Exponential Growth & Decay

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SYLLABUS (6042) REQUIREMENTS

- Explain exponential growth and decay
- Discussing examples of exponential growth and decay in life
- Modelling situations to solve problems involving exponential growth and decay

Exponential Function

Definition: An exponential growth or decay function is a function that grows or shrinks at a constant percentage growth rate.

- The equation is written in the form:

$$f(x) = a(1 + r)^x$$

OR

$$f(x) = ab^x \text{ where } b = 1 + r$$

Where

$a =$ initial starting value of the function,

$r =$ percentage growth or decay rate written as a decimal

$b =$ growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit b to positive values

- Exponential functions are used to model a wide range of real-life situations such as:

Decay

- Depreciation
- Population decay
- Radioactive decay
- Dating fossils.
- Newton's Law of Cooling
- A warm object as it cools to a constant surrounding temperature e.g. cooling of a kettle after the heat is off
- Slowing down.
- A cool object as it warms

Growth

- Population Growth
- Wages (increasing)
- Inflation
- Cell Growth
- Investments (Compound interest, Interest rates, Present value, and Discount rates.)

Typical growth factors

- Doubles $\Rightarrow b = 2$
- Reduces by half $\Rightarrow b = 0.5$
- Increases by 5% $\Rightarrow b = 1 + 0.05 = 1.05$
- Decreases by 20% $\Rightarrow b = 1 - 0.2 = 0.8$

Exponential Growth

Definition: An exponential growth function is a function that grows at a constant percentage growth rate.

- The equation can be written in the form:

$$P_n = P_0(1+r)^n \qquad (Annual\ Growth)$$

Where

$P_0 =$ initial value (the amount before measuring growth or decay)

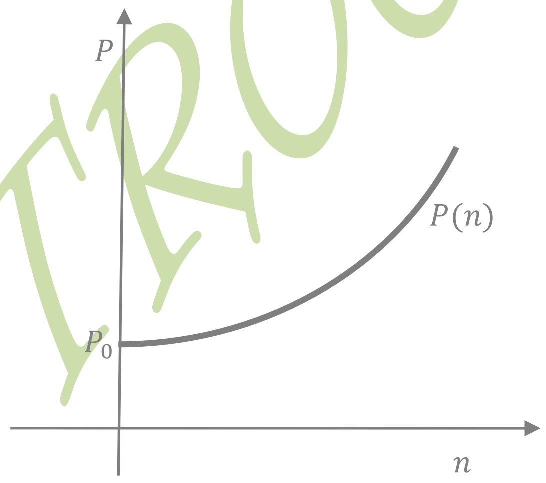
$r =$ growth rate (most often represented as a percentage and expressed as a decimal)

$n =$ time (number of time intervals that have passed)

P_n = the new value after time n.

NB: For continuous growth $P_n = P_0e^{rt}$

Diagram



- In exponential growth, the quantity increases, slowly at first, and then very rapidly. The rate of change increases over time. The rate of growth becomes faster as time passes. This rapid growth is what is meant by the expression "increases exponentially".

Illustration

ZIMSEC JUNE 2020 PAPER 1

The population of a city is 500 000. The population grows at a rate of 5% every year.

Find in terms of n, the population at the end of the n^{th} year

Suggested Solution

Current Population $(P_0) = 500\ 000$

$$P_1 = 500\ 000 + \frac{5}{100}(500\ 000)$$

$$= 500\ 000\left(1 + \frac{5}{100}\right)$$

$$= 500\ 000(1.05)^1$$

$$P_2 = 500\ 000(1.05)^1 + \frac{5}{100}[500\ 000(1.05)^1]$$

$$= 500\ 000(1.05)^1\left(1 + \frac{5}{100}\right)$$

$$= 500\ 000(1.05)^1(1.05)^1$$

$$= 500\ 000(1.05)^2$$

$$\Rightarrow P_n = 500\ 000(1.05)^n$$

Now:

$$P_n = 500\ 000(1.05)^n$$

$$\Rightarrow P_n = P_0(1 + 0.05)^n$$

$$\therefore P_n = P_0(1 + r)^n$$

Exponential Decay

Definition: An exponential decay function is a function that shrinks at a constant percentage growth rate.

