Differentiation

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06 APRIL 2020

SYLLABUS (6042) REQUIREMENTS

- differentiate from first principles excluding logarithmic and exponential functions
- differentiate polynomials, rational functions, natural logarithms, exponentials and trigonometrical functions
- differentiate sums, differences, products, quotients and composite functions
- carry out implicit and parametric differentiation
- locate stationary points
- distinguish between maxima, minima and point of inflexion
- solve problems involving differentiation

Differentiation

Definition

It is all about finding the rate of change of one quantity compared to another or simply the process of finding the derivative of a function at any given point

Notations used

$f'(x)$ or y' or $\frac{dy}{dx}$ or $\frac{df}{dx}$ or $\frac{d[f(x)]}{dx}$ or $\frac{d}{dx}\{f(x)\}$

Notes

- The derivative of $f(x)$ is called the first derivative.
- Now, we denote the second derivative of a function to be the derivative of $f'(x)$, denoted by $f''(x)$ or $\frac{d[f'(x)]}{dx}$ or $\frac{d}{dx}\{f'(x)\}$ or $\frac{d^2y}{dx^2}$.
- Thus, to differentiate higher order derivatives we simply find the derivative of the preceding function e.g. $\frac{d^{n+1}y}{dx^{n+1}}$ we simply find the derivative of the n^{th} term $\frac{d}{dx}\left\{\frac{d^n y}{dx^n}\right\}$

Differentiation from 1st principles

Definition

It is the process of finding the derivative function using the definition:

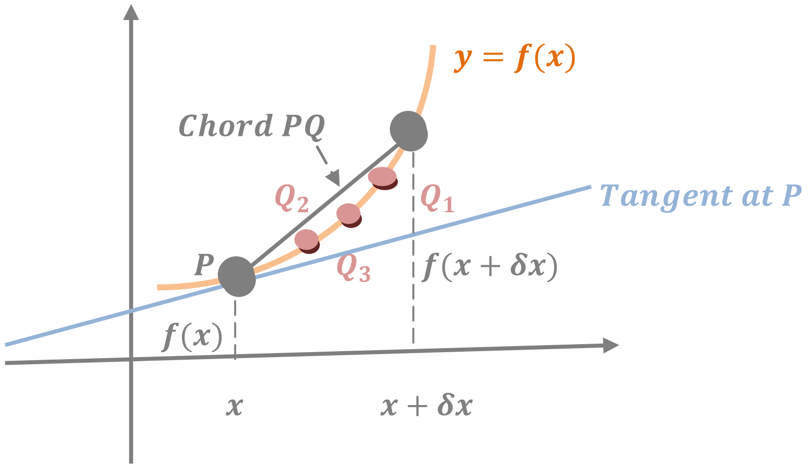
$$f'(x) = lim_{\delta x \to 0}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\} \; ; \; \delta x \neq 0.$$

Notes

- A curve does not have a constant gradient, it changes on every point. At any point on a curve, the gradient is equal to the gradient of the tangent at that point
- A tangent line to a function at a point is the line that best approximates the function at that point better than any other line or a tangent to a curve is a line touching the curve at one point only or a straight line that just touches a curve
- The gradient of a curve $y = f(x)$ at a given point is defined to be the gradient of the tangent at that point or simply the gradient of a curve at a point is the gradient of the tangent line to the curve at that point.
- The gradient of a curve shows the rate at which a quantity changes on a graph.

Geometrical Illustration

- Let the equation of the function be $y = f(x)$.
- Suppose x is the initial point with a slight increase δx, thus substituting x and $x + \delta x$ into $f(x)$ yield $f(x)$ and $f(x + \delta x)$, respectively.
- The expression δx refers to a minute (tiny/very small) increase in x and δ means delta.



Estimating the Gradient of PQ

To find the gradient of the Chord PQ we use the basic formula:

$$gradient = \frac{change\ in\ y}{change\ in\ x}$$

Remember the points P and Q have the following coordinates:

$$\boldsymbol{P[x; f(x)]\text{and } Q[x + \delta x, f(x + \delta x)]}$$

$$\Longrightarrow gradient\ of\ chord\ PQ = \frac{change\ in\ y}{change\ in\ x} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - y_1}}$$

$$= \frac{\boldsymbol{f(x + \delta x) - f(x)}}{\boldsymbol{(x + \delta x) - x}}$$

$$= \frac{\boldsymbol{f(x + \delta x) - f(x)}}{\boldsymbol{\delta x}}$$

Estimating the Gradient of the tangent at P

- Now to find the gradient of P, we try to change the positions of Q, denoted Q_1, Q_2 etc. The gradient at P is closer to the gradient of PQ_3 than the gradient of PQ , PQ_2 and PQ_1 since the chord is shorter.
- The gradient of the chord will become closer to the gradient of the curve at P as we move closer to point P along the curve

- Eventually, when the chord becomes so short that it is almost the tangent, the gradient of the graph will equal the gradient of this tangent.

  Now the gradient of the tangent $\left(\text{the derivative } \frac{dy}{dx}\right)$ is obtained when the chord is at its limit. This is as $\delta x \to 0$ (as delta x tends towards zero)

NB: δx is never equal to zero (we cannot divide by zero).

Gradient of the tangent (from first principles)

⇒ The gradient of the tangent at P is given by:

$$\frac{dy}{dx} = Lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

Example

(a) Differentiate, from first principles, the functions:

(i) $y = x^2$

(ii) $y = x^3$

(iii) $y = x$

(b) Hence, deduce the formula for differentiating $y = x^n$

Solution

a) (i) $f(x) = x^2$

$\Longrightarrow f(x + \delta x) = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2$

Now

$$\frac{dy}{dx} = Lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \left\{ \frac{[x^2 + 2x\delta x + (\delta x)^2] - [x^2]}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \left\{ \frac{2x\delta x + (\delta x)^2}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \left\{ \frac{2x\delta x}{\delta x} + \frac{(\delta x)^2}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \{2x + \delta x\}$$

$$= \ 2x + 0$$

$$\therefore \frac{dy}{dx} \ = \ 2x.$$

a) (ii) $f(x) = x^3$

$$\Longrightarrow f(x + \delta x) = (x + \delta x)^3 = (x + \delta x)(x + \delta x)^2$$

$$= (x + \delta x)[x^2 + 2x\delta x + (\delta x)^2]$$

$$= x^3 + 2x^2\delta x + x(\delta x)^2 + x^2\,\delta x + 2x(\delta x)^2 + (\delta x)^3$$

$$= x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$$

Now

$$\frac{dy}{dx} = Lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \left\{ \frac{[x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3] - [x^3]}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \left\{ \frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \left\{ \frac{3x^2\delta x}{\delta x} + \frac{3x(\delta x)^2}{\delta x} + \frac{(\delta x)^3}{\delta x} \right\}$$

$$= \ Lim_{\delta x \to 0} \{3x^2 + 3x\delta x + (\delta x)^2\}$$

$$= \ 3x^2 + 0 + 0$$

$$\therefore \frac{dy}{dx} = 3x^2.$$

a) (iii) $f(x) = x$

$$\Longrightarrow f(x+\delta x) = x + \delta x$$

Now

$$\frac{dy}{dx} = Lim_{\delta x \to 0}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}$$

$$= Lim_{\delta x \to 0}\left\{\frac{[x+\delta x] - [x]}{\delta x}\right\}$$

$$= Lim_{\delta x \to 0}\left\{\frac{\delta x}{\delta x}\right\}$$

$$= Lim_{\delta x \to 0}\{1\}$$

$$\therefore \frac{dy}{dx} = 1.$$

b) $f(x) = x^1$ $\quad f'(x) = 1 = 1x^0 (= \mathbf{1x^{1-1}})$

$f(x) = x^2$ $\quad f'(x) = 2x = 2x^1 (= \mathbf{2x^{2-1}})$

$f(x) = x^3$ $\quad f'(x) = 3x^2 (= \mathbf{3x^{3-1}})$

Now:

$f(x) = x^n$ $\quad f'(x) = nx^{n-1}$

Follow up exercise

Differentiate, from first principles, the functions:

Question 1

$y = x^2 + 7x$

$Answer: \frac{dy}{dx} = 2x + 7$

Question 2

$y = 1 - 2x^3$

$Answer: \frac{dy}{dx} = -6x^2$

Question 3

$y = 4x - 3$

$Answer: \frac{dy}{dx} = 4$

Question 4

$y = \frac{1}{x}$

$Answer: \frac{dy}{dx} = -\frac{1}{x^2}$

Basic Differentiation

Differentiation of functions of the form $y = x^n$

If $y = a(constant \text{ i.e } ax^0)$ then $\frac{dy}{dx} = (0)ax^{0-1} = 0$

If $y = x^1$(‘a $unit$ $power$’) then $\frac{dy}{dx} = 1x^{1-1} = x^0 = 1$

If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$

NB: When differentiating polynomials we differentiate each term separately.

