7.2.1 Polar Form of a complex number

Consider the complex number $z = x + iy$. We can express z in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$, with $r \in IR$. This is known as the polar form.

Let the point P $(x; y)$ represent the complex number $z = x + iy$ and θ be the angle OP makes with the positive x-axis direction as shown in the diagram.



$\cos\theta = \frac{x}{r}$ => $x = r\cos\theta$

$\sin\theta = \frac{y}{r}$ => $y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

We can now write $z = x + iy$ as

$z = r\cos\theta + ir\sin\theta$

that is $z = r(\cos\theta + i\sin\theta)$

Note that θ, the argument, is not unique. The argument of z could also be $\theta \pm 2\pi$, $\theta \pm 4\pi$ etc. To avoid duplication of θ, we usually quote θ in the range $-\pi < \theta \leq \pi$ and refer to it as the principal argument, $arg\ z$.

Example 1

Express the following complex numbers in the form

$z = r(\cos\theta + \text{i}\sin\theta)$, where

$r > 0$ and $-\pi < \theta \leq \pi$

(i) $z = 1 + i$ (ii) $z = \sqrt{3} - i$

"I tell you, with complex numbers you can do anything" -John Derbyshire

Solution

(i) $z = 1 + i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg z = tan^{-1}1 = \frac{\pi}{4}$

Hence $z = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

(ii) $z = \sqrt{3} - i$

$|z| = \sqrt{\sqrt{3}^2 + (-1)^2} = 2$

$\arg z = -tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

Hence $z = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$= 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

7.2.3 Complex Exponentials

It is often very useful to write a complex number as an exponential with a complex argument.

The complex number $z = r(\cos\theta + i\sin\theta)$ in exponential form is written as $z = re^{\theta i}$ where r is the modulus of z and θ is the argument.

In the same way, the conjugate of z, $\bar{z} = r(\cos\theta - i\sin\theta) = re^{-\theta i}$.

Example

Express the following in the form $re^{\theta i}$, where $-\pi < \theta \leq \pi$

(a) $z = 2 - 3i$ (b) $z = 2\left(cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)$

Solution

(a) $z = 2 - 3i$

$r = \sqrt{2^2 + 3^2} = \sqrt{13}$ $\theta = -tan^{-1}\left(\frac{3}{2}\right) = -0.98$

$\therefore z = \sqrt{13}e^{-0.98i}$.

(b) $z = 2\left(cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)$

Re-write as $z = 2\left(cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

So $r = 2$ and $\theta = -\frac{2\pi}{3}$

$\therefore z = 2e^{-\frac{2\pi}{3}i}$

Exercise

1 Express the following in the form $x + iy$

(a) $e^{\frac{\pi}{4}i}$ (b) e^{i} (c) $e^{\pi i}$

2 Express the following complex numbers in exponential form

(a) $z = 1 - i$ (b) $z = 2 + 3i$ (c) $z = -6$

7.2.3 Multiplication of a complex number by i

Example 1

Express the following in the form $a + bi$

(a) $\frac{(1-i)^4}{(1+i)^3}$ (b) $\frac{(7+8i)^{10}}{(-8+7i)^{11}}$

Solution

(a) $\frac{(1-i)^4}{(1+i)^3}$

Observe that $1 - i = -i(1 + i)$

Hence $\frac{(1-i)^4}{(1+i)^3} = \frac{(-i)^4(1+i)^4}{(1+i)^3}$

$= \frac{(1+i)^4}{(1+i)^3}$

$= 1 + i$

(b) $\frac{(7+8i)^{10}}{(-8+7i)^{11}}$

$\frac{(7+8i)^{10}}{(-8+7i)^{11}} = \frac{(8i+7)^{10}}{(-8+7i)^{11}}$

Observe that $-8 + 7i = i(8i + 7)$

Hence $\frac{(8i+7)^{10}}{(-8+7i)^{11}} = \frac{(8i+7)^{10}}{i^{11}(8i+7)^{11}}$

$= \frac{(8i+7)^{10}}{-i(8i+7)^{11}}$

$$= \frac{1}{-i(7+8i)}$$

$$= \frac{1}{(8-7i)}$$

$$= \frac{1}{(8-7i)} \text{ x } \frac{8+7i}{(8+7i)}$$

$$= \frac{8}{113} + \frac{7}{113}i$$

Exercise

Express the following in the form $a + bi$

1 $\quad \frac{(1+i)^5}{(1-i)^7}$ $\qquad$ 2 $\quad \frac{(1+i)^4}{(2-2i)^3}$ $\qquad$ 3 $\quad \frac{(-2-5i)^9}{(5+2i)^7}$

4 $\quad \frac{(7-3i)^6}{(3+7i)^4}$ $\qquad$ 5 $\quad \frac{(2+3i)^2}{(-3+2i)^4}$

7.2.4 Product and quotient of two complex numbers in polar form

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1); \;\; z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\}$$
$$= r_1 r_2\{cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)\}$$

i.e. *To find the product of two complex numbers,* ***multiply their moduli,*** *and* ***add their arguments***

Taking the case of the quotient,

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\,\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{cos^2\theta_2 + sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\,\{cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\}$$

i.e. *to find the quotient of two complex numbers.* ***Divide their moduli*** *and* ***subtract their arguments.***

And so we have established that:

$$z_1 z_2 = r_1 r_2\{cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)\}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\{cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\}$$

It is important to remember this result

Example 2

Simplify

(a) $3(\cos 2\theta - i\sin 2\theta)\text{x}5(\cos\theta + i\sin\theta)$ (b) $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$

Solution

(a) $3(\cos 2\theta - i\sin 2\theta)\text{x}5(\cos 3\theta + i\sin 3\theta)$

$$= 3(\cos(-2\theta) + i\sin(-2\theta))\text{x}5(\cos 3\theta + i\sin 3\theta)$$

$$= 15(\cos(-2\theta + 3\theta) + i\sin(-2\theta + 3\theta))$$

$$= 15(\cos\theta + i\sin\theta)$$

(b) $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta} = \frac{\cos 3\theta + i\sin 3\theta}{\cos(-2\theta) + i\sin(-2\theta)}$

$$= \cos(3\theta - -2\theta) + i\sin(3\theta - -2\theta)$$

$$= \cos(5\theta) + i\sin(5\theta)$$

Exercise

1 Express the following complex numbers in the form $z = r(\cos\theta + \sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$

(a) $z = -1 - i$ (b) $z = -1 + \sqrt{3}i$

(c) $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ (d) $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

(e) $z = -5$ (f) $z = 3i$

2 Express each of the following in the form $x + yi$

(a) $z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(b) $z = 6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(c) $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(d) $z = \sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

3 Simplify the following

(a) $\frac{1}{\cos\theta + i\sin\theta}$ (b) $\frac{\cos 4\theta - i\sin 4\theta}{\cos 2\theta + i\sin 2\theta}$

4 Express $\frac{3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in the form $x + iy$.

7.2.5 The de Moivre's Theorem and its applications

Let $z = r(\cos\theta + i\sin\theta)$

$$z^2 = z \text{ x } z = r(\cos\theta + i\sin\theta) \text{ x } r(\cos\theta + i\sin\theta)$$

$$= r^2(\cos(\theta+\theta) + i\sin(\theta+\theta))$$

$z_1z_2 = r_1r_2\{cos(\theta_1+\theta_2) + isin(\theta_1+\theta_2)\}$

$$= r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = z^2 \text{ x } z = r^2(\cos 2\theta + i\sin 2\theta) \text{ x } r(\cos\theta + i\sin\theta)$$

$$= r^3(\cos(2\theta+\theta) + i\sin(2\theta+\theta))$$

$z_1z_2 = r_1r_2\{cos(\theta_1+\theta_2) + isin(\theta_1+\theta_2)\}$

$$= r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^4 = z^2 \text{ x } z^2 = r^2(\cos 2\theta + i\sin 2\theta) \text{ x } r^2(\cos 2\theta + i\sin 2\theta)$$

$$= r^4(\cos(2\theta+2\theta) + i\sin(2\theta+2\theta))$$

$z_1z_2 = r_1r_2\{cos(\theta_1+\theta_2) + isin(\theta_1+\theta_2)\}$

$$= r^4(\cos 4\theta + i\sin 4\theta)$$

And so, in general

$$z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$

This is the de Moivre's theorem

> The de Moivre's theorem states that, for a positive integer n,
>
> $$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

It is possible to prove the de Moivre's theorem by Mathematical induction for $n \in Z^+$.

