Graphs & Transformations

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SYLLABUS (6042) REQUIREMENTS

- illustrate relationship amongst the graphs of $y = f(x)$ and $y = af(x), y = f(x) + a, y = f(x + a), y = f(ax)$ and $y = af(x + b)$
- sketch graphs of the form y = f (x), where $f(x) = kx^n$ and $n = \frac{1}{2}$ or n is an integer, and where f(x) is a quadratic or cubic polynomial

Common Functions

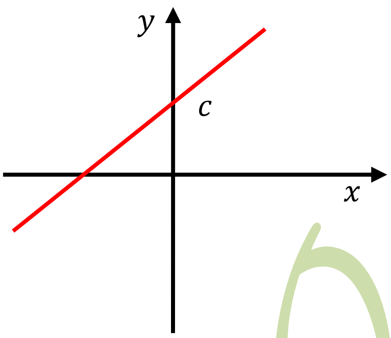
The different types of functions to be considered are: Linear; Quadratic; Power; Rational; Exponential & Logarithmic

General Graphs of Functions

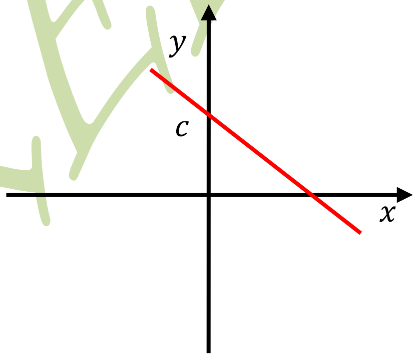
(a) Linear

Linear functions are of the form $y = mx + c$, where m is the gradient and $+c$ is the y −axis intercept.

When m is positive



When m is negative

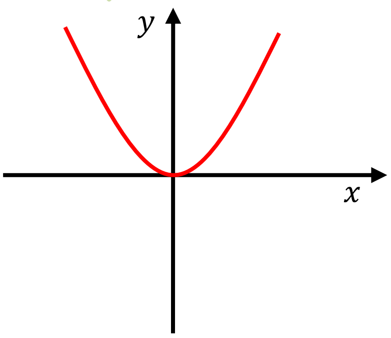


(b) Quadratic

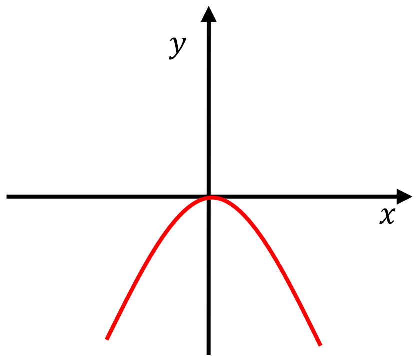
Quadratic functions are of the form $y = ax^2 + bx + c$.

Example: $y = ax^2$

When a is positive



When a is negative

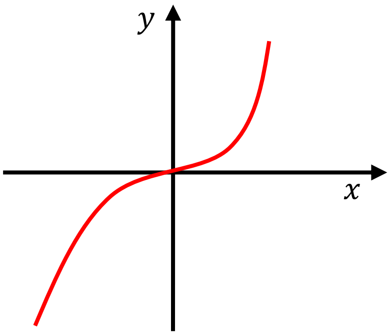


(c) Power

Power functions are of the form $y = ax^n$.