Example

Differentiate the functions:

(i) $y = x^5$

(ii) $y = \frac{1}{x} - \sqrt{x}$

(iii) $y = x^2 + 3$

Solution

(i) $y = x^5$ then $\frac{dy}{dx} = 5x^{5-1} = 5x^4$

(ii) $y = \frac{1}{x} - \sqrt{x} \Longrightarrow y = x^{-1} - x^{-\frac{1}{2}}$ then $\frac{dy}{dx} = -1x^{-1-1} + \frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{x^2} + \frac{1}{2}x^{-\frac{3}{2}}$

(iii) $y = x^2 + 3$ then $\frac{dy}{dx} = 2x^{2-1} + 0 = 2x.$

Differentiation of functions of the form $y = ax^n$ and $(ax+b)^n$

If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$

If $y = (ax+b)^1$ {'a unit power'} then $\frac{dy}{dx} = 1 \times a(ax+b)^{1-1} = a(ax+b)^0 = a$

If $y = (ax+b)^n$ then $\frac{dy}{dx} = na(ax+b)^{n-1}$

NB: When differentiating polynomials we differentiate each term separately.

Example

Differentiate the functions:

(i) $y = \frac{3}{2}x^4$

(ii) $y = \frac{1}{\sqrt{2x+4}}$

(iii) $y = 6x^2 - 11x$

Solution

(i) $y = \frac{3}{2}x^4$ then $\frac{dy}{dx} = \frac{3}{2} \times 4 \times x^{4-1} = 6x^3$

(ii) $y = \frac{1}{\sqrt{2x+4}} \Longrightarrow y = \frac{1}{(2x+4)^{\frac{1}{2}}} \Longrightarrow y = (2x+4)^{-\frac{1}{2}}$

Now $\frac{dy}{dx} = -\frac{1}{2} \times 2 \times (2x+4)^{-\frac{1}{2}-1} = -(2x+4)^{-\frac{3}{2}}$

(iii) $y = 6x^2 - 11x$ then $\frac{dy}{dx} = 2 \times 6 \times x^{2-1} - 11 = 12x - 11.$

The Chain Rule

If y is a function of v, and v is a function of x, then y is also a function of x.
Thus:

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx}$$

Example

Differentiate the functions:

$y = \sqrt{x^3 + 2x}$

Solution

$y = \sqrt{x^3 + 2x}$

Let $v = x^3 + 2x \Longrightarrow \frac{dv}{dx} = 3x^2 + 2$

$\Longrightarrow\ y = \sqrt{v} = v^{\frac{1}{2}}$

Now $\frac{dy}{dv} = \frac{1}{2}v^{\frac{1}{2}-1} = \frac{1}{2}v^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx} = (3x^2 + 2) \times \left[\frac{1}{2}v^{-\frac{1}{2}}\right]$$
$$= \left[\frac{(3x^2 + 2)}{2}v^{-\frac{1}{2}}\right]$$
$$= \frac{(3x^2 + 2)}{2v^{\frac{1}{2}}}$$
$$= \frac{(3x^2 + 2)}{2\sqrt{v}}$$

But $v = x^3 + 2x$

$$\therefore \frac{dy}{dx} = \frac{(3x^2 + 2)}{2\sqrt{x^3 + 2x}}$$

Follow up exercise

Differentiate the functions:

Question 1

$y = (x^2 + 1)^7$

$Answer: \frac{dy}{dx} = 14x(x^2 + 1)^6$

Question 2

$y = (3x^2 - 4x + 5)^8$

$Answer: \frac{dy}{dx} = (48x - 32)(3x^2 - 4x + 5)^7$

Question 3

$y = \sqrt{x^2 + 1}$

$Answer: \frac{dy}{dx} = x(x^2 + 1)^{-\frac{1}{2}}$

Question 4

$y = (4x + x^{-5})^{\frac{1}{3}}$

$Answer: \frac{dy}{dx} = \frac{4x^6 - 5}{3x^{\frac{8}{3}}(4x^6 + 1)^{\frac{2}{3}}}$

Question 5

$y = (3x + 1)^2$

$Answer: \frac{dy}{dx} = 6(3x + 1)$

Exponential Functions and Natural Logarithms

Differentiation of functions of the form $y = e^{f(x)}$ and $In[f(x)]$

If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x) \times e^{f(x)}$

If $y = In[f(x)]$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

NB: When differentiating polynomials we differentiate each term separately.

Example

Differentiate the functions:

(i) $y = 2e^{3x-1}$

(ii) $y = \frac{7}{e^{2-5x}}$

(iii) $y = In(x^2 - 3x)$

(iv) $y = e^{2In(x-1)}$

Solution

(i) $y = 2e^{3x-1}$ then $\frac{dy}{dx} = 2 \times 3 \times e^{3x-1} = 6e^{3x-1}$

(ii) $y = \frac{7}{e^{2-5x}} \Rightarrow y = 7(e)^{-(2-5x)} \Rightarrow y = 7e^{5x-2}$ then $\frac{dy}{dx} = 7 \times 5 \times e^{5x-2} = 35e^{5x-2}$

(iii) $y = In(x^2 - 3x)$ then $\frac{dy}{dx} = \frac{2x-3}{x^2-3x}$.

(iv) $y = e^{2In(x-1)} \Rightarrow y = e^{In(x-1)^2} \Rightarrow y = (x-1)^2$

Now $\frac{dy}{dx} = 2(x-1)^{2-1} = 2(x-1) = 2x - 2$

Follow up exercise

Differentiate the functions:

Question 1

$y = In(2x^3)$

$Answer: \frac{dy}{dx} = \frac{3}{x}$

Question 2

$y = In(x^3 e^x)$

$Answer: \frac{dy}{dx} = 1 + \frac{3}{x}$

Question 3

$y = e^{x^7}$

$Answer: \frac{dy}{dx} = 7x^6 e^{x^7}$

Question 4

$In(e^x + x^3)$

$Answer: \frac{dy}{dx} = \frac{e^x + 3x^2}{e^x + x^3}$

Question 5

$y = In\left(\frac{x^2 + 1}{x^3 - x}\right)$

$Answer: \frac{dy}{dx} = \frac{2x}{x^2 + 1} - \frac{x^2 - 1}{x^3 - x}$

Question 6

$y = In(11x^7)$

$Answer: \frac{dy}{dx} = \frac{7}{x}$

Question 7

$y = e^{x^2+x^3}$

$Answer: \frac{dy}{dx} = (2x + 3x^2)e^{x^2+x^3}$

Question 8

$y = In(3x^2 + 5)$

$Answer: \frac{dy}{dx} = \frac{6x}{3x^2 + 5}$

Question 9

$y = In\left(\frac{6}{x^2}\right)$

$Answer: \frac{dy}{dx} = -\frac{2}{x}$

Question 10

$y = e^{2In\left(\frac{6}{x^2}\right)}$

$Answer: \frac{dy}{dx} = -\frac{144}{x^5}$

Trigonometric Functions

Differentiation of Trigonometrical functions

A thorough understanding of trigonometric functions is needed in order to master this concept properly. $\frac{dy}{dx}$

(i) If $y = sinx$, then $\frac{dy}{dx} = cosx$

(ii) If $y = cosx$, then $\frac{dy}{dx} = -sinx$

(iii)If $y = tanx$, then $\frac{dy}{dx} = sec^2x$

(iv)If $y = secx$, then $\frac{dy}{dx} = secx\ tanxx$

(v) If $y = cosecx$, then $\frac{dy}{dx} = -cosecx\ cotx$

(vi)If $y = cotx$, then $\frac{dy}{dx} = -cosec^2x$

- Learners need to commit to memory the derivatives of sin, cos and tan and use the rules of calculus to find the derivatives of the reciprocal functions $sec, cosec$ and cot. The derivatives of $sec, cosec$ and cot need not be committed to memory.
- Derivatives for (iii), (iv), (v) and (vi) are derived using the Quotient rule as illustrated below under Quotient rule section below.
- We can use the product, quotient and chain rules to differentiate functions that are combinations of the trigonometric functions.
- We can use the Power × Trigonometric function × Algebra (PTA) method or the chain rules to differentiate functions that are combinations of the trigonometric functions.