Proof

We want to prove that for any positive integer n

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

"I tell you, with complex numbers you can do anything" -John Derbyshire

For $n = 1$ LHS = $[r(\cos\theta + i\sin\theta)]^1 = r(\cos\theta + i\sin\theta)$

RHS = $r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$

LHS = RHS, hence true for $n = 1$.

Assume that the theorem is true for $n = k$, where $k \in Z^+$ that is we assume that

$[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$

If the theorem is true for $n = k$, we show that it is also true for $n = k + 1$, that is we will show that

$[r(\cos\theta + i\sin\theta)]^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

Taking the LHS

$[r(\cos\theta + i\sin\theta)]^{k+1} = [r(\cos\theta + i\sin\theta)]^k$ x $r(\cos\theta + i\sin\theta)$ By Laws of indices

$= r^k(\cos k\theta + i\sin k\theta)$ x $r(\cos\theta + i\sin\theta)$ By assumption

$= r^{k+1}(\cos k\theta \text{ x } \cos\theta - \sin k\theta \text{ x} \sin\theta + i(\sin k\theta \text{ x} \cos\theta + \cos k\theta \text{ x} \sin\theta))$

$= r^{k+1}[cos(k\theta + \theta) + i(sin(k\theta + \theta))]$ By addition formulae for trigs

$= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

Which is what we intended to show

Since the theorem is true for $n = 1$, and if true for $n = k$ implies it is true for $n = k + 1$, it follows from the Principle of Mathematical Induction that it is true for any positive integer n.

The de Moivre's theorem also holds when n is a negative integer or a fraction.

In exponential form, if $z = re^{\theta i}$, the de Moivre's theorem is $z^n = \left(re^{\theta i}\right)^n = r^n e^{n\theta i}$

“I tell you, with complex numbers you can do anything” -John Derbyshire

Example 1

Use the de Moivre's theorem to simplify

(a) $(\cos\theta + i\sin\theta)^7$ (b) $\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^6$

(c) $\left(1+\sqrt{3}i\right)^{10}$

Solution

(a) $(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$

(b) $\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right)^6$

$= \left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)^6$ Using $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$= \cos\left(\frac{-6\pi}{4}\right) + i\sin\left(\frac{-6\pi}{4}\right)$ By the De Moivre's theorem

$= \cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)$

$= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$ 2π added to express in polar form

$= i$

Note that it is apparent from this example that $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$. It is very important to realise that this is a **deduction from** the de Moivre's theorem and it must not be quoted **as** the theorem.

(c) $\left(1+\sqrt{3}i\right)^{10}$

Let $u = 1 + \sqrt{3}i$

=> $|u| = \sqrt{1^2 + \sqrt{3}^2} = 2$

"I tell you, with complex numbers you can do anything" -John Derbyshire

$$\arg(u) = tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

Hence in polar form

$$(1+\sqrt{3}i)^{10} = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{10}$$

$$= 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$$ By the De Moivre's theorem

$$= 2^{10}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$ 2π subtracted to express in polar form

$$= 2^{10}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= 512(-1+\sqrt{3}i)$$

Example 2

Simplify

(a) $\frac{(\cos 3\theta + i\sin 3\theta)^9}{(\cos 2\theta + i\sin 2\theta)^7}$ (b) $\frac{\left[2\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)\right]^4}{\left[5\left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)\right]^6}$ (c) $\frac{1}{(1+\sqrt{3}i)^5}$

Solution

(a) $\frac{(\cos 3\theta + i\sin 3\theta)^9}{(\cos 2\theta + i\sin 2\theta)^7}$

$$= \frac{\cos 27\theta + i\sin 27\theta}{\cos 14\theta + i\sin 14\theta}$$ By the De Moivre's theorem

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$$= \cos(27\theta - 14\theta) + i\sin(27\theta - 14\theta)$$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\}$

$$= \cos 13\theta + i\sin 13\theta$$

(b) $\frac{\left[2\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)\right]^4}{\left[3\left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)\right]^5}$

$$= \frac{\left[2\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)\right]^4}{\left[3\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)\right]^5}$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$$= \frac{16\left(\cos\frac{28\pi}{3} + i\sin\frac{28\pi}{3}\right)}{243\left(\cos\left(-\frac{20\pi}{3}\right) + i\sin\left(-\frac{20\pi}{3}\right)\right)}$$

By the De Moivre's theorem

$$= \frac{16}{243}\left[\cos\left(\frac{28\pi}{3} - \frac{20\pi}{3}\right) + i\sin\left(\frac{28\pi}{3} - \frac{20\pi}{3}\right)\right]$$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\}$

$$= \frac{16}{243}\left[\cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right)\right]$$

$$= \frac{16}{243}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

2π subtracted to express in polar form

$$= \frac{16}{243}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \frac{8}{243}\left(-1 + \sqrt{3}i\right)$$

(c) $\frac{1}{(1+\sqrt{3}i)^5} = \left(1 + \sqrt{3}i\right)^{-5}$

$$= \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{-5}$$

Polar form

$$= 2^{-5}\left(\cos\frac{-5\pi}{3} + i\sin\frac{-5\pi}{3}\right)$$

By the De Moivre's theorem

$$= 2^{-5}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

2π added to express in polar form

$$= \frac{1}{32}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

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$$= \frac{1}{64}\left(1+\sqrt{3}i\right)$$

Example 3

(i) Use the De Moivre's theorem to show that

$$\cos 6\theta = 32cos^6\theta - 48cos^4\theta + 18\cos^2\theta - 1.$$

(ii) Deduce that $\cos\frac{\pi}{12}$ is a root of the equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$ and write down the other four roots in a similar form.

Solution

(i) $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$= cos\,^6\theta + 6cos^5\theta(i\sin\theta) + 15cos^4\theta(i\sin\theta)^2 + 20cos^3\theta(i\sin\theta)^3 + 15\cos^2\theta(i\sin\theta)^4 +$
$+6\cos\theta\,(i\sin\theta)^5 + (i\sin\theta)^6.$

Pascal's triangle used to expand RHS

$$= cos\,^6\theta + 6icos^5\theta\sin\theta - 15cos^4\theta sin^2\theta - 20icos^3\theta sin^3\theta + 15\cos^2\theta\, sin^4\theta + 6i\cos\theta\, sin^5\theta - sin^6\theta$$

$\therefore\ cos\,6\theta = cos\,^6\theta - 15cos^4\theta sin^2\theta + 15\cos^2\theta\, sin^4\theta - sin^6\theta$

After equating real parts

$$= cos\,^6\theta - 15cos^4\theta(1-cos^2\theta) + 15\cos^2\theta\,(1-cos^2\theta)^2 - (1-cos^2\theta)^3$$

$$= cos\,^6\theta - 15cos^4\theta + 15cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$$

$$= 32cos^6\theta - 48cos^4\theta + 18\cos^2\theta - 1$$

Notice that by equating imaginary parts of the equation, we will obtain an expression for $\sin 6\theta$.

(ii) When $\theta = \frac{\pi}{12}$, $\cos\left(6\text{x}\frac{\pi}{12}\right) = \cos\frac{\pi}{2} = 0$ hence $\cos\frac{\pi}{12}$ is a root.

The other roots of $\cos 6\theta = 0$ are $\pm\frac{3\pi}{12},\ \pm\frac{5\pi}{12},\ \pm\frac{7\pi}{12},\ \pm\frac{9\pi}{12}$ etc..

So the other roots of the equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$ are

$\cos\frac{3\pi}{12}$, $\cos\frac{5\pi}{12}$, $\cos\frac{7\pi}{12}$, $\cos\frac{9\pi}{12}$, $\cos\frac{11\pi}{12}$.