- The equation can be written in the form:

$$P_n = P_0(1-r)^n \quad (Annual\ Decline)$$

Where

$P_0 =$ initial value (the amount before measuring growth or decay)

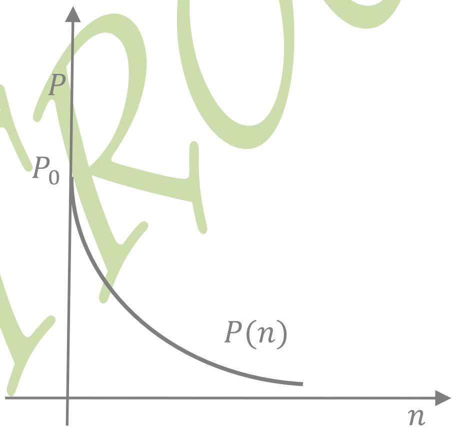
$r =$ decay rate (most often represented as a percentage and expressed as a decimal)

$n =$ time (number of time intervals that have passed)

P_n = the new value after time n.

NB: For continuous decline $P_n = P_0 e^{-rt}$

Diagram



- In exponential decay, the quantity decreases very rapidly at first, and then more slowly. The rate of change decreases over time. The rate of decay becomes slower as time passes.

Illustration

The current value of a car is $50\ 000$. This value depreciates at a rate of 5% every year. Find in terms of n, the value of the car at the end of the n^{th} year

Suggested Solution

Current Value $(P_0) = 50\ 000$

$$P_1 = 50\ 000 - \frac{5}{100}(50\ 000)$$

$$= 50\ 000\left(1 - \frac{5}{100}\right)$$

$$= 50\ 000(0.95)^1$$

$$P_2 = 50\ 000(0.95)^1 - \frac{5}{100}[50\ 000(0.95)^1]$$

$$= 50\ 000(0.95)^1\left(1 - \frac{5}{100}\right)$$

$$= 50\ 000(0.95)^1(0.95)^1$$

$$= 50\ 000(0.95)^2$$

$$\Rightarrow P_n = 50\ 000(0.95)^n$$

Now:

$$P_n = 50\ 000(0.95)^n$$

$$\Rightarrow P_n = P_0(1 - 0.05)^n$$

$$\therefore P_n = P_0(1 - r)^n$$

Compound Interest

Definition: It is the addition of interest to the principal sum of a loan or deposit, or in other words, interest on interest. Interest is calculated once per period on the current amount borrowed or invested. Each period, the interest becomes a part of the principal

- The effect of compounding depends on:
  - The nominal interest rate which is applied and
  - The frequency interest is compounded i.e. annually ($n = 1$), semi-annually ($n = 2$), quarterly ($n = 4$), monthly ($n = 12$) and daily ($n = 365$)
- The total accumulated value, including the principal sum P plus compounded interest I, is given by the formula:

$$P_n = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

Where:

P_0 is the principal (initial amount invested)

P_n is the new balance

r is the rate (in decimal form)

n is the number of times it is compounded

t is the time in years

- The total compound interest generated is the final value minus the initial principal

$$I = P_0\left(1 + \frac{r}{n}\right)^{nt} - P_0$$

NB: If the interest is compounded continuously (compounded every instant) for t years at a rate of r per year then the compounded amount is given by $P_n = P_0e^{rt}$

SOLVED PROBLEMS

Question 1

The value of a new car is \$30 000. If the car depreciates by 5% in value each year, show that the value of the car 13 *years* later will be \$15 400.26.

Suggested Solution

$New\ Value\ (P_n) = P_0(1-r)^n$ (For decay)

Where:

- P_0 is the initial value.
- r is the decay/growth rate in decimal form and
- n is the time

$$P_0 = \$30\ 000,\ r = 0.05, n = 13 \text{ and } R = 1 - r = \frac{95}{100}$$

$P_n = P_0R^n$

Now

$$P_n = (\$30\ 000)\left(\frac{95}{100}\right)^{13}$$

$$= \$15\ 400.2625$$

$$= \$15\ 400.26$$

The new value of the car 13 *years* later is \$15 400.26 (*As required*).

Question 2

In 1990, a small town had a population of 15 000 people. Since 1990, the town has increased by 3% per year. What is the population in 2005

Suggested Solution

$New\ Population\ (P_n) = P_0(1+r)^n$ (For growth)

$$P_0 = 15\ 000,\ r = 0.03, n = 15 \text{ and } R = 1 + r = \frac{103}{100}$$

$P_n = P_0 R^n$

Now

$P_n = (15\ 000)\left(\frac{103}{100}\right)^{15}$

$= 23\ 369.51125$

$= 23\ 370$

The population in 2005 is approximately 23 370.