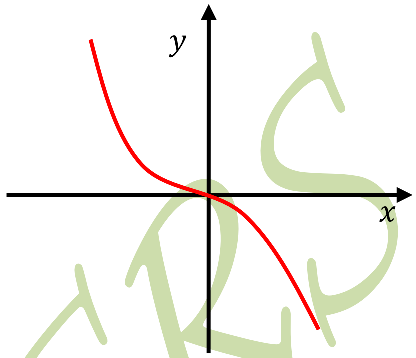
When n is positive

Example: $y = ax^3$



When n is negative

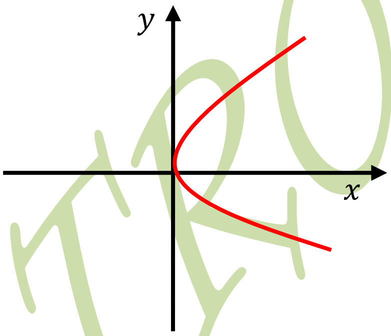
Example: $y = ax^{-3}$



NB: 'a' is positive in both cases

When n is fractional

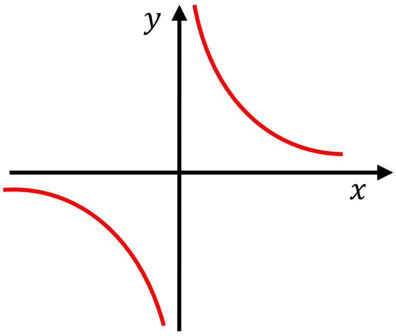
Example: $y = \sqrt{x}$



(d) Rational

Rational functions are of the form $\frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$.

Example: $y = \frac{1}{x}$

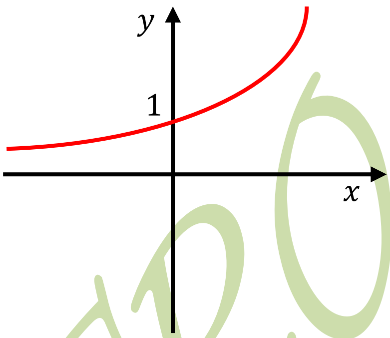


NB: The axes becomes the asymptotes

(e) Exponential

Exponential functions are of the form $y = ab^x$.

Example: $y = e^x$

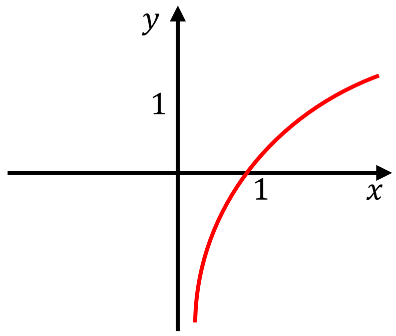


NB: $e^{-\infty} = 0$

(f) Logarithmic

Logarithmic functions are of the form $y = aIn(x) + b$.

Example: $y = Inx$



NB: $In(0) = N/A$ and $In(negatives) = N/A$

Sketching Graphs

- Sketching is different from plotting
- When sketching graphs we only show critical points i.e. the points where the graphs cuts the axes (intercepts)
  (a) The graph cuts the x −axis when $y = 0$
  (b) The graph cuts the y −axis when $x = 0$
- When asked to show the turning/stationary points we apply differentiation i.e.

  (a) At turning points $\frac{dy}{dx} = 0$

  (b) When $\frac{d^2y}{dx^2} > 0$ we have a minimum turning point

  (c) When $\frac{d^2y}{dx^2} < 0$ we have a maximum turning point

Sketch Graphs

Graphs of Factorised functions

Question One

Sketch the graphs of the following functions. [No need to show the turning points on (b), (c) and (d)]

(a) $f(x) = (x-1)(x+2)$
(b) $f(x) = x(x+3)$
(c) $f(x) = (x+1)(x+2)(x+4)$
(d) $f(x) = x(1-x)(3-x)$

Solution

NB:

(i) Every graph cuts the $x - axis$ when $f(x)$ or $y = 0$ and the $y - axis$ when $x = 0$.
(ii) If 0 is a critical value then use any value between the critical values to inspect whether $f(x)$ is negative or positive.

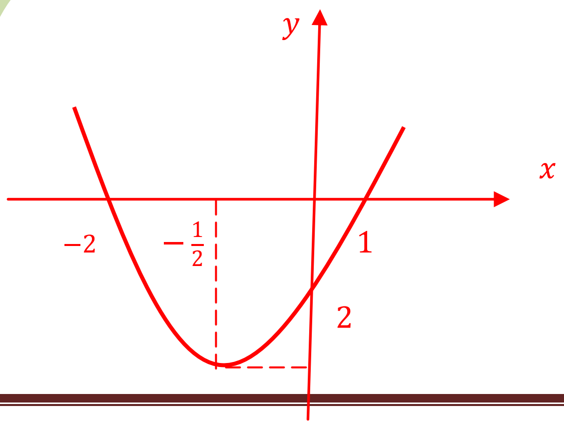
(a) $f(x) = (x-1)(x+2) = x^2 + x - 2$

When $y = 0; x = 1 \; or -2$ and when $x = 0, y = -2$.