Example 4

Using the de Moivre's theorem, show that $\tan 5\theta = \frac{5\tan\theta - 10tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta}$.

Solution

Consider $\cos 5\theta + i\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5$

We will expand in the same way we did in example 3. We will let $\cos\theta$ be c and $\sin\theta$ be s to avoid too much writing.

$\cos 5\theta + i\sin 5\theta = (\mathrm{c} + \mathrm{is})^5 = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$.

Equating the real parts in the equation $\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$.

Equating the imaginary parts in the equation $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$

Now, $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$

Dividing numerator and denominator of RHS by c^5 we have

$\tan 5\theta = \frac{5\tan\theta - tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5\,tan^4\,\theta}$

Exercise

1 Express $\frac{\sin 6\theta}{4\sin\theta}$ in terms of $\cos\theta$.

2 Use the de Moivre's theorem to show that

$\cos 7\theta = 64cos^7\theta - 112cos^5\theta + 56cos^3\theta - 7cos\theta$

Hence obtain the roots of the equation $128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$ in the form $\cos q\pi$ where q is a rational number.

7.2.6 More Applications of the de-Moivre's theorem

If $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$= (\cos(-\theta) + i\sin(-\theta))$ By the De Moivre's theorem

$= \cos\theta - i\sin\theta$ Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Now, $z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

Notice also that

If $z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

By the De Moivre's theorem

$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= (\cos(-n\theta) + i\sin(-n\theta))$

By the De Moivre's theorem

$= \cos n\theta - i\sin n\theta$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Now, $z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

And so we have established that:

$z + \frac{1}{z} = 2\cos\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$

$z - \frac{1}{z} = 2i\sin\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$

Example 1

Express $8cos^4\theta$ in terms of multiple angles of $\cos\theta$.

Solution

$(2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4$

since $2\cos\theta = z + \frac{1}{z}$

⇨ $16cos^4\theta = z^4 + 4z^3.\frac{1}{z} + 6z^2.\frac{1}{z^2} + 4z.\frac{1}{z^3} + \frac{1}{z^4}$

Expansion of RHS using Pascal's Δ

⇨ $16cos^4\theta = z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

⇨ $16cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$

Using $z^n + \frac{1}{z^n} = 2\cos n\theta$

Dividing both sides by 2 gives

$8cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3.$

Exercise

1 Using applications of the de Moivre's theorem prove that

$cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$, and hence find $\int cos^5\theta\, d\theta$.

2 Show that $32cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

Hence find $\int_0^{\pi/6} cos^6\theta\, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are constants

3 (a) Express $\left(z^2 + \frac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$.

(b) Hence or otherwise show that $cos^3 2\theta = a\cos 6\theta + b\cos 2\theta$,

where a and b are constants

(c) Hence or otherwise show that $\int_0^{\pi/6} cos^3 2\theta\, d\theta = k\sqrt{3}$, where k is

a constant.

4 Simplify the following in terms of multiples of θ

(a) $2z^3 + 5z^4 + \frac{2}{z^3} - \frac{5}{z^4} + 6$

(b) $z^7 - 7z^5 + \frac{7}{z^5} - 35z + \frac{35}{z} - \frac{21}{z^3} + 21z^3 - \frac{1}{z^7}$

7.2.7 Solution of Complex numbers using the de Moivre's Theorem

Recall the forms in which the complex number *z* may
be written:

$$z = x + iy = r(\cos\theta + i\sin\theta\ ) = re^{i\theta}$$

where *r* is a *real* number with *r* > 0, and θ is an angle such that $-\pi$< θ ≤ π.

We need to add multiples of 2π to obtain other angles that are co-terminal to θ and therefore have the same cosine and sine. In other words we obtain the same complex number.

$$z = re^{i\theta} = re^{i(\theta+2k\pi)} = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)] \quad \text{for } k = 0, 1, 2, \dots, n-1.$$

Also recall the de Moivre's theorem which states

$$z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

This follows immediately from the properties of complex exponentials:

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

Example 1

Find all the complex solutions of the equation $z^4 = -256$ and illustrate your solutions on an argand diagram.

Solution

$z^4 = -256 + 0i$ ==> $|z^4| = 256$ and $\arg(z^4) = \pi$

$$z^4 = 256e^{(\pi+2k\pi)i}$$

$$z = \left[256e^{(\pi+2k\pi)i}\right]^{1/4}$$

$$z_k = \sqrt[4]{256}e^{\left(\frac{\pi+2k\pi}{4}\right)i}$$

By the De Moivre's theorem

$$z_k = \sqrt[4]{256}\left(\cos\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)\right)$$

Polar form

$$z_k = 4\left(\cos\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)\right) \quad \text{for } k = 0, 1, 2, 3.$$

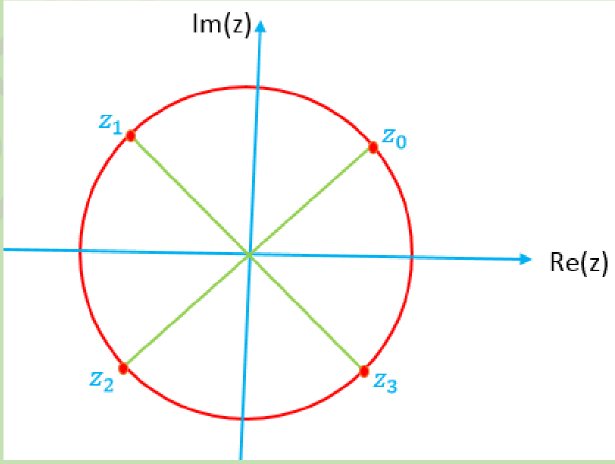
For

$k = 0$: $z_0 = 4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = 2\sqrt{2}+2\sqrt{2}i$

$k = 1$: $z_1 = 4\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right) = -2\sqrt{2}+2\sqrt{2}i$

$k = 2$: $z_2 = 4\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right) = -2\sqrt{2}-2\sqrt{2}i$

$k = 3$: $z_3 = 4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right) = 2\sqrt{2}-2\sqrt{2}i$



Notice that the pairs (z_0 , z_3) and (z_1 , z_2) are conjugates having the form $x \pm iy$.

When a polynomial equation has real coefficients, then it is always the case that the complex solutions occur in conjugate pairs.

Example 2

Find all cubic roots of $z = -1 + i$ and illustrate your solutions on an Argand diagram

Solution

We want to solve $z^3 = -1 + i$

Let $u = -1 + i$ ==> $|u| = \sqrt{2}$ and $\arg(u) = \frac{3}{4}\pi$

$$u = \sqrt{2}e^{\left(\frac{3}{4}\pi + 2k\pi\right)i}$$

$$z = u^{1/3}$$

$$z_k = \sqrt[3]{2}e^{\left(\frac{\frac{3}{4}\pi + 2k\pi}{3}\right)i}$$

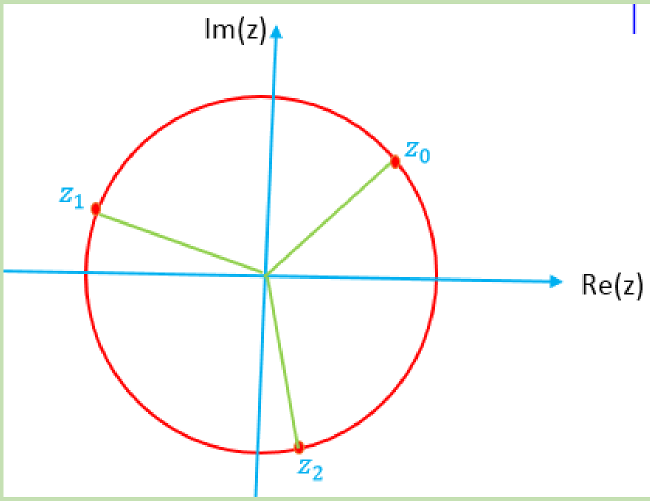
$$z_k = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right)$$

$$z_k = 2^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right) \quad \text{for } k = 0, 1, 2.$$