Question 3

Suppose that a population of bacteria grows naturally. Suppose that the initial population is 100 bacteria, and, after 1 $hour$, the population has grown to 120.

(a) How large will the population be after 2 $hours$?

(b) How long will it take for the population to reach 1 000 000 bacteria?

Suggested Solution

$New\ Population\ (P_n) = P_0(1 + r)^n$ (For growth)

$$P_0 = 100,\ r = 0.2,\ n = 2 \text{ and } R = 1 + r = \frac{120}{100}$$

(a) $P_n = P_0 R^n$

Now

$P_n = (100)\left(\frac{120}{100}\right)^2 = 144$

(b) $P_n = P_0 R^n$

Now

$P_n = (100)\left(\frac{120}{100}\right)^n = 1\ 000\ 000$

$$\left(\frac{120}{100}\right)^n = 1\,0\,000$$

$$In\left(\frac{120}{100}\right)^n = In(1\,0\,000)$$

$$n = \frac{In(1\,0\,000)}{In\left(\frac{120}{100}\right)}$$

$$= 50.51701255$$

∴The population will reach 1 000 000 after 51 *hours*

Question 4

Suppose a principal amount of $1,500 is deposited in a bank paying an annual interest rate of 4.3%, compounded quarterly. Find the balance after 6 years.

Suggested Solution

$$P_n = P_0\left(1+\frac{r}{n}\right)^{nt}$$

Where:

P_0 is the principal (initial amount invested)

P_n is the new balance

r is the rate (in decimal form)

n is the number of times it is compounded

t is the time in years

$$P_n = P_0\left(1+\frac{r}{n}\right)^{nt}$$

$$P_0 = 1500,\ r = 0.043\ (4.3\%),\ n = 4 \text{ and } t = 6$$

$$P_n = 1500\left(1+\frac{0.043}{4}\right)^{4(6)}$$

$$P_n = 1500 \times 1.292557881$$

$$= 1938.836822$$

∴ The new principal after 6 years is approximately $1,938.84.

Question 5

The world population was 4.8 billion in 1985 with an estimated growth rate of 1.8% per year. Find the projected population in 2020. Round to 2 $significant\ fgures$.

Suggested Solution

$New\ Population\ (P_n) = P_0(1+r)^n$ (For growth)

$$P_0 = 4.8\ billion,\ r = 0.018, n = 35 \text{ and } R = 1 + r = \frac{509}{500}$$

$P_n = P_0 R^n$

Now

$$P_n = (4\,800\,000\,000)\left(\frac{509}{500}\right)^{35}$$

$$= 8\,962\,175\,980$$

∴ The projected population in 2020 is approximately 9 000 000 000

Question 6

The initial cost of a car is $35 000 and it depreciates with 10% after every 6 $months$. How much is the car worth in 7 $years$.

Suggested Solution

$New\ Population\ (P_n) = P_0(1-r)^n$ (For decay)

$$P_0 = \$35\,000,\ r = 0.1, n = 7 \times 2 = 14 \text{ and } R = 1 - r = \frac{90}{100}$$

$P_n = P_0 R^n$

Now

$$P_n = (\$35\,000)\left(\frac{90}{100}\right)^{14}$$

$$= \$8\,006.877359$$

$= \$8\ 007$

$\therefore$ The new value in 7 $years$ is approximately \$8 007.

Question 7

Trockers received \$1500 as a gift from his mother after graduating from a local university. Trockers chose to invest the money with an annual interest rate of 3% compounded quarterly

a) Write down an exponential function to model this relation.

b) How much will Trockers have invested after 5 $years$?

Suggested Solution

$$P_n = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

$$P_0 = 1500,\ r = 0.03\ (3\%),\ n = 4 \text{ and } t = 5$$

(a) $P_n = P\left(1 + \frac{r}{n}\right)^{nt}$

$$P_n = 1500\left(1 + \frac{0.03}{4}\right)^{4t}$$

$$= 1500(1.0075)^{4t}$$

(b) $P_n = 1500(1.0075)^{4(5)}$

$$= 1\ 741.776\ 213\ 4548$$

$\therefore$ Trockers invested \$1 742 in 5 $years$

Question 8

A population of 3 000 people doubles in size every 10 $years$. Determine the population after 50 $years$.