Example

Differentiate the functions:

(i) $y = \sin^3(4x - 1)$

(ii) $y = \cos(x^2 + 2x)$

(iii) $y = \tan^5(x^2)$

Solution

(i) $y = \sin^3(4x - 1)$

Method 1

Using the PTA method:

$$\frac{dy}{dx} = 3[\sin^{3-1}(4x-1)] \times \frac{d}{dx}[\sin(4x-1)] \times \frac{dy}{dx}(4x-1)$$

$$= 3[\sin^2(4x-1)] \times \cos(4x-1) \times 4$$

$$= 12\sin^2(4x-1)\cos(4x-1)$$

Method 2

Using the Chain rule method:

$y = \sin^3(4x - 1)$

Let $u = 4x - 1 \Rightarrow \frac{du}{dx} = 4$

$\Rightarrow y = \sin^3 u = (\sin u)^3$

Let $v = \sin u \Rightarrow \frac{dv}{du} = \cos u$

$\Rightarrow y = v^3$

$\frac{dy}{dv} = 3v^2$

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx} = (3v^2) \times \cos u \times 4$$

$$= 12v^2\cos u$$

But $v = \sin u$

$$\frac{dy}{dx} = 12\sin^2 u\cos u$$

But $u = 4x - 1$

$$\frac{dy}{dx} = 12sin^2(4x-1)\cos(4x-1)$$

(ii) $y = \cos(x^2 + 2x)$

Method 1

Using the PTA method:

$$\frac{dy}{dx} = [\cos(x^2+2x)] \times \frac{dy}{dx}(x^2+2x)$$

$$= -\sin(x^2+2x) \times (2x+2)$$

$$= -(2x+2)\sin(x^2+2x)$$

Method 2

Using the Chain rule method:

$y = \cos(x^2 + 2x)$

Let $u = x^2 + 2x \Rightarrow \frac{du}{dx} = 2x + 2$

$\Rightarrow\ y = \cos u$

$$\frac{dy}{du} = -\sin u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (-\sin u) \times (2x+2)$$

$$= (2x+2)\sin u$$

But $v = x^2 + 2x$

$$\therefore \frac{dy}{dx} = -(2x+2)\sin(x^2+2x)$$

(iii)$y = tan^5(x^2)$

Method 1

Using the PTA method:

$$\frac{dy}{dx} = 5[tan^{5-1}(x^2)] \times \frac{d}{dx}[tan\,(x^2)] \times \frac{dy}{dx}(x^2)$$

$$= 5[tan^4(x^2)] \times sec^2(x^2) \times 2x$$

$$= 10x\, tan^4(x^2)\, sec^2(x^2)$$

Method 2

Using the Chain rule method:

$y = \tan^5 (x^2)$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow\; y = tan^5\, u = (tanu)^5$

Let $v = \tan u \Rightarrow \frac{dv}{du} = \sec^2 u$

$\Rightarrow\; y = \mathrm{v}^5$

$\frac{dy}{dv} = 5v^4$

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx} = (5v^4) \times \sec^2 u \times 2x$$

$$= 10xv^4 \sec^2 u$$

But $v = \tan u$

$$\frac{dy}{dx} = 10xtan^4 u \cos u$$

But $u = x^2$

$$\frac{dy}{dx} = 10xtan^4(x^2)\cos(x^2)$$

Follow up exercise

Differentiate the functions using both methods:

Question 1

$y = sin^5\,(6x^2 + 4x - 1)$

$\frac{dy}{dx} = (60x + 20)[\sin^4(6x^2 + 4x - 1)\cos(6x^2 + 4x - 1)]$

Question 2

$y = \cos(7x - 8)$

$\frac{dy}{dx} = -7\sin(7x - 8)$

Question 3

$y = tan^3\,(x - 9)$

$\frac{dy}{dx} = 3\tan^2(x - 9)\sec^2(x - 9)$

Question 4

$y = \sin\,(x^2)$

$\frac{dy}{dx} = 2x\cos(x^2)$

Question 5

$y = cos^4\,(8 - 13x)$

$\frac{dy}{dx} = 52\cos^3(8 - 13x)\sin(8 - 13x)$

Question 6

$y = tan\,(3x^3 + 1)$

$\frac{dy}{dx} = 9x^2\sec^2(3x^3 + 1)$

The Product Rule

- The Product Rule must be utilized when the derivative of the product of two functions is to be taken.
- If u and v are two functions of x then the derivative of the product uv is given by:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The Product Rule in Words

The Product Rule says that the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

NB: The product rule does NOT say that the derivative of a product is the product of the derivatives.

Example 1

Find the derivative of $y = e^{2x}\tan x$

Solution

$y = e^{2x}\tan x$

Let $u = e^{2x}$ then $\frac{du}{dx} = 2e^{2x}$ and $v = \tan x$ then $\frac{dv}{dx} = sec^2 x$

Now

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$= (e^{2x})(sec^2x) + \tan x(2e^{2x})$

$= e^{2x}sec^2x + 2e^{2x}\tan x$

$= e^{2x}(sec^2x + 2\tan x)$

Example 2

Find the derivative of $y = x^3In(2x + 1)$

Solution

$y = x^3In(2x + 1)$

Let $u = x^3$ then $\frac{du}{dx} = 3x^2$ and $v = In(2x + 1)$ then $\frac{dv}{dx} = \frac{2}{2x+1}$

Now

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x^3)\left(\frac{2}{2x + 1}\right) + In(2x + 1)(3x^2)$$

$$= \frac{2x^3}{2x + 1} + 3x^2In(2x + 1)$$

$$= \frac{2x^3 + 3x^2(2x + 1)In(2x + 1)}{2x + 1}$$

$$= \frac{2x^3 + (6x^3 + 3x^2)In(2x + 1)}{2x + 1}$$

Follow up exercise

Question 1

Find $\frac{dy}{dx}$ if $y = (x - 2)sin2x$.

Answer: $\frac{dy}{dx} = 2cos2x(x - 2) + sin2x$

Question 2

Find $\frac{dy}{dx}$ if $y = (3x - 2x^2)(5 + 4x)$.

Answer: $\frac{dy}{dx} = -24x^2 + 4x + 15$

Question 3

Find $\frac{dy}{dx}$ if $y = 2x(x^2 + 3x)$.

Answer: $\frac{dy}{dx} = 6x^2 + 12x$

Question 4

Find $\frac{dy}{dx}$ if $y = 3(x^4 - 1)(5x^2 + 2x)$.

Answer: $\frac{dy}{dx} = 3(30x^5 + 10x^4 - 10x - 2)$

The Quotient Rule

- A quotient is just a fraction
- The Quotient Rule must be utilized when the derivative of the quotient of two functions is to be taken.
- If u and v are two functions of x then the derivative of the product uv is given by:

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The Quotient Rule in Words

The Quotient Rule says that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

NB (i): It doesn't matter if you reverse the terms in the product rule, but it

does matter in the quotient rule.

NB (ii): $y = \frac{u}{v}$ can be written as $y = uv^{-1}$ and apply the product rule.

Example 1

Show that if $y = cosecx$, then $\frac{dy}{dx} = -cosecx\ cotx$

Solution

Remember the '**C**' '**S**' relationship: $\frac{1}{Sinx} = \boldsymbol{Cosecx}$ and $\frac{1}{Cosx} = \boldsymbol{Secx}$.

Thus:

$$Cosecx = \frac{1}{Sinx}$$

Let $u = 1$ then $\frac{du}{dx} = 0$ and $v = sinx$ then $\frac{dv}{dx} = cosx$

Now

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(sinx)(0) - (1)(\cos x)}{sin^2x} = \frac{-\cos x}{sin^2x}$$

$$= \frac{-\cos x}{sinx}\left(\frac{1}{sinx}\right) = -cosecx\ cotx \quad (as\ required)$$

Example 2

Find the derivative of the function $y = \frac{3x^2}{Sinx}$

Solution

$$y = \frac{3x^2}{Sinx}$$

Let $u = 3x^2$ then $\frac{du}{dx} = 6x$ and $v = sinx$ then $\frac{dv}{dx} = cosx$

Now

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(sinx)(6x) - (3x^2)(\cos x)}{sin^2x}$$

$$= \frac{6x\text{sin}x - 3x^2\cos x}{sin^2x}$$

Example 3

Find the gradient at the point where the curve $y = \frac{1+x^2}{1+e^{2x}}$ crosses the $y - axis$

Solution

$$y = \frac{1 + x^2}{1 + e^{2x}}$$

$$\frac{dy}{dx} = \frac{(1 + e^{2x})(2x) - (1 + x^2)(2e^{2x})}{(1 + e^{2x})^2}$$

The curve crosses the $y - axis$ when $x = 0$

Now

$$\frac{dy}{dx} = \frac{\left(1 + e^{2(0)}\right)(2 \times 0) - (1 + \{0\}^2)\left(2e^{2\{0\}}\right)}{(1 + e^{2\{0\}})^2}$$

$$\frac{dy}{dx} = \frac{(1 + 1)(0) - (1 + 0)(2)}{(1 + 1)^2}$$

$$\frac{dy}{dx} = -\frac{2}{4}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}$$

Follow up exercise

Question 1

Show that if $y = tanx$, then $\frac{dy}{dx} = sec^2x$

Question 2

Show that if $y = secx$, then $\frac{dy}{dx} = secx\ tanxx$

Question 3

Show that if $y = cotx$, then $\frac{dy}{dx} = -cosec^2x$

Question 4

Differentiate the following function:

$$f(x) = \frac{Cos(2x^3)}{3x}$$

Suggested Solution

$$y = \frac{Cos(2x^3)}{3x}$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{3x[-Sin(2x^3) \times 6x^2] - Cos(2x^3)[3]}{(3x)^2}$$

$$= \frac{-18x^3Sin(2x^3) - 3Cos(2x^3)}{9x^2}$$

$$= \frac{-6x^3Sin(2x^3) - Cos(2x^3)}{3x^2}$$

Implicit Differentiation

Implicit differentiation is a method for finding the slope of a curve, when the equation of the curve is not given in “explicit" form $y = f(x)$, but in \implicit" form by an equation $g(x;\ y) = 0$.