$k = 0$: $z_0 = 2^{\frac{1}{6}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$k = 1$: $z_1 = 2^{\frac{1}{6}}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

$k = 2$: $z_2 = 2^{\frac{1}{6}}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$



"I tell you, with complex numbers you can do anything" -John Derbyshire

Example 3

Find the fourth root of $-16i$ and illustrate your solutions on an Argand diagram

Solution

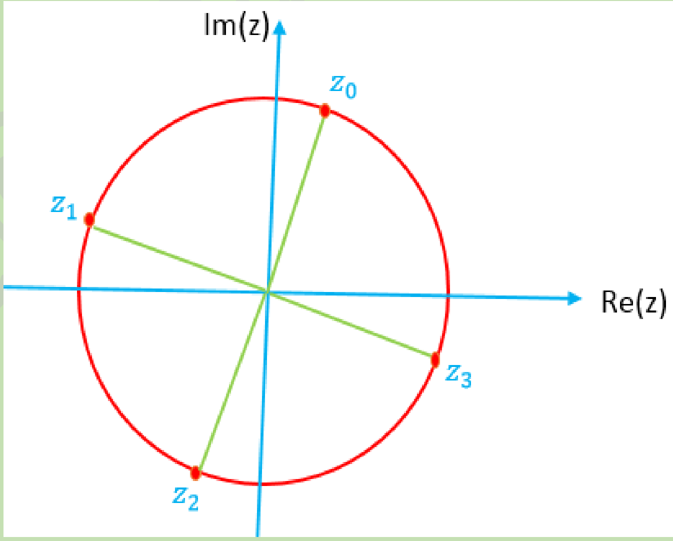
We want to solve $z^4 = -16i$

Let $z^4 = 0 - 16i$ ==> $|z^4| = 16$ and $\arg(z^4) = \frac{3}{2}\pi$

$$z^4 = 16e^{\left(\frac{3}{2}\pi+2k\pi\right)i}$$

$$z_k = 16^{\frac{1}{4}}e^{\frac{1}{4}\left(\frac{3}{2}\pi+2k\pi\right)i}$$

$$z_k = 2e^{\left(\frac{3}{8}\pi+\frac{1}{2}k\pi\right)i} \quad \text{for } k = 0, 1, 2, 3.$$



$$z_0 = 2\left(\cos\frac{3}{8}\pi + i\sin\frac{3}{8}\pi\right)$$

$$z_1 = 2\left(\cos\frac{7}{8}\pi + i\sin\frac{7}{8}\pi\right)$$

$$z_2 = 2\left(\cos\frac{11}{8}\pi + i\sin\frac{11}{8}\pi\right)$$

$$z_3 = 2\left(\cos\frac{15}{8}\pi + i\sin\frac{15}{8}\pi\right)$$

Example 4

Using the substitution $w = z^4$, solve the equation $z^8 - z^4 - 6 = 0$ where z is a complex number.

Solution

$w^2 - w - 6 = 0$

$(w-3)(w+2) = 0$

$w = 3$ or -2

$z^4 = 3$

Taking $w = 3 + 0i$ ==> $|w| = 3$ and $\arg(w) = 0$

$z_k = \sqrt[4]{3}e^{(2k\pi)\frac{1}{4}i}$

$z_k = \sqrt[4]{3}e^{\frac{k\pi}{2}i}$ for $k = 0, 1, 2, 3.$

$z_0 = \sqrt[4]{3}\left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right) = \sqrt[4]{3}$

$z_1 = \sqrt[4]{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \sqrt[4]{3}i$

$z_2 = \sqrt[4]{3}(\cos\pi + i\sin\pi) = -\sqrt[4]{3}$

$z_3 = \sqrt[4]{3}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -\sqrt[4]{3}i$

Considering $z^4 = -2$

Taking also $w = -2 + 0i$ ==> $|w| = 2$ and $\arg(w) = \pi$

$z_k = \sqrt[4]{2}e^{\frac{1}{4}(\pi+2k\pi)i}$

$z_k = \sqrt[4]{2}e^{\left(\frac{1}{4}\pi+\frac{1}{2}k\pi\right)i}$ for $k = 0, 1, 2, 3.$

$z_0 = \sqrt[4]{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{2^{3/4}}{2}(1+i)$

$$z_1 = \sqrt[4]{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{2^{3/4}}{2}(-1 + i)$$

$$z_2 = \sqrt[4]{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -\frac{2^{3/4}}{2}(1 + i)$$

$$z_3 = \sqrt[4]{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{2^{3/4}}{2}(1 - i)$$

Exercise

1 Solve the equation $z^3 = 2 + 2i$ leaving your answers in exponential form.

2 Express $3\sqrt{3} - 3i$ in the form $re^{\theta i}$. Hence find the fourth root of $3\sqrt{3} - 3i$ giving your answers correct to 2 decimal places

3 Solve the equation $z^4 + 9i = 0$, giving your answers in the form $re^{\theta i}$, where $r > 0$ and $-\pi < \theta \leq \pi$

4 Leaving your answers in the form $r(\cos\theta + i\sin\theta)$, find all the cube roots of $z = -1 + i$.

5 Leaving your answers in the form $r(\cos\theta + i\sin\theta)$, solve the equation $z^4 + 8 + i8\sqrt{3} = 0$

6 Solve the following equation giving your answers in the form $a + bi$ correct to 2dp.

(a) $z^5 + 32 = 0$ (b) $z^4 = 8 - 8\sqrt{3}i$

7 Solve the following equations, expressing your answers for z in the form $r(\cos\theta + i\sin\theta)$, $r > 0$ and $-\pi < \theta \leq \pi$.

(a) $z^3 = -16\sqrt{3} - 16i$ (b) $z^4 + 2\sqrt{3}i = 2$ (c) $z^7 - 8 - 8i = 0$

8 Leaving your answers in exact form in the form $a + bi$, solve the following equation $z^3 = 1$. Show your solutions on an Argand diagram.

7.2.8 nth Roots of unity

The solutions of the equation $z^n = 1$ are called the **nth roots of unity**.

Notice that the modulus of z^n is always 1, and the argument of z^n is always zero.

Therefore nth roots of unity can be expressed in the form.

$z_k = e^{\frac{2k\pi}{n}i}$ Or $z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ for $k = 0, 1, 2, 3, \ldots, n-1$

Example 1

Find the fifth roots of unity and show your solutions on an Argand diagram.

Solution

We want to solve $z^5 = 1$.

We consider $z_k = e^{\frac{2k\pi}{5}i}$ for $k = 0, 1, 2, 3, 4$.

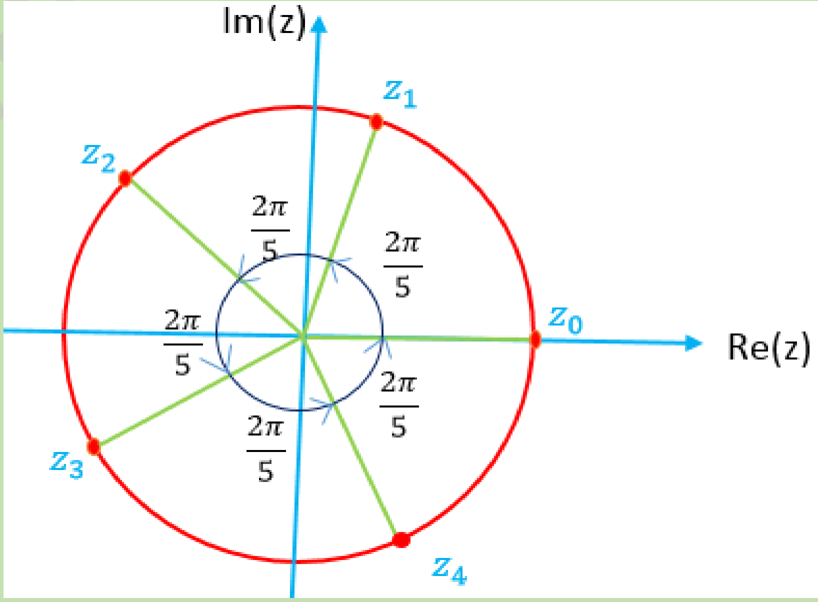
$z_0 = e^0 = 1$

$z_1 = e^{\frac{2\pi}{5}i} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_2 = e^{\frac{4\pi}{5}i} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_3 = e^{\frac{6\pi}{5}i} = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$z_4 = e^{\frac{8\pi}{5}i} = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$



7.2.9 Roots of complex numbers

When a polynomial equation has real coefficients, then it is always the case that the complex solutions occur in conjugate pairs.