Finding turning points

At turning points $\frac{dy}{dx} = 0 \Longrightarrow 2x + 1 = 0 \therefore x = -\frac{1}{2}$.

Now when $x = -\frac{1}{2}$, $y = -\frac{9}{4} \therefore$ the coordinates of the turning points are $\left(-\frac{1}{2}; -\frac{9}{4}\right)$.



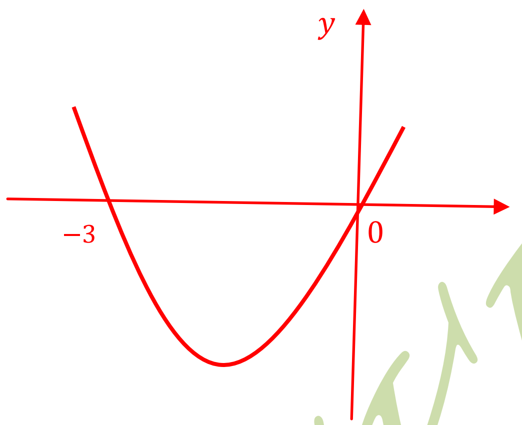
$\left(-\frac{1}{2}; -\frac{9}{4}\right)$ $-\frac{9}{4}$

NB: You also need to observe symmetry.

(b) $f(x) = x(x + 3)$

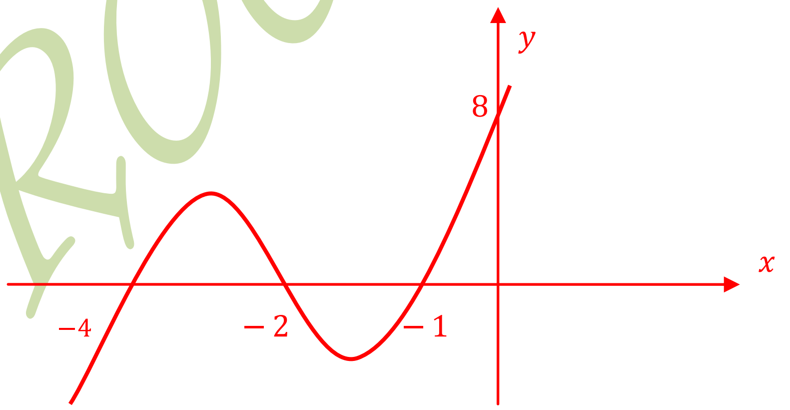
When $y = 0; x = 0 \; or - 3$ and when $x = 0, y = 0$.

Since 0 is a critical point, thus consider any value between 0 and -2, say -1: $x = -1, y < 0$.



c) $f(x) = (x + 1)(x + 2)(x + 4)$

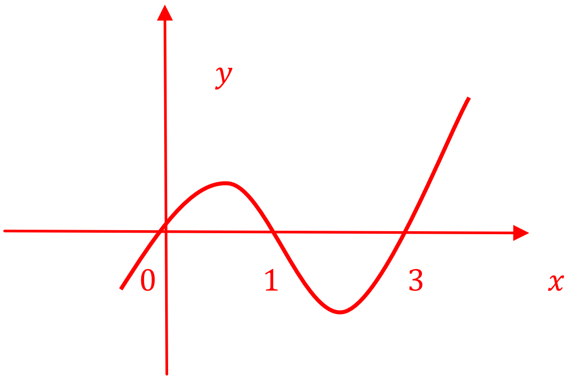
When $y = 0; x = -1, -2 \; or - 4$ and when $x = 0, y = 8$.



d) $f(x) = x(1 - x)(3 - x)$

When $y = 0; x = 0, 1 \; or \; 3$ and when $x = 0; y = 0$.

Since 0 is a critical point, thus consider any value between 0 and 1 or between 0 and 1, say $\frac{1}{2}$: $x = \frac{1}{2}, y > 0$.



TRANSFORMATIONS

To transform means to change in shape (size) and position.

There are three types of transformations which are: Translation, Reflection and Stretch.

[1] TRANSLATION

(i) $\boldsymbol{x - axis}$

$f(x - a)$: This is translation along the $x - axis$ with translation vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ or simply moving a units to the right side.