Example 1

Find $\frac{dy}{dx}$ by implicit differentiation given that

$$x^2 + y^2 = 25.$$

Solution

$x^2 + y^2 = 25$

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{2y}$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$

Example 2

Find $\frac{dy}{dx}$ by implicit differentiation given that

$$x\,y^2 = \sin(y^3)$$

Solution

$x\,y^2 \;= \sin(y^3)$

$\frac{d}{dx}(x\,y^2) \;=\; \frac{d}{dx}[\sin(y^3)]$

$x\frac{d}{dx}(y^2\,) + y^2\frac{d}{dx}(\,x) \;=\; 3y^2cos(y^3)\frac{dy}{dx}$

$2xy\frac{dy}{dx} + y^2 \;=\; 3y^2cos(y^3)\frac{dy}{dx}$

$2xy\frac{dy}{dx} - 3y^2cos(y^3)\frac{dy}{dx} = -y^2$

$[2xy - 3y^2cos(y^3)]\frac{dy}{dx} = -y^2$

$\frac{dy}{dx} = \frac{-y^2}{2xy - 3y^2cos(y^3)}$

$\therefore \frac{dy}{dx} = \frac{y^2}{3y^2cos(y^3) - 2xy}$

Example 3

Find the equation of the tangent line to the curve

$$x^2y - y^3 \;=\; x - \;7 \text{ at } (1;\;2).$$

Solution

$x^2y - y^3 \;=\; x - \;7$

$\frac{d}{dx}(x^2y - y^3) \;=\; \frac{d}{dx}(x - \;7)$

$\frac{d}{dx}(x^2y\,) - \frac{d}{dx}(\,y^3) \;=\; 1$

$x^2\frac{d}{dx}(y) + y\,\frac{d}{dx}(x^2) - 3y^2\frac{dy}{dx} = 1$

$$x^2\frac{dy}{dx} + 2xy - 3y^2\frac{dy}{dx} = 1$$

$$(x^2 - 3y^2)\frac{dy}{dx} = 1 - 2xy$$

$$\therefore \frac{dy}{dx} = \frac{1 - 2xy}{(x^2 - 3y^2)}$$

At $(1; 2)$

$$\frac{dy}{dx} = \frac{1 - 2(1)(2)}{(1)^2 - 3(2)^2}$$

$$\frac{dy}{dx} = \frac{1 - 4}{1 - 12}$$

$$\therefore \frac{dy}{dx} = \frac{-3}{-11}$$

$$\Longrightarrow \frac{y - 2}{x - 1} = \frac{3}{11}$$

$$\Longrightarrow 11(y - 2) = 3(x - 1)$$

$$\Longrightarrow 11y - 22 = 3x - 3$$

$\therefore$ the equation is $3x - 11y + 19 = 0$

Example 4

Differentiate the following functions:

$$x^3 + 2y^3 = 3xy$$

Suggested Solution

$$3x^2 + 6y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$\Rightarrow 3x^2 - 3y = 3x\frac{dy}{dx} - 6y^2\frac{dy}{dx}$$

$$\Rightarrow x^2 - y = (x - 2y^2)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y}{x - 2y^2}$$

Example 4

Find the gradient at the point where the curve $2x^3 + 5xy + y^3 = 8$ crosses the $y - axis$

Solution

$$2x^3 + 5xy + y^3 = 8$$

$$6x^2 + 5x\frac{dy}{dx} + 5y + 3y^2\frac{dy}{dx} = 0$$

$$(5x + 3y^2)\frac{dy}{dx} = -6x^2 - 5y$$

$$\frac{dy}{dx} = -\frac{6x^2 + 5y}{5x + 3y^2}$$

The curve crosses the $y - axis$ when $x = 0$

i.e $2(0)^3 + 5(0)y + y^3 = 8$

$$\Rightarrow y^3 = 8$$

$$\therefore y = 2$$

Now

$$\frac{dy}{dx} = -\frac{6(0)^2 + 5(2)}{5(0) + 3(2)^2}$$

$$\frac{dy}{dx} = -\frac{10}{12}$$

$\therefore \frac{dy}{dx} = -\frac{5}{6}$

Horizontal and Vertical Tangents

(i) When the tangent is horizontal we equate $\frac{dy}{dx}$ to 0 i.e. $\frac{dy}{dx} = 0$

(ii) When the tangent is Vertical we equate the denominator of $\frac{dy}{dx}$ to 0 i.e. $dx = 0$

Example

A curve has equation $4x^2 + 8x + 9y^2 - 36y + 4 = 0$.

(i) Find $\frac{dy}{dx}$.

(ii) Write down the equation(s) of the tangent(s) of the curve that are parallel to:

(a) x −axis

(b) $y - a$xis

Solution

(i) $4x^2 + 8x + 9y^2 - 36y + 4 = 0$

$$\frac{d}{dx}(4x^2 + 8x + 9y^2 - 36y + 4) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(4x^2) + \frac{d}{dx}(8x) + \frac{d}{dx}(9y^2) - \frac{d}{dx}(36y) + \frac{d}{dx}(4) = 0$$

$$8x + 8 + 18y\frac{dy}{dx} - 36\frac{dy}{dx} + 0 = 0$$

$$8x + 8 = 36\frac{dy}{dx} - 18y\frac{dy}{dx}$$

$$8x + 8 = (36 - 18y)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{8x + 8}{36 - 18y}$$

(ii) Finding the equation(s) of the tangent(s) of the curve that are parallel to:

(a) x −axis

Along the x-axis $\frac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{8x+8}{36-18y} = 0$

$\Rightarrow 8x + 8 = 0 \;\therefore x = -1$

Now finding the value(s) of y:

$4x^2 + 8x + 9y^2 - 36y + 4 = 0$

$4(-1)^2 + 8(-1) + 9y^2 - 36y + 4 = 0$

$4 - 8 + 9y^2 - 36y + 4 = 0$

$9y^2 - 36y = 0$

$9y(y-4) = 0$

$\therefore y = 0$ or 4

Now the points are $(-1; 0)$ and $(-1; 4)$

The equations are $y = 0$ and $y = 4$.

(b) $y - a$xis

Along the x-axis $dx = 0$

$\Rightarrow dx = 36 - 18y = 0$

$\Rightarrow 36 - 18y = 0 \therefore y = 2$

Now finding the value(s) of x:

$4x^2 + 8x + 9y^2 - 36y + 4 = 0$

$4x^2 + 8x + 9(2)^2 - 36(2) + 4 = 0$

$4x^2 + 8x - 32 = 0$

$x^2 + 2x - 8 = 0$

$x^2 + 4x - 2x - 8 = 0$

$x(x+4) - 2(x+4) = 0$

$(x+4)(x-2) = 0$

$\therefore x = -4 \; or \; 2)$

Now the points are $(-4; 2) \;\; and \; (2; 2)$

The equations are $x = -4$ or $x = 2$.

Follow up exercise

Question 1

Differentiate the functions:

(i) $x^3 + y^3 = 4$

$Answer: \frac{dy}{dx} = -\frac{x^2}{y^2}$

(ii) $y - Iny = 10x^3 - 6x^2 + 4$

$Answer: \frac{dy}{dx} = -\frac{30x^2y - 12xy}{y - 1}$

(iii) $tany = cosx$

$Answer: \frac{dy}{dx} = -sinxcos^2y$

(iv) $y - \sqrt{y} = Inx$

$Answer: \frac{dy}{dx} = \frac{2\sqrt{y}}{2x\sqrt{y} - x}$

(v) $\cos(xy) + x^2 = -7y$

$Answer: \frac{dy}{dx} = \frac{ysin(xy) - 2x}{7 - xsin(xy)}$

Question 2

If $x^2 + y^3 - 3y = 4$, find $\frac{dy}{dx}$. Find the coordinates on the curve for which the tangent is

(i) parallel to $x - a$xis
(ii) parallel to $y - a$xis

$Answer: \frac{dy}{dx} = \frac{-2x}{3y^2 - 3}$

(i)$(0; 2.1958233)$

(ii)$(1; \sqrt{6}), (1; -\sqrt{6}), (-1; -\sqrt{2})$& $(-1; \sqrt{2})$.