Example 1

(a) Show that $2-3i$ is a root of the equation $2z^3-9z^2+30z-13=0$.

(b) Hence find the other roots.

Solution

(a) Let $f(z)=2z^3-9z^2+30z-13$

$$f(2-3i)=2(2-3i)^3-9(2-3i)^2+30(2-3i)-13$$

$$=2[8+3\text{x}4(-3i)+3\text{x}2(-3i)^2+(-3i)^3]-9(4-12i-9)+60-90i-13$$

$$=2(8-36i-54+27i)-36+108i+81+60-90i-13$$

$$=16-72i-108+54i-36+108i+81+60-90i-13$$

$$=0$$

Since $f(2-3i)=0$, it means $2-3i$ is a root of the equation.

(b) If $2-3i$ is a root then its conjugate $2+3i$ is another root.

So $z-2+3i$ and $z-2-3i$ are factors of the complex number.

Hence a quadratic factor is $(z-2+3i)(z-2-3i)=z^2-4z+13$

Using long division to find the third factor we get

$2z-1$ as the third factor

Taking $2z - 1 = 0$ means $\frac{1}{2}$ is the third root.

Example 2

Given that $u = 1 + 2i$ is a root of the equation $z^4 + pz^3 + qz^2 - 6z + 65 = 0$ find the values of p and q.

Solution

Let $f(z) = z^4 + pz^3 + qz^2 - 6z + 65$

If $u = 1 + 2i$ is a root then $f(u) = 0$, that is

$$(1 + 2i)^4 + p(1 + 2i)^3 + q(1 + 2i)^2 - 6(1 + 2i) + 65 = 0$$

Now, $(1 + 2i)^4 = 1 + 4\text{x}2i + 6\text{x}(2i)^2 + 4\text{x}(2i)^3 + (2i)^4 = -7 - 24i$

$$(1 + 2i)^3 = 1 + 3\text{x}2i + 3\text{x}(2i)^2 + (2i)^3 = -11 - 2i$$

$$(1 + 2i)^2 = -3 + 4i$$

Thus, $-7 - 24i + p(-11 - 2i) + q(-3 + 4i) - 6(1 + 2i) + 65 = 0$

$$-7 - 24i - 11p - 2pi - 3q + 4qi - 6 - 12i + 65 = 0$$

$$(-7 - 11p - 3q - 6 + 65) + (-24 - 2p + 4q - 12)i = 0$$

Equating the real parts we have

$$52 - 11p - 3q = 0 \ldots\ldots\ldots\ldots (1)$$

Equating the imaginary parts we have

$$-36 - 2p + 4q = 0 \ldots\ldots\ldots\ldots (2)$$

From (2), $p = -18 + 2q$

∴ Substituting into (1) => $52 - 11(-18 + 2q) - 3q = 0$

"I tell you, with complex numbers you can do anything" -John Derbyshire

$$52 + 198 - 22q - 3q = 0$$

$$q = 10 \quad \text{and } p = 2$$

Exercise

1 (a) The equation $3z^3 - 10z^2 + 20z - 16 = 0$ has $1 - \sqrt{3}i$ as one of its roots.

(i) Find the other roots.

(ii) Sketch the roots in an Argand diagram.

2 Show that $-2 + i$ is one root of the quartic equation

$z^4 + 2z^3 + 2z^2 + 10z + 25 = 0$, and find the other roots.

3 Verify that $z = i$ satisfies the equation $z^4 + 4z^3 + 6z^2 + 4z + 5 = 0$. Find the other roots.

4 ki is a root of the equation $2z^3 - z^2 + 18z - 9 = 0$, where k is a real number.

Find the values of k and the three roots of the equation $2z^3 - z^2 + 18z - 9 = 0$.

5 Given that $1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where p and q are constants, find

(a) the other two roots,

(b) the values of p and q.

6 (a) Factorise $z^2 - 4$ and, hence or otherwise, solve the equation $z^2 - 4 = 0$.

(b) Show that $z^2 - 4$ is a factor of $z^3 + (3 + i)z^2 - 4z - 4(3 + i)$.

(c) Find the three roots of the equation $z^3 + (3 + i)z^2 - 4z - 4(3 + i) = 0$.

7.2.10 Loci

A locus is a path traced out by a point subjected to certain restrictions.

CASE 1: $|z - z_1| = k$

This is a circle centre z_1 and radius k.

Example 1

Describe and sketch on an Argand diagram the locus of

(i) $|z| = 2$

(ii) $|z - 1| = 1$

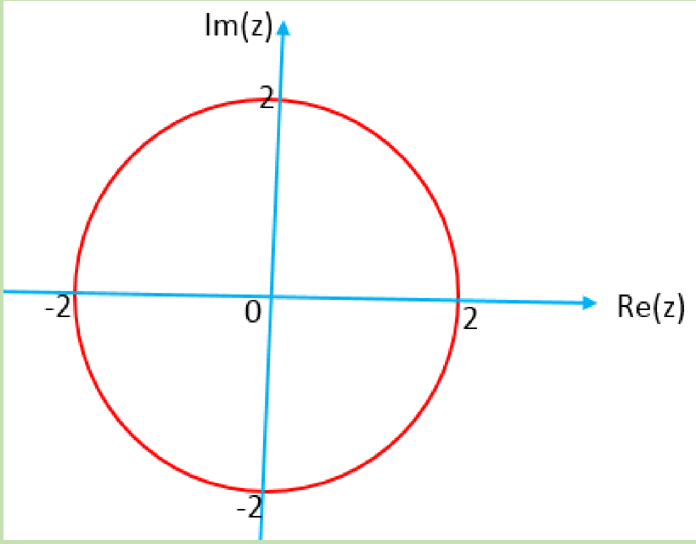
(iii) $|z - 1 - i|^2 = 9$

Solution

(i) $|z| = 2$

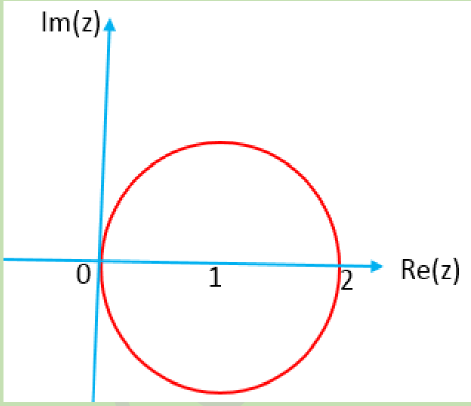
Re-write as $|z - (0 + 0i)| = 2$

It is a circle centre $(0, 0)$ radius 2.