$f(x + a)$: This is translation along the $x - axis$ with translation vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ or simply moving a units to the left side.

(ii) $\boldsymbol{y - axis}$

$f(x) - a$: This is translation along the $y - axis$ with translation vector $\begin{pmatrix} 0 \\ -a \end{pmatrix}$ or simply moving a units downwards.

$f(x) + a$: This is translation along the $y - axis$ with translation vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$ or simply moving a units upwards.

[2] REFLECTION

(i) $\boldsymbol{x - axis}$

$-f(x)$: This is reflection along the $x - axis$

(ii) $\boldsymbol{y - axis}$

$f(-x)$: This is reflection along the $y - axis$

[3] STRETCH

(i) $\boldsymbol{x - axis}$

$f(ax)$: This is stretch along the $x - axis$ with factor $\left(\frac{1}{a}\right)$ or simply divide every $x - coordinate$ by a.

(ii) $\boldsymbol{y - axis}$

$af(x)$: This is stretch along the $y - axis$ with factor a or simply multiply every $y - coordinate$ by a.

EXAMPLES

Question One

Given that $f(x) = (x-2)(x+4): x \in \mathbb{R}$. Sketch on separate diagrams, without showing turning points, graphs of the following functions:

(a) $f(x)$

(b) $|f(x)|$

(c) $f(x+1)$

(d) $f(2x)$

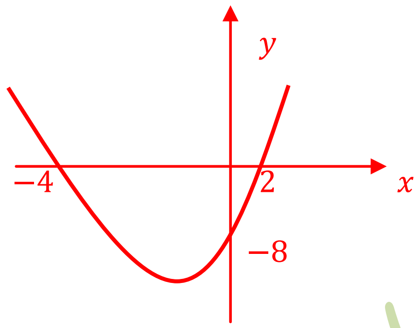
(e) $f(x) - 1$

(f) $\boldsymbol{2 - f(2x - 1)}$

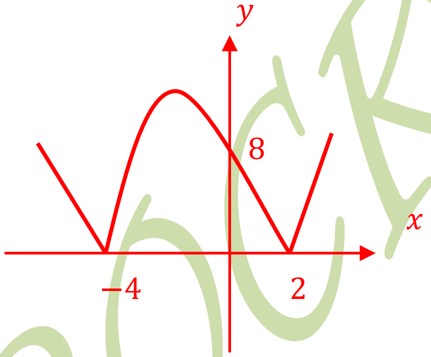
Suggested solution

$f(x) = (x-2)(x+4): x \in \mathbb{R}$.

a) When $x = 0$, $f(x) = -8$ and when $f(x) = 0, x = 2 \; or -4$

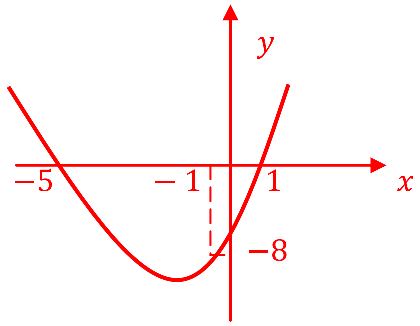


b) $|f(x)|$



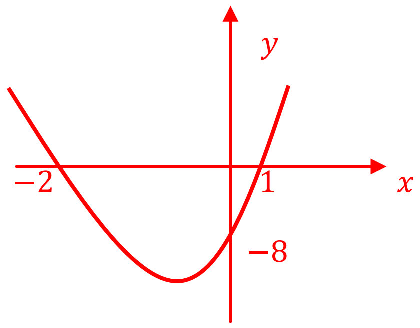
c) $f(x+1)$

Translation, $x - axis$ with translation vector $= \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, or simply moving 1 unit leftwards



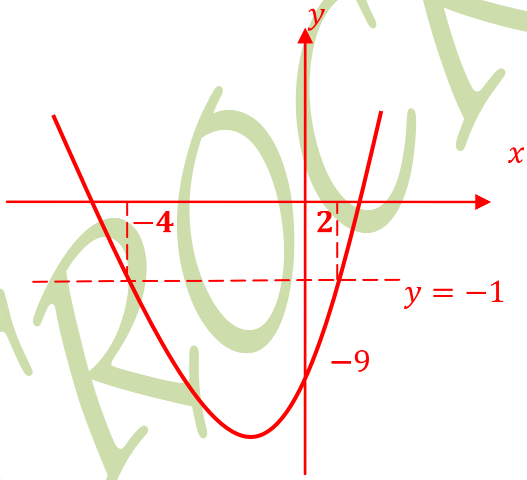
d) $f(2x)$

Stretch, $x - axis$ with factor $\frac{1}{2}$, or simply dividing every $x - coordinate$ by 2.



e) $f(x) - 1$

Translation, $y - axis$ with translation vector $= \binom{0}{-1}$, or simply moving 1 unit downwards.