Question 3

Find the points on the curve $x^2 - axy + y^2 = a$.

(i) Find $\frac{dy}{dx}$.

(ii) Write down the equation(s) of the tangent(s) of the curve that are parallel to $y - a$xis

Answer: (i) $\frac{dy}{dx} = \frac{ay-2x}{2y-ax}$

(ii) the equation of the tangent is: $x = \pm\sqrt{\frac{4a}{4-a^2}}$.

Past Examination Questions

ZIMSEC November 2004 Paper 1

A curve is given by the equation $3x^2 - 7xy + 4y^2 = 16$.

(i) Show that $\frac{dy}{dx} = \frac{6x-7y}{7x-8y}$. **[3]**

(ii) Hence show that the gradient of the curve cannot be equal to 1. **[3]**

ZIMSEC 2017 June Paper 2

Find the equation of the normal to the curve $x^2 - 3xy + y^2 = 5$, at a point $(1; 4)$. **[5]**

Answer: $\frac{dy}{dx} = \frac{3y-2x}{2y-3x}$ and the equation of the normal is: $x + 2y = 9$.

ZIMSEC November 2018 Paper 1

A curve has equation $y^2 = 3xy + x^2 - 3$.

Find the equation of the tangent to the curve at the point where $y = 1$ and $x > 0$. **[8]**

Answer: $\frac{dy}{dx} = \frac{3y+2x}{2y-3x}$ and the equation of the tangent is: $y = -5x + 6$.

ZIMSEC November 2015 Paper 1

Given that $Iny = xy$, where $y > 0$'

(i) Show that $\frac{dy}{dx} = \frac{y^2}{1-xy}$

(ii) find $\frac{d^2y}{dx^2}$. **[5]**

Answer: $\frac{d^2y}{dx^2} = \frac{2y\frac{dy}{dx} - xy^2\frac{dy}{dx} + y^3}{(1-xy)^2}$

Parametric Differentiation

- Parametric equations are two equations, in which one relates x with the third variable/quantity, say t, and one relates y with the same third variable/quantity, say t. This third variable/quantity is called a parameter. So in this case, the function $f(x)$, is not defined explicitly in terms of the independent variable x but it both x and y are defined explicitly in terms of the parameter.
- To differentiate parametric equations, we must use the chain rule.
- If $x = f(t)$ and $y = g(t)$ are two functions of t then the derivative is given by:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Example 1

Show that the equation of the tangent line to the semicircle with parametric equations

$$x = cos\, t \qquad y = sin\, t \qquad 0 \leq t \leq \pi$$

at $t = \frac{\pi}{4}$ is $y = -x + \sqrt{2}$.

Solution

$x = \cos t\,;\ y = \sin t$

Now $x = \cos t$ then $\frac{dx}{dt} = -\sin t$ and $y = \sin t$ then $\frac{dy}{dt} = \cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$$

At $\cos t = \frac{\pi}{4}$

$$\frac{dy}{dx} = -\frac{\cos t}{\sin t} = -\frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = -\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$$

Now $x = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$; $y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Thus the equation is given by:

$$\frac{y - \frac{\sqrt{2}}{2}}{x - \frac{\sqrt{2}}{2}} = -1$$

$$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$

$$y - \frac{\sqrt{2}}{2} = -x + \frac{\sqrt{2}}{2}$$

$$y = -x + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$\therefore y = -x + \sqrt{2}\ (as\ required).$

Example 2

If $x = 2a\,t^2$ and $y = 4a\,t$ find $\frac{dy}{dx}$.

Solution

$x = 2a\,t^2$; $y = 4a\,t$

Now $x = 2a\,t^2$ then $\frac{dx}{dt} = 4at$ and $y = 4a\,t$ then $\frac{dy}{dt} = 4\text{a}$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{4\text{a}}{4a\,t} = \frac{1}{t}$$

Example 3

Find the derivative of the parametric function $x = t^2 + t$ and $y = 2t - 1$.

Hence find the equation of the normal to the curve when $t = 1$. Express your answer in the form $ax + by + c = 0$.

Solution

$x = t^2 + t$; $y = 2t - 1$

Now $x = t^2 + t$ then $\frac{dx}{dt} = 2t + 1$and $y = 2t - 1$ then $\frac{dy}{dt} = 2$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{2}{2t+1}$$

At cos $t = 1$

$$\frac{dy}{dx} = \frac{2}{2t+1} = \frac{2}{2(1)+1} = \frac{2}{3}$$

Normal gradient $= -1 \div \frac{2}{3} = -\frac{3}{2}$

Now $x = (1)^2 + 1 = 2$; $y = 2(1) - 1 = 1$

Thus the equation is given by:

$$\frac{y-1}{x-2} = -\frac{3}{2}$$

$2y - 2 = -3x + 6$

$\therefore 3x + 2y - 8 = 0$

Example 4

Cambridge Question

The parametric equations of a curve are:$x = 1 + tan\theta$ and $y = sec\theta$, for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

a) Show that $\frac{dy}{dx} = sin\theta$.

b) Find the coordinates of the point on the curve at which the gradient of the curve is $\frac{1}{2}$. **[3]**

Answer

(a) $x = 1 + tan\theta$

$$\frac{dx}{d\theta} = sec^2\theta$$

$$= \frac{1}{cos^2\theta}$$

$y = sec\theta$

$$= \frac{1}{cos\theta}$$

$$\frac{dy}{d\theta} = \frac{cos\theta(0) - 1(-sin\theta)}{cos^2\theta}$$

$$= \frac{sin\theta}{cos^2\theta}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{sin\theta}{cos^2\theta} \div \frac{1}{cos^2\theta}$$

$$= \frac{sin\theta}{cos^2\theta} \times \frac{cos^2\theta}{1}$$

$= sin\theta$ (as required)

(b) $\frac{dy}{dx} = \frac{1}{2}$

$\Rightarrow sin\theta = \frac{1}{2}$

$\Rightarrow \theta = sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \theta = 30°$

Now:

$x = 1 + tan30°$

$$= 1 + \frac{\sqrt{3}}{3}$$

$$= \frac{\sqrt{3} + 3}{3}$$

$$y = sec\theta$$

$$= \frac{1}{cos30°}$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{2}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}$$

∴ The coordinates are $\left(\frac{\sqrt{3}+3}{3}; \frac{2\sqrt{3}}{3}\right)$

Follow up exercise

Question 1

For the sets of parametric equations find $\frac{dy}{dx}$:

(i) $x = 3t^2;\ y = 4t^3$

Answer: $\frac{dy}{dx} = 2t$

(ii) $x = 4 - t^2; y = t^2 + 4t$

Answer: $\frac{dy}{dx} = 1 - \frac{2}{t}$

(iii) $x = t^2 e^t;\ \ y = t$

Answer: $\frac{dy}{dx} = \frac{e^{-t}}{2t + t^2}$

Question 2

Find the equation of the tangent line to the curve

$$x = 1 + 3\sin t;\ y = 2 - 5\cos t \text{ at } t = \frac{\pi}{6}.$$

Answer: $\frac{dy}{dx} = \frac{5}{3}\tan t\ ,\ y = \frac{5\sqrt{3}}{9}x + 2 - \frac{35\sqrt{3}}{9}$

Past Examination Questions

ZIMSEC November 2015 Paper 1

The parametric equations of a circle are

$$x = cosec\Phi$$

$$y = cot\Phi, \text{where } 0 < \Phi < 2\pi.$$

(i) Show that $\frac{dy}{dx} = sec\Phi$. **[3]**

(ii) Find the equation of the normal to the curve at the point where $\Phi = \frac{\pi}{6}$ in the form $y = mx + c$. **[4]**

Answer: The equation of the tangent is: $y = -\frac{\sqrt{3}}{2}x + 2\sqrt{3}$.