"I tell you, with complex numbers you can do anything" -John Derbyshire

(ii) $|z-1|=1$

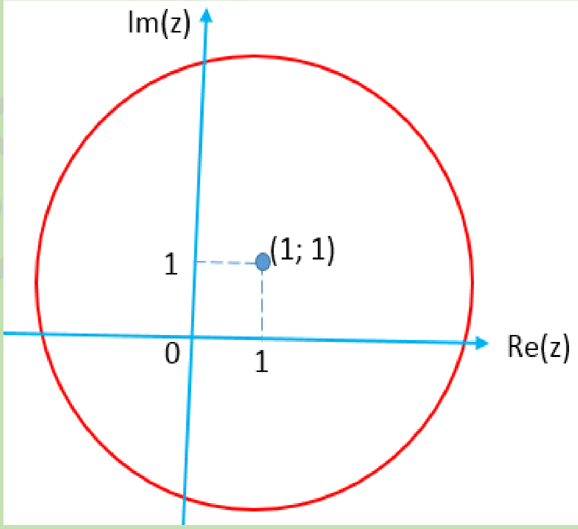


Re-write as $|z-(1+0i)|=1$

It is a circle centre $(1, 0)$ and radius 1

Notice that the vertical axis is a tangent to the circle

(iii) $|z-1-i|^2=9$



Taking square roots of both sides we get

$|z-1-i|=3$

Re-write as $|z-(1+i)|=3$

It is a circle centre $(1, 1)$ and radius 3

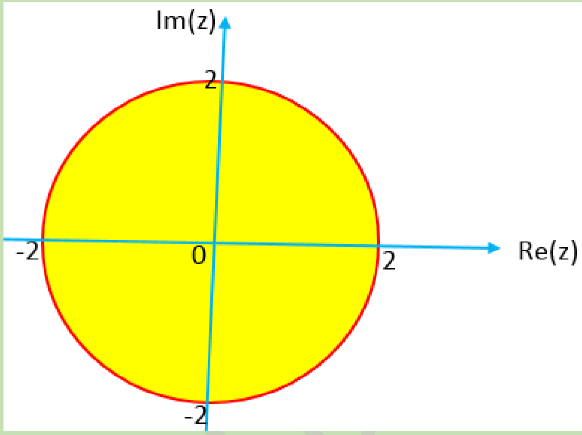
Example 1

Describe and sketch on an Argand diagram the locus of

(i) $|z|\leq 2$

(ii) $2<|z|\leq 3$

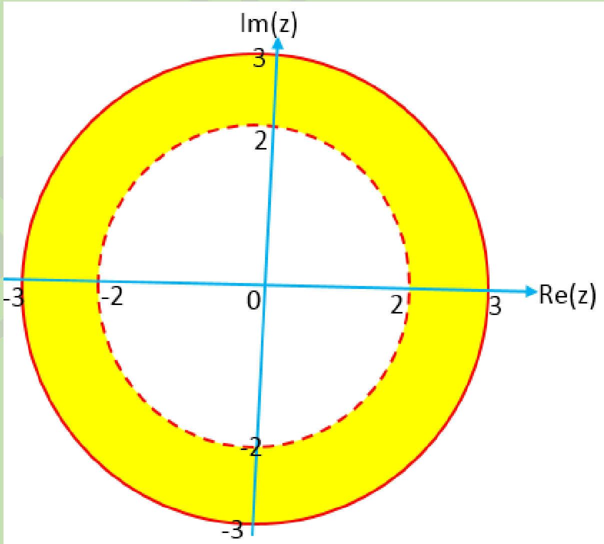
Solution



(i) $|z| \leq 2$

Re-Write as $|z-(0+0i)| \leq 2$

This is the circumference and inside of a circle centre $(0;0)$ and radius 2



(ii) $2 < |z| \leq 3$

Re-write as $2 < |z-(0+0i)| \leq 3$

This is the region between the two circles, one with centre (0;0) and radius 2 and the other with centre (0;0) and radius 3.

Notice that the circumference of the smaller circle is a broken line.

CASE 2: $|z-z_1| = |z-z_2|$

Perpendicular bisector of the line segment joining complex numbers z_1 and z_2.

Example 1

Sketch on an Argand diagram the locus of

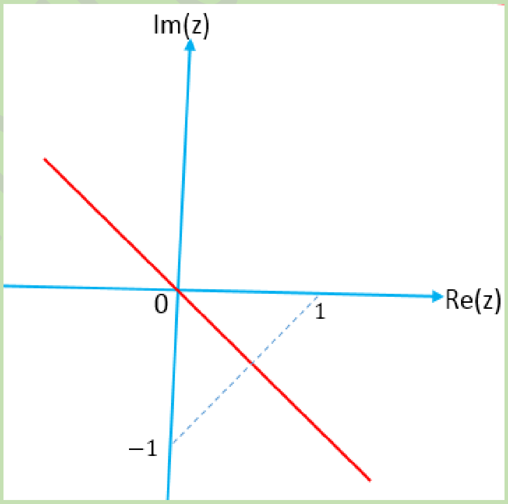
$$|z+i| = |z-1|$$

Solution

$$|z+i| = |z-1|$$

Re-write as $|z-(0-i)| = |z-(1+0i)|$

Perpendicular bisector of the line joining $(0,-1)$ and $(1,0)$



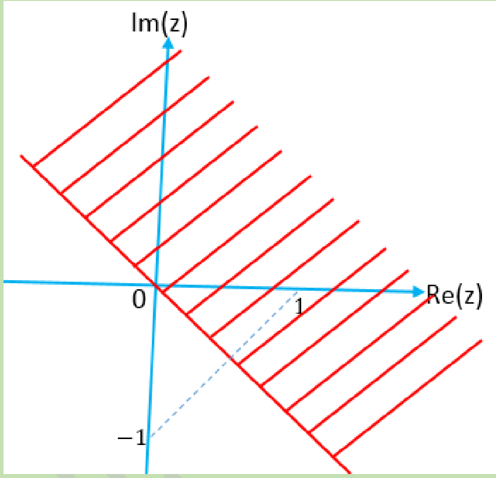
Example 2

Sketch on an Argand diagram and describe the locus represented by

$$|z+i| \geq |z-1|$$

"I tell you, with complex numbers you can do anything" -John Derbyshire

Solution



It is a half plane to the right of the perpendicular bisector joining $(0,-1)$ and $(1,0)$.

Notice that this locus consists of points closer to (1; 0) than $(0,-1)$.

CASE 3: $\text{Arg}(z-z_1) = \alpha$

This is a half line having one end at z_1 and making an angle α with the (positive) horizontal direction.

Example

On an Argand diagram, represent the following loci

(i) $\text{Arg}(z) = \frac{\pi}{6}$

(ii) $\text{Arg}(z+1-i) = \frac{2\pi}{3}$

(iii) $\text{Arg}(z-2-i) = -\frac{\pi}{4}$

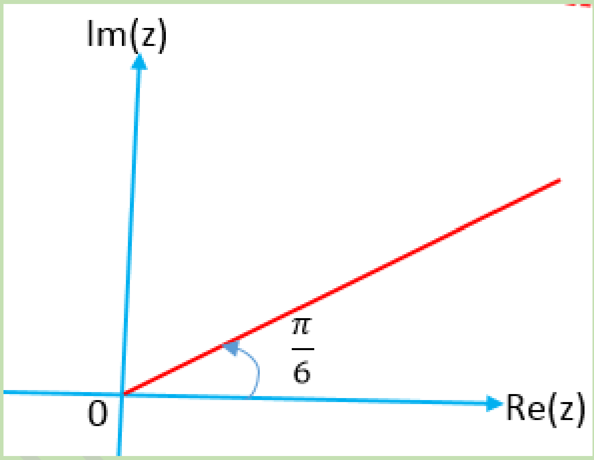
(iv) $\frac{\pi}{3} \leq \arg(z-2) \leq \frac{\pi}{2}$

Solution

(i) $\text{Arg}(z) = \frac{\pi}{6}$

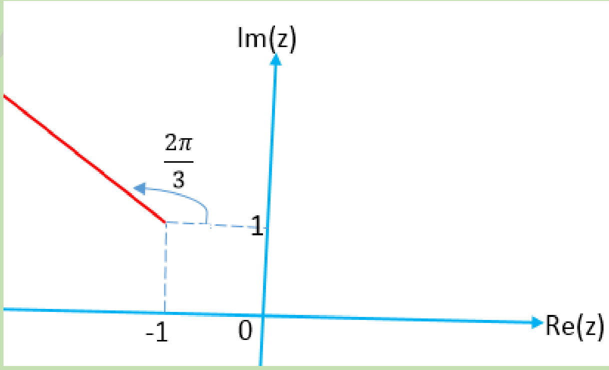
Re-write as $\text{Arg}(z - (0 + 0i)) = \frac{\pi}{6}$

This is a half line having one end at O and making an angle $\frac{\pi}{6}$ with the (positive) horizontal direction.