Suggested Solution

$New\ Population\ (P_n) = P_0(1 + r)^n$ (For growth)

$$P_n = 3\ 000,\ r = 1,\ n = 5 \text{ and } R = 1 + r = 2$$

$P_n = P_0R^n$

Now

$P_n = (3\,000)(2)^5$

$= 96\,000$

$\therefore$ The population after 50 $years$ is approximately 96 000.

Question 9

ZIMSEC JUNE 2020 PAPER 1

The population of a city is 500 000. The population grows at a rate of 5% every year.

Find

(a) in terms of n, the population at the end of the n^{th} year, **[2]**

(b) the population, to the nearest thousand, after 10 years, **[2]**

(c) the year in which the population first exceeds 1 000 000. **[3]**

Suggested Solution

Current Population $(P_0) = 500\,000$

New Population $(P_n) = P_0(1+r)^n$

$r = 0.05$ and $R = 1 + r = 1.05$

(a) $P_n = P_0R^n$

$= 500\,000(1.05)^n$

(b) $P_{10} = 500\,000(1.05)^{10}$

$= 814\,447.3134$

$= 814\,000$ ($to\ the\ nearest\ thousand$)

(c) $P_n > 1\,000\,000$

$\Rightarrow 500\,000(1.05)^n > 1\,000\,000$

$\Rightarrow (1.05)^n > \dfrac{1\,000\,000}{500\,000}$

$\Rightarrow In(1.05)^n > In2$

$\Rightarrow nIn(1.05) > In2$

$$\Rightarrow n > \frac{In2}{In(1.05)}$$

$\Rightarrow n > 14.206699082$

$\Rightarrow n = 15$

∴ The population will first exceed 1 000 000 in year 15.

Question 10

If \$6 000 is invested in a bank which pays 3% per year compounded continuously. How much will be in the account after 8 years?

Suggested Solution

$P_n = P_0 e^{rt}$

$$P_0 = 6000,\ r = 0.03\ (3\%), \text{and } t = 8$$

$P_n = 6000e^{0.03(8)}$

$P_n = 6000 \times 1.271249150$

$= 7\ 627.49$

∴ The new principal after 6 years is approximately \$7 627.49.

Question 11

The growth rate of a country rises by15%. Assuming continuous growth, find the new population in 5 years if the current population is 100 000.

Suggested Solution

$P_n = P_0 e^{rt}$

$$P_0 = 100\ 000,\ r = 0.15\ (15\%), \text{and } t = 5$$

$P_n = 100\ 000e^{(5\times 0.15)}$

$P_n = 100\ 000 \times 2.117000017$

$= 211\ 700.002$

∴ The new population in 5 years will be 211 700

Question 12

The cost of a calculator was \$16 in 2003. In 2018 it was \$40.

(a) Find r, the annual growth rate.

(b) Find the cost of the calculator in 2023.

(c) After when will the cost of the calculator be \$32?

Suggested Solution

(a) $P = P_0(1+r)^n$

$\Rightarrow 40 = 16(1+r)^{15}$

$\Rightarrow \dfrac{40}{16} = (1+r)^{15}$

$\Rightarrow In\left(\dfrac{40}{16}\right) = In(1+r)^{15}$

$\Rightarrow In\left(\dfrac{40}{16}\right) = 15In(1+r)$

$\Rightarrow \dfrac{1}{15}In\left(\dfrac{40}{16}\right) = In(1+r)$

$\Rightarrow e^{\frac{1}{15}In\left(\frac{40}{16}\right)} = e^{In(1+r)}$

$\Rightarrow 1 + r = e^{\frac{1}{15}In\left(\frac{40}{16}\right)}$

$\Rightarrow r = e^{\frac{1}{15}In\left(\frac{40}{16}\right)} - 1$

$\Rightarrow r = 0.062990379$

$\Rightarrow r = 0.063$ or 6.3% $(to\ 2\ s.f.)$

(b) $P = P_0(1+r)^n$

$P = 16(1+0.063)^{20}$

$= \$54.298180030$

$= \$54\ \ (to\ 2\ s.f.)$

(c) $P = P_0(1+r)^n$

$32 = 16(1+0.063)^n$

$\Rightarrow \dfrac{32}{16} = (1+0.063)^n$

$\Rightarrow In(2) = In(1 + 0.063)^n$

$\Rightarrow In(2) = nIn(1 + 0.063)$

$\Rightarrow \frac{In(2)}{In(1 + 0.063)} = n$

$\Rightarrow n = 11.345381018$

$\Rightarrow n = 11\ years\ (to\ 2\ s.f.)$

PRACTICE PROBLEMS

Question 1

In 1990, a small town had a population of 15 000 people. Since 1990, the town has increased by 3% per year. What was the population in 2005?