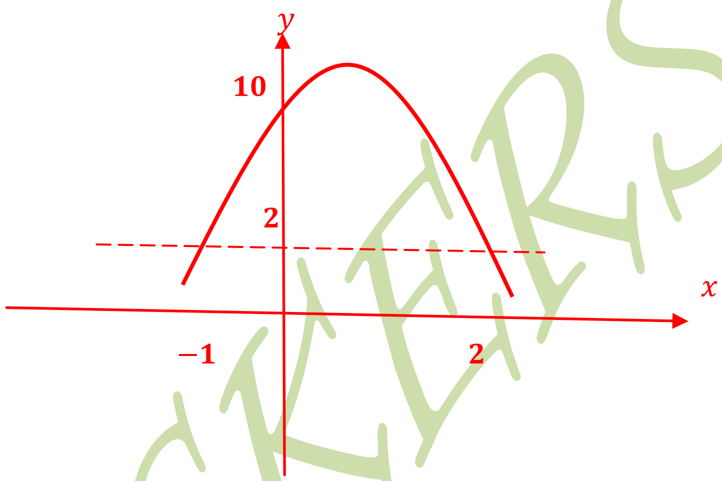
f) Combined Transformations: $2 - f(2x - 1)$

(i) Stretch, $x - axis$ with factor $\frac{1}{2}$, or simply dividing every $x - coordinate$ by 2. **[$f(2x)$]**

(ii) Translation, $x-axis$ with translation vector $=\binom{1}{0}$, or simply moving 1 unit rightwards. $[\boldsymbol{h}(\boldsymbol{x}-\mathbf{1})]$

(iii)Reflection along the $x-axis$. $[-\boldsymbol{g}(\boldsymbol{x})]$

(iv)Translation, $y-axis$ with translation vector $=\binom{0}{2}$, or simply moving 1 unit downwards. $[\boldsymbol{k}(\boldsymbol{x})+\mathbf{2}]$



<u>Question Two</u>

State the sequence of transformation which the graph of $f(x)=\frac{1}{x}, x\in\mathbb{R}: x\neq 0$ undergoes to produce the graph of $f(x)=\frac{x-4}{x-2}, x\in\mathbb{R}: x\neq 2$. Hence sketch the graph of

$f(x)=\frac{x-4}{x-2}, x\in\mathbb{R}: x\neq 2.$

<u>Suggested solution</u>

$$f(x)=\frac{x-4}{x-2}, x\in\mathbb{R}: x\neq 2$$

Performing the long division for $\frac{x+1}{x-2}$ yields $1-\frac{2}{x-2}$.

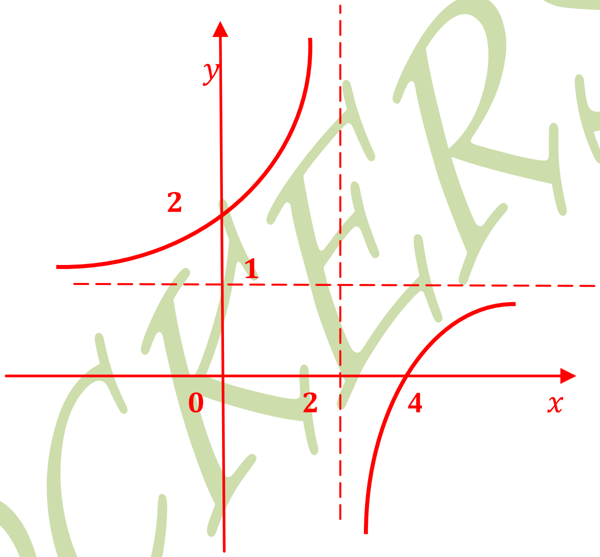
The resultant transformations are:

(i) Translation, $x - axis$ with translation vector $= \binom{2}{0}$, or simply moving 2 unit rightwards. $[\boldsymbol{h}(\boldsymbol{x}-\mathbf{2})]$

(ii) Stretch, $y - axis$ with factor 2, or simply multiply every $y - coordinate$ by 2. $[\mathbf{2}\boldsymbol{f}(\boldsymbol{x})]$

(iii)Reflection along the $x - axis$. $[-\boldsymbol{g}(\boldsymbol{x})]$

(iv)Translation, $y - axis$ with translation vector $= \binom{0}{1}$, or simply moving 1 unit downwards. $[\boldsymbol{k}(\boldsymbol{x})+\mathbf{1}]$



Question Three

Express the $2x^2 - 4x + 6$ in the form $a(x+b)^2 + c$

Hence state the sequence of transformation which the graph of $f(x) = x^2, x \in \mathbb{R}$ undergoes to produce the graph of $f(x) = 2x^2 - 4x + 6, x \in \mathbb{R}$.

Suggested solution

$2x^2 - 4x + 6 = 2(x^2 - 2x + 3) = 2[(x-1)^2 - 1 + 3] = 2(x-1)^2 + 4$

Now $f(x) = 2(x-1)^2 + 4$

The resultant transformations are:

(i) Translation, $x - axis$ with translation vector $= \begin{pmatrix}1\\0\end{pmatrix}$, or simply moving 1 unit rightwards. $[\boldsymbol{f}(\boldsymbol{x}-\mathbf{1})]$

(ii) Stretch, $y - axis$ with factor 2, or simply multiply every $y - coordinate$ by 2. $[\mathbf{2}\boldsymbol{g}(\boldsymbol{x})]$

(iii)Translation, $y - axis$ with translation vector $= \begin{pmatrix}0\\4\end{pmatrix}$, or simply moving 4 unit upwards. $[\boldsymbol{h}(\boldsymbol{x}) + \mathbf{4}]$

Question Four

The graph of $y = x^2$ undergoes the following transformations:

(i) Translation along the $x - axis$ with translation vector $= \begin{pmatrix}-2\\0\end{pmatrix}$,

(ii) Stretch, $y - axis$ with factor 2 and

(iii)Translation, $y - axis$ with translation vector $= \begin{pmatrix}0\\1\end{pmatrix}$

Write down the equation of the resultant graph.

Suggested solution

The resultant transformations are:

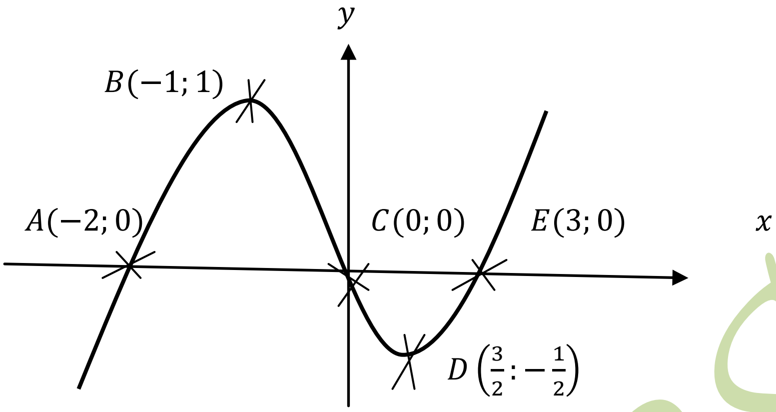
(i) $[\boldsymbol{f}(\boldsymbol{x}+\mathbf{2})] = (x+2)^2$

(ii) $[\mathbf{2}\boldsymbol{f}(\boldsymbol{x}+\mathbf{2})] = 2(x+2)^2$

(iii)$[\mathbf{2}\boldsymbol{f}(\boldsymbol{x}+\mathbf{2}) + \mathbf{1}] = 2(x+2)^2 + 1.$

$\therefore$ The resultant graph is $2(x^2 + 4x + 4) + 1 = 2x^2 + 8x + 8 + 1 = 2x^2 + 8x + 9.$

Question Five



Sketch on separate diagrams, graphs of the following functions, showing clearly coordinates of the intercepts and turning points:

(i) $f(x-1)$

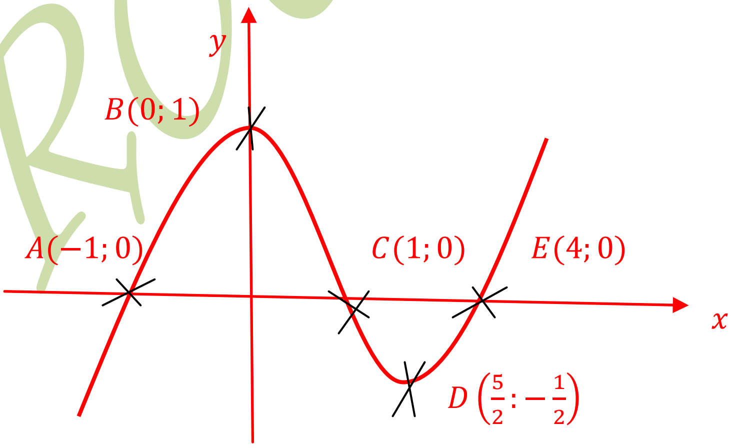
(ii) $f(2x)$

(iii) $1-f(x)$

(iv) $|f(x)|$

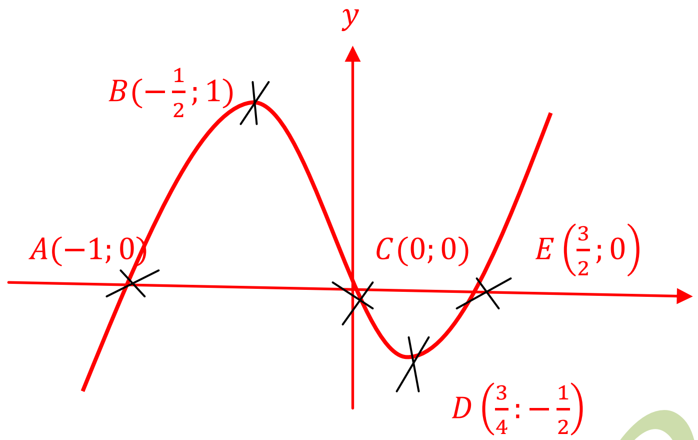
Suggested solution

(i) $f(x-1)$



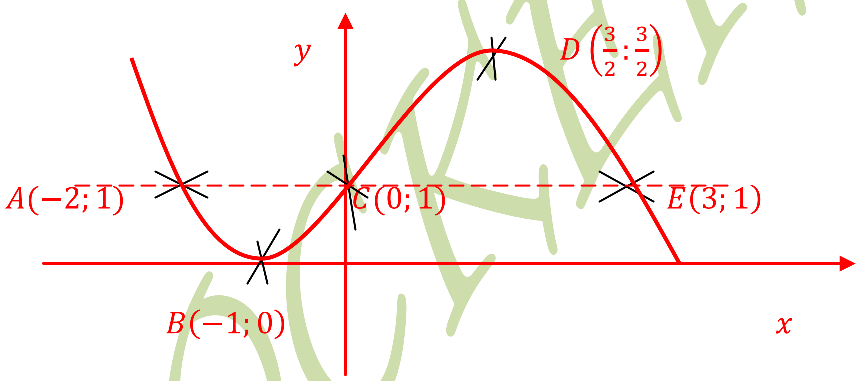
Translation x-axis moving 1 unit rightwards

(ii) $f(2x)$



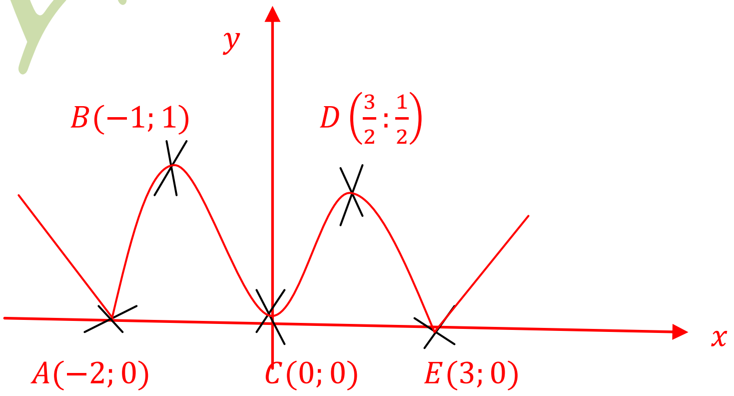
Stretch x-axis with factor $\frac{1}{2}$.