(ii) $\text{Arg}(z + 1 - i) = \frac{2\pi}{3}$

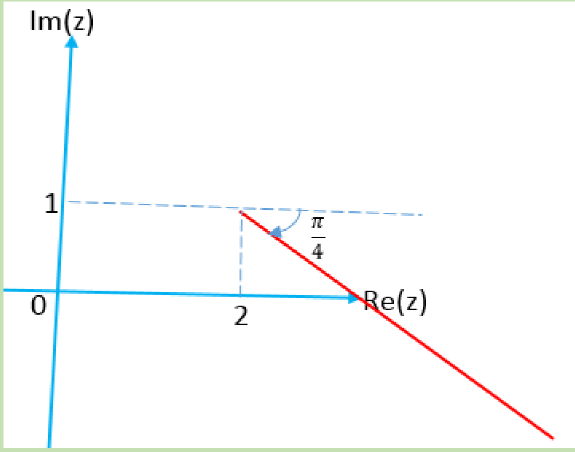
This is a half line having one end at $(-1, 1)$ and making an angle $\frac{2\pi}{3}$ with the (positive) horizontal direction.



(iii) $\text{Arg}(z - 2 - i) = -\frac{\pi}{4}$

Re-write as $\text{Arg}(z - (2 + i)) = -\frac{\pi}{4}$

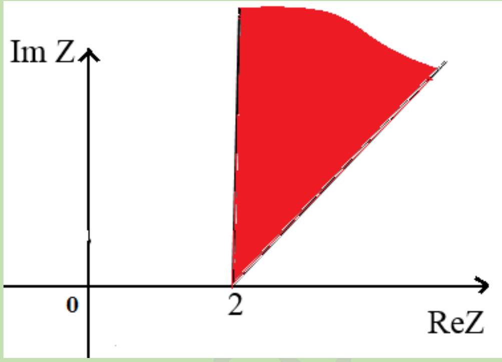
This is a half line having one end at $(2, 1)$ and making an angle $\frac{\pi}{4}$ below the (positive) horizontal direction.



(iv) $\frac{\pi}{3} \leq \arg(z-2) \leq \frac{\pi}{2}$

It is the region bounded by the half lines

$\boldsymbol{\arg(z-2) = \frac{\pi}{3}}$ **and** $\boldsymbol{\arg(z-2) = \frac{\pi}{2}}$



CASE 4: $Arg\left(\frac{z-z_1}{z-z_2}\right) = \theta$

This locus is an arc of a circle in which the chord made by joining points z_1 and z_2 subtends an angle of θ on the arc.

Example

Sketch the locus defined by $arg\left(\frac{z-2-2i}{z-6i}\right) = \frac{\pi}{4}$

Solution

$arg\left(\frac{z-2-2i}{z-6i}\right) = \frac{\pi}{4}$

Re write as $\arg(z-2-2i) - \arg(z-6i) = \frac{\pi}{4}$

Let l_1 be the half line satisfying $\arg(z-2-2i) = \theta$

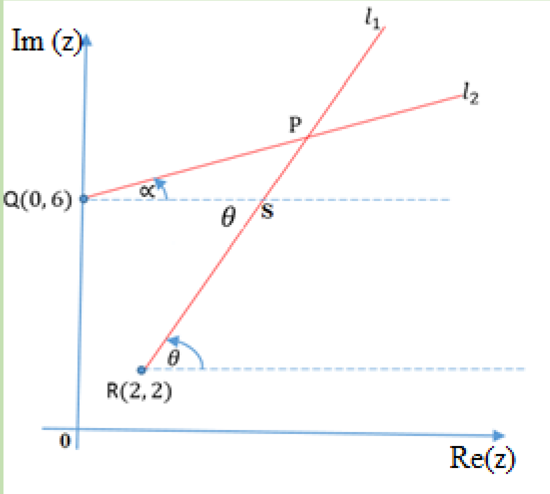
and l_2 be the half line satisfying $\arg(z-6i) = \propto$

This implies that $\theta - \propto = \frac{\pi}{4}$

All points on l_1 satify the locus $\arg(z-2-2i) = \theta$

All points on l_2 satify the locus $\arg(z-6i) = \propto$

Point P is at the intersection of l_1 and l_2

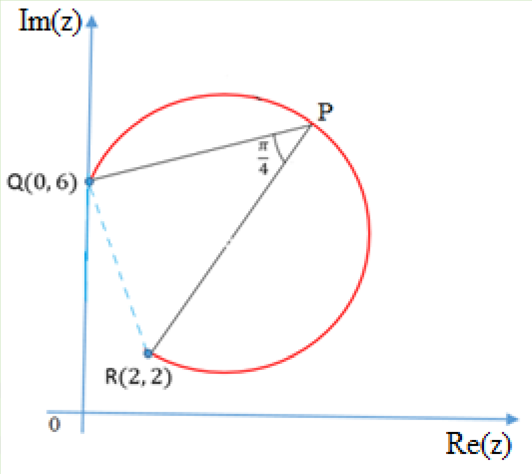


"I tell you, with complex numbers you can do anything" -John Derbyshire

Notice that $Q\hat{S}R = \theta$ is an exterior angle for ΔQPS

Hence $Q\hat{P}S = \theta - \propto = \frac{\pi}{4}$

The locus is an arc of a circle in which the chord made by joining $(2;2)$ and $(0;6)$ subtends an angle of $\frac{\pi}{4}$ on the arc.



7.2.11 Cartesian Equations

Example 1

Using algebraic methods find the Cartesian equations represented by the following loci

(i) $|z-2| = |z+2i|$

(ii) $|Z-3+i| = 5$

(iii) $\text{Arg}(z-2-i) = -\frac{\pi}{4}$

Solution

(i) $|z-2| = |z+2i|$

Let $z = x + iy$

$|x+iy-2| = |x+iy+2i|$

$|(x-2)+i(y+0)| = |(x+0)+i(y+2)|$ Grouping real and imaginary parts

$(x-2)^2 + (y+0)^2 = (x+0)^2 + (y+2)^2$ By property of modulus

$x^2 - 4x + 4 + y^2 = x^2 + y^2 + 4y + 4$

$y = -x$

So the locus is a straight line with equation $y = -x$.

(ii) $|Z - 3 + i| = 5$

Let $z = x + iy$

$|x + iy - 3 + i| = 5$

$|(x-3) + (y+1)i| = 5$ Grouping real and imaginary parts

$(x-3)^2 + (y+1)^2 = 5$ By property of modulus

$$x^2 + y^2 - 6x + 2y + 5 = 0$$

This is a circle centre $(3; -1)$ and radius 5

(iii) $\operatorname{Arg}(z - 2 - i) = -\frac{\pi}{4}$

Let $z = x + iy$

$$\operatorname{Arg}(x + iy - 2 - i) = -\frac{\pi}{4}$$

$$\operatorname{Arg}\big((x - 2) + (y - 1)i\big) = -\frac{\pi}{4}$$

$$\frac{y-1}{x-2} = tan\left(-\frac{\pi}{4}\right)$$

By property of argument

$$\frac{y-1}{x-2} = -1$$

$$y = -x + 3$$

The locus is a straight line with equation $y = -x + 3$

Example 2

Show that the locus represented by $|z - 2 - i| = 3|z + 6 + 3i|$ is a circle and find its centre and radius.

Solution

$$|z - 2 - i| = 3|z + 6 + 3i|$$

"I tell you, with complex numbers you can do anything" -John Derbyshire

Let $z = x + iy$

$|(x-2)+(y-1)i| = 3|(x+6)+(y+3)i|$

$(x-2)^2 + (y-1)^2 = 3^2[(x+6)^2 + (y+3)^2]$

$x^2 + y^2 - 4x - 2y + 5 = 9(x^2 + y^2 + 12x + 6y + 45)$

$x^2 + y^2 - 4x - 2y + 5 = 9x^2 + 9y^2 + 108x + 54y + 405$

$8x^2 + 8y^2 + 112x + 56y + 400 = 0$

$x^2 + y^2 + 14x + 7y + 50 = 0$

$(x+7)^2 + (y+\frac{7}{2})^2 = \frac{45}{4}$ By completing the square

This is a circle centre $\left(-7; -\frac{7}{2}\right)$ and radius $\sqrt{\frac{45}{4}}$

Example 2

Given that the complex number $\frac{z+2}{z-2}$ is completely imaginary, show that the locus representing z in the Argand plane is a circle centre $(0; 0)$ and radius 2.