Answer:23 370

Question 2

The total gallons of juice each person consumes yearly had decreased 4.1% each year. In 1981, each person drank 16.5 gallons of juice on average. What was the approximate amount consumed per person in 2010?

Answer:4.9

Question 3

Nyasha put $100 into a bank savings account. The account accrues 9% interest rate compounded annually. How much money is in the account after 12 years?

Answer:281.27

Question 4

The foundation of a house has about 1 200 termites. The termites grow at a rate of about 2.4% per day. What will be the population of the termites in 5 days?

Answer:1 351

Question 5

A Samsung galaxy $S10$ costs $2 100. Its value decreases by 45% annually. What will be its value in 3 years?

Answer:$349.39

Question 6

Based on census data, the population of Norton was 43 230 in year 2000 and has been growing at an average annual rate of 3.5%. What is the estimated population in 2015?

Answer:72 425

Question 7

A certain type of bacteria, given a favourable growth medium doubles in population every 5 hours. Given that there were approximately 10 000 bacteria to start with, estimate the number of cells that will be in the culture in 24 hours?

Answer:278 576

Question 8

If $10 000 is invested in a bank which pays 5% per year compounded continuously. How much will be in the account

(a) after 6 years?

(b) after 30 years?

Answer: a) $13 498.59   b) $44 817

Question 9

The current population of a country is 26 700.

(a) Calculate the population in 8 years time if it is increasing at 1.4% per year.

(b) Calculate the population in 6 years time if it is decreasing at 0.4% per year.

Answer: a) 29 841   b) 26 066

Question 10

The current sales of a company in $275 000.

(a) Calculate the sales in 4 years time if they are increasing at 12% per year.

(b) Calculate the sales in 2 years time if they are increasing at 3.2% per year.

Answer: a) \$432 717.82 b) \$292 881.60

Question 11

Patrick plans to purchase a car that depreciates at a rate of 12% per year. The initial costs of the car is \$19 000. How much money will the car be worthy after 3 years?

Answer:\$12 947.97

Question 12

Trockers is opening a savings account at his local bank. He invests an initial amount of \$5 000. His bank gives him $5\frac{1}{4}$% interest of his account each year.

(a) How much money will he have after 25 years?

(b) How much interest has Trockers accumulated over that time?

Answer:\$17 968.95 b) \$12 968.95

Question 14

How many days will it take for an insect population to double if its growth rate is 5%.

Answer: 14

Question 15

In 1950 the population of Norton was 6 000, by 1990 the population grew to 12 000. In what year will the population reach 30 000?

Answer: 93 $years \Rightarrow$ 2043

Question 16

Patrick wants his initial amount to grow to \$211 700 in 5 years. The interest is compounded continuously at 15%. Calculate the initial amount.

Answer: \$100 000

Question 17

How long does it take for the amount to double at 8.5% compounded

(a) annually,
(b) continuously?

Answer: a) 8.5 b) 8.15

Question 18

The population of a mine city was 250 000 in 2005 and 200 000 in 2015 Assuming the population is decreasing continuously, find the population in 2025

Answer:160 000

Question 19

The cost of a calculator was \$16 in 2003. In 2018 it was \$40. Assuming continuous growth:

(a) Find r.
(b) Find the cost of the calculator in 2023.
(c) After when will the cost of the calculator be \$32?

Answer: a) 6.2% b) \$54.28 c) 11.36

Question 20

At what rate compounded annually will an investment of \$2 100 accumulate to \$3 400 by the end of 6 years.

Answer: 8.36%

PAST EXAMINATION QUESTIONS

Question 1

ZIMSEC NOVEMBER 2002 PAPER 1

The population of a country increases by 3% every year. Find the time in years, it takes for the population to double. **[4]**

Answer:23

Question 2

ZIMSEC JUNE 2014 PAPER 1

A woman measures the height of her child at birth and at monthly intervals afterwards.

The child's height increases by 5% per month. Find the number of measurements she has made before the child's height is twice what it was at birth. **[4]**

Answer:14

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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