ZIMSEC June 2015 Paper 1

A curve has parametric equations

$$y = 1 + \cos\left(\frac{\pi}{3}e^{3\theta}\right)$$

$$x = 2 - sin\left(\frac{\pi}{3}e^{3\theta}\right), \text{for } 0 < \theta < 2\pi.$$

Find

(i) $\frac{dy}{dx}$ in terms of θ in its simplest form and state the exact value of $\frac{dy}{dx}$ when $\theta = 0$, **[3]**

(ii) the Cartesian equation of the curve and describe fully what it represents geometrically. **[4]**

Answer: (i) $\frac{dy}{dx} = \tan\left(\frac{\pi}{3}e^{3\theta}\right)$; $\sqrt{3}$ and the cartesian equation is: $(x-2)^2 + (y-2)^2 = 1$;

(ii) It is an equation of a circle with radius 1 and centre $(2,1)$.

ZIMSEC June 2013 Paper 1

The curve C has parametric equations $y = at, x = \frac{a}{t}$, where $t > 0$.

(i) Write down the cartesian equation of C in terms a. **[2]**

(ii) Given that point P lies on C, show that the equation of the normal to C at P when $t = 2$, $8y = 2x + 15a$. **[4]**

(iii)This normal meets C again at point $Q\left(-8a; \frac{-a}{8}\right)$. Given that PQ is the diameter of the circle, show that the equation of this circle is

$$\left(x + \frac{15a}{4}\right)^2 + \left(y - \frac{15a}{16}\right)^2 = \frac{4913a^2}{256}.$$ [**4**]

Answer: $\frac{dy}{dx} = \frac{a^2}{x}$

ZIMSEC June 2013 Paper 1

A curve has parametric equations $x = 1 + t^2$ and $y = 4t - t^3$ where $t > 0$.

Find $\frac{dy}{dx}$ in terms of t, hence show that the coordinates of the turning points are $\left(\frac{7}{3}; \frac{16}{3\sqrt{3}}\right)$. **[7]**

Answer: $\frac{dy}{dx} = \frac{4-3t^2}{2t}$

ZIMSEC November 2009 Paper 1

A curve is represented parametrically by

$$x = 2 + \sin\theta \text{ , } y = 6\cos\theta$$

(i) Find $\frac{dy}{dx}$ in terms of θ. **[3]**

(ii) By expressing $\sin\theta$ in terms of x and $\cos\theta$ in terms of y, use an appropriate identity to show that

$$y^2 + 36x^2 - 144x + 108 = 0$$ [**3**]

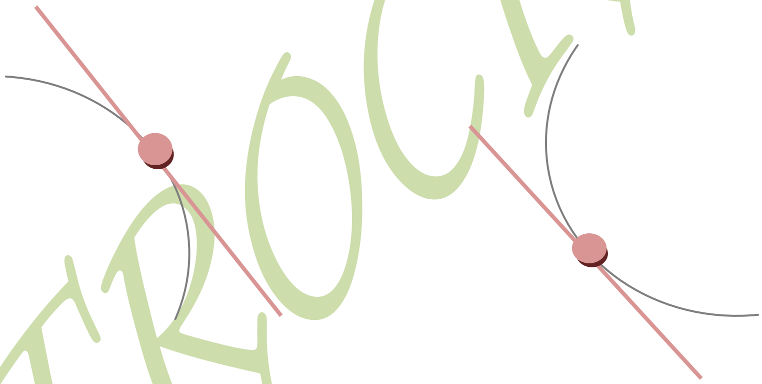
Answer: $\frac{dy}{dx} = -6\tan\theta$

Stationary Points

Increasing and Decreasing Functions

A function, $f(x)$ is said to be increasing if its gradient is positive, $\frac{dy}{dx} > 0$

A function, $f(x)$ is said to be decreasing if its gradient is negative, $\frac{dy}{dx} < 0$



Example

Find the values of x which make the function, $f(x) = x^3 - x^2 - x + 7$,

(a) an increasing function,

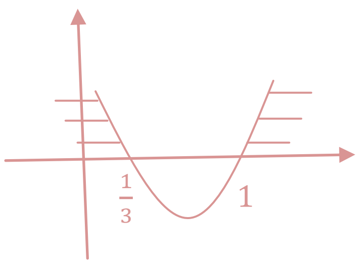
(b) a decreasing function.

Solution

$f\,(x) \;=\; x^3 - x^2 - x \;+\; 7$

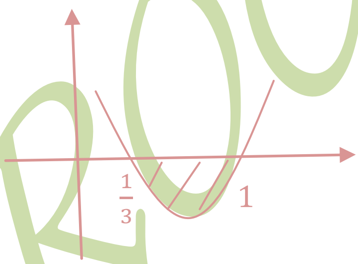
$f'\,(x) = \; 3x^2 - 2x - 1 = (3x-1)(x-1)$

(a) If the function is increasing then $f^{'}\,(x) > 0$
$(3x-1)(x-1) > 0$
Critical values are $x = \frac{1}{3}$ or $x = 1$.



There the solution set is: $x \in \Re$: $x < \frac{1}{3}$ or $x > 1$.

(b) If the function is decreasing then $f^{'}\,(x) < 0$
$(3x-1)(x-1) < 0$
Critical values are $x = \frac{1}{3}$ or $x = 1$.



There the solution set is: $\in \Re$: $\frac{1}{3} x < 1$.

Stationary/Turning Points

- A graph of a function of x is said to have a turning point (or points) if its derivative/ gradient can take a value of zero for some value(s) of i.e. $\frac{dy}{dx} = 0$.
- The gradient at any point on the curve is zero when the tangent to the graph is horizontal

- Turning points are also known as stationary points.
- All quadratic graphs have one turning point, i.e. one occurrence of a zero derivative.
- There are 3 types of stationary points: maximum points, minimum points, and points of inflexion.

NB: Not all points where $\frac{dy}{dx} = 0$ are turning points i.e. not all stationary points are turning points.

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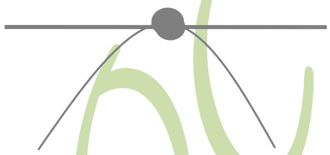
Local maxima and minima

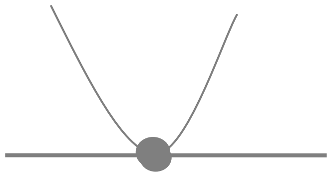
- Local maxima and minima are called turning points.
- The gradient at a local maximum or minimum is 0.

Maximum point



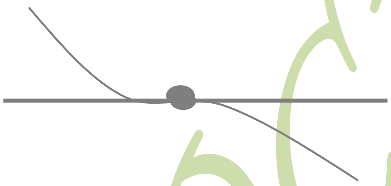
- Just before a maximum point the gradient is positive.
- At the maximum point the gradient is zero
- Just after the maximum point it is negative.
- The value of $\frac{dy}{dx}$ or $f'(x)$is decreasing, so the rate of change of $\frac{dy}{dx}$ or $f'(x)$with respect to x is negative. This means $\frac{d^2y}{dx^2}$ or $f''(x) < 0$ at the maximum point.

Minimum point



- Just before a minimum point the gradient is negative
- At the minimum the gradient is zero,
- Just after the minimum point it is positive.
- The value of $\frac{dy}{dx}$ or $f'(x)$ is increasing, so the rate of change of $\frac{dy}{dx}$ or $f'(x)$ with respect to x is positive. This means $\frac{d^2y}{dx^2}$ or $f''(x) > 0$ at the maximum point.

Points of inflexion



- It is a point which is neither maximum nor minimum
- At the point of inflexion, $\frac{dy}{dx}$ or $f'(x)$ and $\frac{d^2y}{dx^2}$ or $f''(x) = 0$

Curve Sketching

(i) Find the stationary point(s)

Steps

- Find an expression for $\frac{dy}{dx}$ and put it equal 0
- Solve the resulting equation to find the x coordinate(s) of the stationary point(s).
- Find $\frac{d^2y}{dx^2}$ and substitute each value of x to find the kind of stationary point(s).

- (+ suggests a minimum, – a maximum, 0 could be either of these or a point of inflexion)
- Use the curve's equation to find the y coordinate(s) of the stationary point(s).

(ii) Find the point(s) where the curve meets the axes

Steps

- Remember every graph cuts the x −axis when $y = 0$ and cuts the y −axis when $x = 0$.
- Substitute $x = 0$ in the curve's equation to find the y coordinate of the point where the curve meets the y axis.
- Substitute $y = 0$ in the curve's equation.
- If possible, solve the equation to find the x coordinate(s) of the point(s) where the curve meets the x −axis.

(iii)Sketch the curve.

Example 1

For the curve $x^2 - 5x + 4$:

(a) find the stationary point and what type it is;
(b) find the co-ordinates of the point(s) where the curve meets the x and y axes;
(c) sketch the curve $x^2 - 5x + 4$.