"I tell you, with complex numbers you can do anything" -John Derbyshire

Solution

$\frac{z+2}{z-2}$

Let $z = x + iy$

$\frac{x+iy+2}{x+iy-2}$

$\frac{(x+2)+iy}{(x-2)+iy}$

$\frac{(x+2)+iy}{(x-2)+iy} \text{ x } \frac{(x-2)-iy}{(x-2)-iy}$ Multiplying numerator and denominator by conjugate of denominator

$\frac{(x^2+y^2-4)-4yi}{(x-2)^2+y^2}$

$\frac{(x^2+y^2-4)}{(x-2)^2+y^2} - \left(\frac{4y}{(x-2)^2+y^2}\right)i$

Now, since the the complex number is completely imaginary it means the real part is zero.

That is

$$\frac{(x^2+y^2-4)}{(x-2)^2+y^2} = 0$$

$$x^2 + y^2 - 4 = 0$$

$$(x-0)^2 + (y-0)^2 = 2^2$$

This is a circle centre $(0;0)$ and radius 2.

"I tell you, with complex numbers you can do anything" -John Derbyshire

Exercise

1 Sketch on an Argand diagram the locus of Z, where

$|z+4| = |z-4i|$ [2]

Hence or otherwise state the Cartesian equation of the locus. [1]

2 (i) Sketch an argand diagram of the locus of z where

$|z-1-i| = |z+2+3i|$

(ii) Hence or otherwise state the Cartesian equation of the locus. [5]

3 Sketch in an argand diagram the set of points representing all complex numbers z satisfying both the inequalities

$|z-3-i| \leq 4$ and $\frac{\pi}{3} \leq \arg(z-4-2i) \leq \frac{\pi}{2}$. [3]

4 Indicate by shading on a single Argand diagram the region in which

both of the following inequalities are satisfied:

$\frac{\pi}{4} < \arg(z) < \frac{\pi}{2}$ $|z-3i| \leq 3$ [3]

5 Illustrate on an Argand diagram the set of points representing the complex number z satisfying both:

$|z-1-2i| \leq 3$ and $\arg(z-2-i) = \frac{3\pi}{4}$ [3]

6 If $\arg(z-2) = \frac{\pi}{3}$, sketch the locus of P$(x, y)$ which is represented by z on an Argand diagram. Find the Cartesian equation of this locus.

7 Describe the locus represented by $|2-5i-z|=3$.

8 On the same axes, draw a diagram showing the locus of z in each of the following

1. $|z|<2$ 2. $\frac{\pi}{6}<\arg(z)<\frac{\pi}{3}$

Shade the region which is common to both loci above. [3]

9 Sketch in an Argand diagram the set of points representing all complex numbers z satisfying both the inequalities

$|z-2i|\leq 2$ and $|z-2i|\leq |z|$ [3]

10 The complex number z is such that

$$0\leq \arg(z+1)\leq \frac{2\pi}{3} \quad \text{and} \quad \frac{\pi}{6}\leq \arg(z+3)\leq \pi$$

(i) Sketch on an Argand diagram the region *R* in which *z* must lie.

(ii) Mark on this diagram the point *A* belonging to *R* at which *z* has its least possible value.

11 (a) Shade in, on separate Argand diagrams the region represented by

(i) $|z-4-2i|\leq 2$, (ii) $|z-4|<|z-6|$,

(iii) $0\leq arg(z-2-2i)\leq \frac{\pi}{4}$.

(b) Hence on one Argand diagram shade in the region which satisfies

$|z-4-2i|\leq 2$, $|z-4|<|z-6|$ and

$0\leq arg(z-2-2i)\leq \frac{\pi}{4}$.

12 If $|z-6|=2|z+6-9i|$,

(a) use algebra to show that the locus of z is a circle, stating its centre and its radius.

(b) sketch the locus of z on an Argand diagram.

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13 Sketch the following locus on an Argand diagram:

(a) $\text{Arg}\left(\frac{z-1}{z-4i}\right) = \frac{\pi}{3}$ [4]

(b) $Arg\left(\frac{z-2-2i}{z-6}\right) = \frac{\pi}{6}$ [4]

(c) $\text{Arg}\left(\frac{z}{z-2}\right) = \frac{\pi}{4}$ [4]

(d) $\text{Arg}\left(\frac{z+2i}{z-2i}\right) = \frac{\pi}{4}$ [4]

(e) $\text{Arg}\left(\frac{z-1}{z-3}\right) = \frac{\pi}{3}$ [4]

(f) $\text{Arg}\left(\frac{z}{z+4i}\right) = \frac{\pi}{6}$ [4]

14 If $\arg(z - 2) = \frac{\pi}{3}$, sketch the locus of P$(x, y)$ which is represented by z on an Argand diagram. Find the Cartesian equation of this locus.

15 Given that $\frac{z-8i}{z-6}$ is completely imaginary and $z = x + iy$, where x and y are real, show that the locus of the point representing z in the Argand diagram is a circle. [3]

Hence by first finding its centre and radius, sketch the circle on an Argand diagram. [4]

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Miscellaneous Exercises 2

1 Describe geometrically and sketch the region on the complex plane for which

(a) $\frac{-\pi}{4} \leq \arg(z-2i) \leq \frac{\pi}{3}$

(b) $arg\left(\frac{z-5+7i}{z+1+i}\right) = \frac{\pi}{2}$

(c) $2 < |z-3+i| \leq 5$

(d) $\arg(z+2) - \arg(z-3) = \frac{\pi}{3}$

(e) $|z-2| + |z-3+i| = 0$

(f) $arg\left(\frac{z-5+7i}{z+1+i}\right) = \frac{\pi}{2}$

(g) $arg(z-3)^2 = \frac{\pi}{2}$

2 (a) Sketch on one Argand diagram:

(i) the locus of points satisfying $|z - i| = |z - 2|$;

(ii) the locus of points satisfying $arg(z - i) = \frac{1}{4}\pi$

(b) Shade on your diagram the region in which

$|z - i| \leq |z - 2|$ and $-\frac{\pi}{2} \leq arg(z - i) \leq \frac{\pi}{4}$

2 Simplify $(1+i)^{10} - (1-i)^{10}$.

Given that n is a positive integer show that $(1+i)^{4n} - (1-i)^{4n} = 0$.

3 Find all values of n such that $\left(-\sqrt{3}+i\right)^n + \left(-\sqrt{3}-i\right)^n = 0$

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4 The complex number z satisfies the equation $|z+1| = \sqrt{2}|z-1|$. The point P represents z on an Argand diagram. Show that the locus of P is a circle and find its centre and radius.

5 Use the method of Mathematical induction to prove that

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ when n is a positive integer.

Deduce that the result is also true when n is a negative integer.

6 (i) Use the de Moivre's theorem to show that

$$\left(\sqrt{3} - i\right)^n = 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$$

(ii) Find the least positive integer m for which $\left(\sqrt{3} - i\right)^m$ is real and positive.

(iii) Given that $\left(\sqrt{3} - i\right)$ is a root of the equation

$$z^9 + 16(1+i)z^3 + a + ib = 0,$$

find the values of the real constants a and b.

7 Find all the solutions for each of the following, giving your answers in both polar and cartesian form.

` (a) $z^4 - 4 = 0$ (b) $z^4 + 8 + 8\sqrt{3}i = 0$ (c) $z^6 = 64$.

8 Solve for z if

(a) $z^4 + 16z^2 - 225 = 0$ (b) $z^4 - 18z^2 - 243 = 0$

9 Given that $x_1 = 1 + 2i$ is a root of the equation $x^4 - 4x^3 - 6x^2 + 20x - 75 = 0$, find the other three roots.

10 Given that $2 + 3i$ is a solution of the equation $2z^4 - 3z^3 + 3z^2 + 77z - 39 = 0$, find the other solutions.

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