Answer

(a) $y = x^2 - 5x + 4$

$\frac{dy}{dx} = 2x - 5 = 0 \therefore x = \frac{5}{2}.$

Now :

$$y = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 4$$

$$= \frac{25}{4} - \frac{25}{2} + 4$$

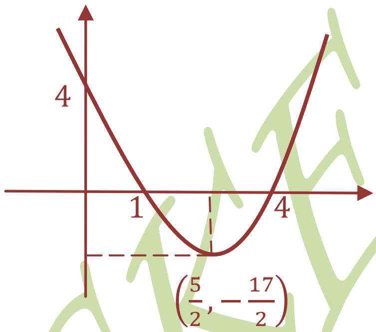
$= -\frac{25}{2} + 4$

$= -\frac{17}{2}$

$\frac{d^2y}{dx^2} = 2 > 0$ ∴ it is minimum at $\left(\frac{5}{2};\ -\frac{17}{2}\right)$

(b) When $x = 0$, $y = 4$ and when $y = 0$, $x = 1$ or 4. Then the curve meets axes at $(0, 4), (4, 0), (1, 0)$.

(c)



Example 2

Cambridge Question

Find the x-coordinate of the stationary points of $y = \frac{4}{3}x^3 + \frac{1}{4x}$ and determine their nature.

[7]

Suggested Solution

At stationary points: $\frac{dy}{dx} = 0$

$y = \frac{4}{3}x^3 + \frac{1}{4}x^{-1}$

$$\frac{dy}{dx} = 4x^2 - \frac{1}{4}x^{-2}$$

$$\Rightarrow 4x^2 - \frac{1}{4}x^{-2} = 0$$

$$\Rightarrow 4x^2 = \frac{1}{4x^2}$$

$$\Rightarrow x^4 = \frac{1}{16}$$

$$\Rightarrow x = \pm\sqrt[4]{\frac{1}{16}}$$

$$\Rightarrow x = \pm\frac{1}{2}$$

Determining the nature of roots:

$$\frac{d^2y}{dx^2} = 8x + \frac{1}{2}x^{-3}$$

$$= 8x + \frac{1}{2x^3}$$

$$@x = \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = 8x + \frac{1}{2x^3}$$

$$= 8\left(\frac{1}{2}\right) + \frac{1}{2\left(\frac{1}{2}\right)^3}$$

$$= 4 + \frac{1}{2\left(\frac{1}{8}\right)}$$

$$= 4 + \frac{1}{\left(\frac{1}{4}\right)}$$

$= 4 + 4$

$= 8 > 0$

$\therefore x = \frac{1}{2}$ is a minimum point.

$@x = -\frac{1}{2}$

$\frac{d^2y}{dx^2} = 8x + \frac{1}{2x^3}$

$= 8\left(-\frac{1}{2}\right) + \frac{1}{2\left(-\frac{1}{2}\right)^3}$

$= -4 + \frac{1}{2\left(-\frac{1}{8}\right)}$

$= -4 + \frac{1}{\left(-\frac{1}{4}\right)}$

$= -4 - 4$

$= -8 < 0$

$\therefore x = -\frac{1}{2}$ is a maximum point.

Example 3

Cambridge Question

A curve has equation $y = e^{-3x} tanx$. Find the x-coordinates of the stationary points on the curve in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Give your answers correct to 3 decimal places. **[6]**

Suggested Guide

$y = e^{-3x} \tan x$

At stationary points $\frac{dy}{dx} = 0$

Now:

$$\frac{dy}{dx} = uv' + vu'$$

$$\Rightarrow e^{-3x}(\sec^2 x) + \tan x(-3e^{-3x}) = 0$$

$$\Rightarrow e^{-3x}(\sec^2 x - 3\tan x) = 0$$

$$\Rightarrow e^{-3x} = 0 \quad or \quad \sec^2 x - 3\tan x = 0$$

$$No\ Solution \quad or \quad \tan^2 x + 1 - 3\tan x = 0$$

$$\tan^2 x - 3\tan x + 1 = 0$$

$$\tan x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$

$$\tan x = \frac{3 \pm \sqrt{5}}{2}$$

$$\tan x = \frac{3 + \sqrt{5}}{2} \text{ or } \tan x \ \frac{3 - \sqrt{5}}{2}$$

$$x = \tan^{-1}\left(\frac{3 + \sqrt{5}}{2}\right) \text{ or } x = \tan^{-1}\left(\frac{3 - \sqrt{5}}{2}\right)$$

$$x = 1.205932499 \text{ or } x = 0.364863828$$

$$\therefore\ x = 0.365 \text{ or } 1.206 \text{ (to 3 decimal places)}$$

Follow up exercise

Questions

For each of the curves whose equations are given below:

a. find each stationary point and what type it is;

b. find the co-ordinates of the point(s) where the curve meets the x and y axes;

c. sketch the curve;

(i) $y = x^2 - 4x$

(ii) $y = x^3 - 3x^2$

(iii) $y = x^3 - 3x - 5$

(iv) $y = x^2 - 6x + 5$

Answers

(i) Minimum at $(2, -4)$, meets axes at $(0, 0)$, $(4, 0)$

(ii) Minimum $(2, -4)$, maximum at $(0, 0)$, meets axes at $(0, 0)$, $(3, 0)$

(iii) Minimum at $(1, -7)$, maximum at $(-1, -3)$, meets axis at $(0, -5)$

(iv) Minimum at $(3, -4)$, meets axes at $(0, 5)$, $(1, 0)$, $(5, 0)$

Past Examination Questions

ZIMSEC November 2009 Paper 1

Show that the curve $y = x^2 Inx$ has turning points at $\left(e^{-\frac{1}{2}}, -\frac{1}{2}e^{-1}\right)$ and that this is a minimum point. **[6]**

ZIMSEC June 2018 Paper 1

Given that $y = \frac{2-x}{x^2-3}$, find the coordinates of the turning points and determine the nature of each turning point of the curve. **[7]**

Answer: $\left(1, -\frac{1}{2}\right)$ is a maximum turning point and $\left(3, -\frac{1}{6}\right)$ is a minimum turning point

ZIMSEC November 2018 Paper 1

The function $f(x)$ is given by $f(x) = \frac{1}{32}x + \frac{1}{2}(x-3)^{-1}, x \neq k$

(i) State the value of k. **[1]**

(ii) Find the coordinates of the turning points of the graph of $y = f(x)$, and investigate their nature. **[9]**

(iii)Show that the graph does not cross the x −axis. **[3]**

Answers:

(i) $k = 3;$

(ii) $\frac{dy}{dx} = \frac{1}{32} - \frac{1}{2}(x-3)^{-2}; x = -1$ or 7. It has a maximum turning point at $\left(-1, -\frac{5}{32}\right)$ and a minim turning point at $\left(7, \frac{11}{32}\right)$;

(iii)$b^2 - 4ac = -55 < 0$ ($no\ real\ roots$) ∴ the graph does not cross the x −axis.

ZIMSEC June 2017 Paper 2

(i) Find the coordinates of the turning point of the curve $y = e^x + 4e^{-2x}$. **[4]**

(ii) Determine the nature of the turning point in **(i)**. **[2]**

Answers:

(i) $(In2; 3)$;

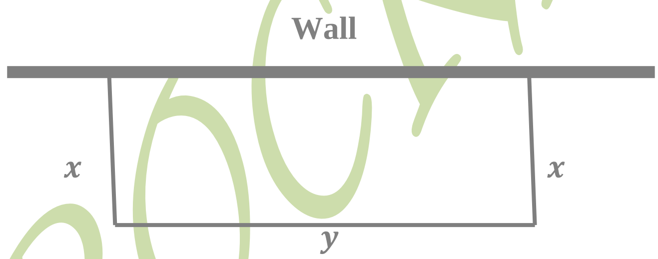
(ii) $\frac{d^2y}{dx^2} = 2 + \frac{16}{4} > 0$. It has a minimum stationary point.

Maximum and minimum problems

- In this case we apply the knowledge on maximum and minimum points to solve any given problems involving maximum/largest/greatest or minimum/smallest/least areas, profits, volumes etc.
- When dealing with surface areas you need to take note of the following cases:
    - (a) When the box is closed :- When calculating surface areas include all faces/edges
    - (b) When the box is open :- When calculating surface areas exclude the upper face

Example 1

A farmer wants to make a rectangular enclosure y metres long and x metres deep. There is a fixed solid wall on one side, and he has 32 metres of fencing available for the other three sides. He wants to make the enclosed area as large as possible.



(i) Show that the enclosed area, A, of the pen can be expressed as $A = x(32 - 2x)\ m^2$.

(ii) For $A = x(32 - 2x)$, find $\frac{dA}{dx}$, and hence find the value of x for which A takes a maximum value.

(iii) Calculate this maximum value of A. Explain algebraically why it is a maximum.

Suggested Solution

(i) $x + x + y = 32$

$2x + y = 32$

$\therefore y = 32 - 2x$

Now $A = l \times w$

$= x \times (32 - 2x)$

$= x(32 - 2x)m^2 \quad (as\ required)$

(ii) $y = 32x - 2x^2$

$\frac{dA}{dx} = 32 - 4x$

If it is a maximum value,

$\frac{dA}{dx} = 32 - 4x = 0$

$\Rightarrow x = 8.$

(iii)$A = x(32 - 2x)$

$= 4(32 - 2 \times 8)$

$= 4 \times 16$

$= 64m^2$

$\frac{d^2A}{dx^2} = -4 < 0 \therefore$ it is a maximum value.

Example 2

An open box has an area of $486cm^2$. If its length is twice the width, find the maximum volume. Show that this volume is a maximum.

Answers

Since the box is open, we calculate the area excluding the upper part.

Let the width be x, therefore the length is $2x$. Let the height be y

area $= 486cm^2$

$(lb) + 2(lh) + 2(bh) = 486$

$\Rightarrow (2x \times x) + 2(x \times y) + 2(2x \times y) = 486$

$\Rightarrow (2x^2) + 6xy = 486$

$\Rightarrow 6xy = 486 - 2x^2$

$\therefore y = \dfrac{486 - 2x^2}{6x}$

Finding the volume

volume $= lbh$

volume $= x \times 2x \times y = (2x^2) \times \dfrac{486 - 2x^2}{6x}$

$= (x) \times \dfrac{486 - 2x^2}{3}$

$= \dfrac{486x - 2x^3}{3}$

Now

$\dfrac{dV}{dx} = 0$

$\Rightarrow \dfrac{486 - 6x^2}{3} = 0$

$\Rightarrow 486 - 6x^2 = 0$

$\Rightarrow 6x^2 = 486$

$\Rightarrow x^2 = 81$

$\Rightarrow x = \pm\sqrt{81}$

$\therefore x = 9$

$V = \dfrac{486x - 2x^3}{3}$

$= \dfrac{486(9) - 2(9)^3}{3}$

$= 972$

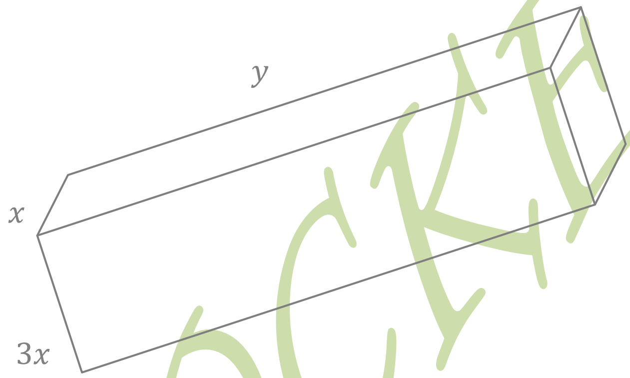
$$\frac{dV}{dx} = \frac{486 - 6x^2}{3}$$

$$\frac{d^2V}{dx^2} = -4x$$

$$= -4 \times 9$$

$$= -36 < 0 \ \therefore \text{its a maximum.}$$

Example 3



The diagram above shows a rectangular box with height ycm, length $3xcm$ and width xcm. Given that the volume is $972cm^3$, show that $y = \frac{324}{x^2}$. **[2]**

Hence find the dimensions of the box if the surface area is to be minimised. **[4]**

Suggested Solution

$\text{volume} = 972cm^3$

$3x \times x \times y = 972$

$$y = \frac{324}{x^2}$$

$Surface\ area\ (SA) = 2(3xy) + 2(xy) + 2(3x^2)$

$Surface\ area\ (SA) = 8xy + 6x^2$

Replacing y with $\frac{324}{x^2}$ in equation (i) above gives:

$SA = \frac{2592}{x} + 6x^2 = 2592x^{-1} + 6x^2$

Finding stationary point(s) of x

$\frac{dSA}{dx} = 0 \Rightarrow \frac{dSA}{dx} = -2592x^{-2} + 12x = 0$ which yields $x = 6$

Finding the dimensions

$y = \frac{324}{x^2} = \frac{324}{6^2}$ $9; x = 6; 3x = 3 \times 6 = 18$

∴ The dimensions are 6; 9; 18.

Example 4

The area of an open box is $486cm^2$. The length of the box is twice its width.

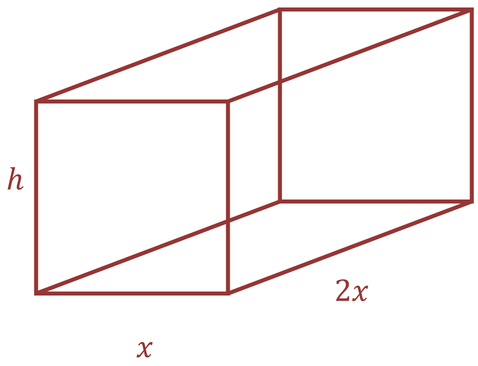
The width is denoted by x.

(a) Show that $h = \frac{243-x^2}{3x}$.

(b) Find the maximum volume and show that it is a maximum.

Suggested Solution

(a)



$Area = 486cm^2$

$\Rightarrow 2(2x \times h) + 2(x \times h) + x \times 2x = 486$

$\Rightarrow 4xh + 2xh + 2x^2 = 486$

$\Rightarrow 6xh = 486 - 2x^2$

$\Rightarrow h = \dfrac{486 - 2x^2}{6x}$

$\therefore h = \dfrac{243 - x^2}{3x}$ $(as\ required)$

(b) $V = lwh$

$\Rightarrow V = x \times 2x \times \dfrac{243 - x^2}{3x}$

$\Rightarrow V = 2x^2 \times \dfrac{243 - x^2}{3x}$

$\Rightarrow V = 2x \times \dfrac{243 - x^2}{3}$

$V = 162x - \dfrac{2x^3}{3}$

$\dfrac{dV}{dx} = 162 - 2x^2 = 0$

$\Rightarrow 81 - x^2 = 0$

$\Rightarrow x = \sqrt{81}$

$\therefore x = 9$

$width = x$

$= 9$

$height = \frac{243 - x^2}{3x}$

$= \frac{243 - (9)^2}{3(9)}$

$= 6$

$Length = 2x$

$= 2(9)$

$= 18$

Now:

$V = lwh$

$\Rightarrow V = 9 \times 18 \times 6$

$= 972cm^3$

$\frac{dV}{dx} = 162 - 2x^2$

$\frac{d^2V}{dx^2} = -4x$

$= -4(9)$

$= -36 < 0$

$\therefore$ It is a maximum point.

Follow up exercise

Question

A particle moves in a straight line which passes through the fixed point O. The particle's displacement, s, from O is given by $s = 12t^2 - 2t^3$ where t is the time in seconds and $0 \leq t \leq 6$.

(i) Find an expression for the velocity of the particle in metres per second at time t seconds.

(ii) Find the particle's displacement when $t = 4$, and show that this value is a maximum.

(iii) At what time does the particle have zero acceleration?

Answers:

(i) $v = \frac{ds}{dt} = 24t - 6t^2$.

(ii) $a = -24$.

(iii) The particle has zero acceleration after 2 seconds.

Past Examination Questions

ZIMSEC November 2017 Paper 1

A rectangular wooden block has base length $3x\ metres$, width $2x\ metres$, height $h\ metres$, total surface area of Acm^2 and volume $144m^3$.

(a) Express in terms of x the

(i) height, h, **[1]**

(ii) total surface area, A, **[2]**

(b) Given that x can vary, find the stationary value of the total surface area A and determine its nature. **[5]**

Answers

(i) $h = \frac{24}{x^2}$; $A = 12x^2 + \frac{240}{x}$

(ii) $x = 2.15(minimum); A = 167m^2$

ZIMSEC November 2013 Paper 1

A closed tin of oil is in the shape of a right circular cylinder. Given that its capacity is $300\ ml$,

(i) write down an expression for its total surface are, A in terms of r,

(ii) Calculate the radius and height of the tin that minimises A. **[5]**

Answers

(iii)$A = 2\pi r^2 + \frac{600}{r}$

(iv)$r = \left(\frac{150}{\pi}\right)^{\frac{1}{3}}$; $r = \left(\frac{1200}{\pi}\right)^{\frac{1}{3}}$

ZIMSEC June 2013 Paper 2

A farmer intends to build a rectangular pen whose width is $x\ m$ for his goats. He has $56\ m$ of mesh wire.

(i) Write down an expression for the area A, of the pen in terms of x. **[1]**

(ii) Hence, find the maximum area which can be enclosed by the mesh wire, verifying that it is a maximum. **[3]**

Answers

(i) Area of pen= $28x - x^2$

(ii) Maximum area= $14 \times 14 = 196$; $\frac{d^2A}{dx^2} = -2(maximum)$

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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