(iii)$1-f(x)$



Reflection along the x-axis and Translation y-axis moving 1 unit upwards

(iv)$|f(x)|$



Question Six

Sketch the graph of $y = e^x$.

Hence on separate diagrams, sketch the graphs of the following functions and state the resultant transformation.

(i) $y = e^{-x}$

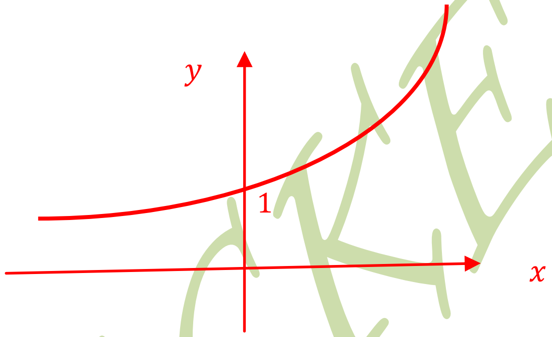
(ii) $y = -\frac{1}{4}e^x + 1$

(iii) $y = e^x - 2$

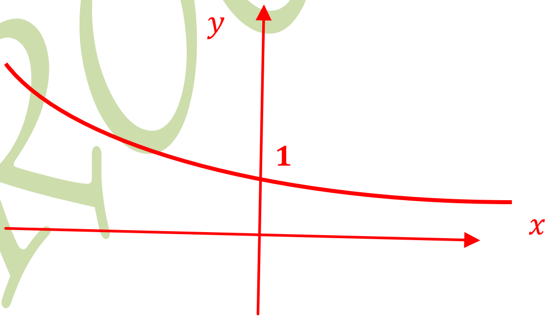
(iv) $y = e^{x+1}$

Suggested solution

$y = e^x$

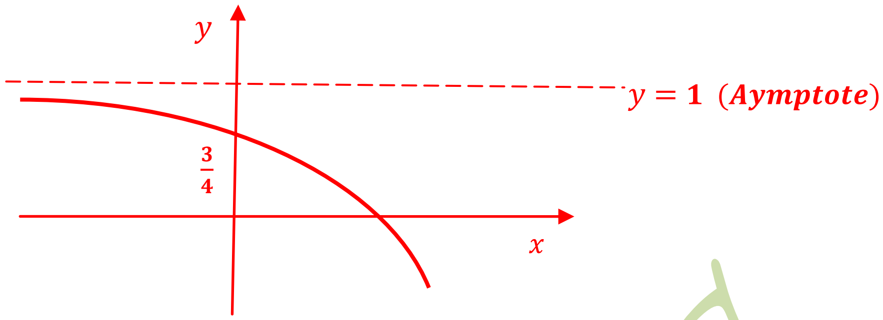


i) $y = e^{-x}$



Reflection along the $y - axis$

ii) $y = -\frac{1}{4}e^{x} + 1$

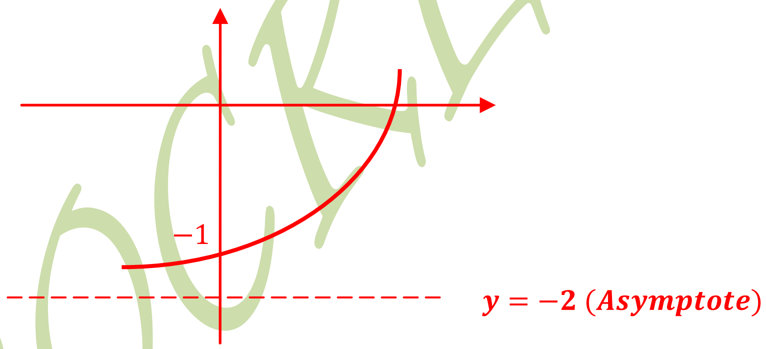


Stretch along the $y - axis$ with factor $\frac{1}{4}$

Reflection along the x-axis

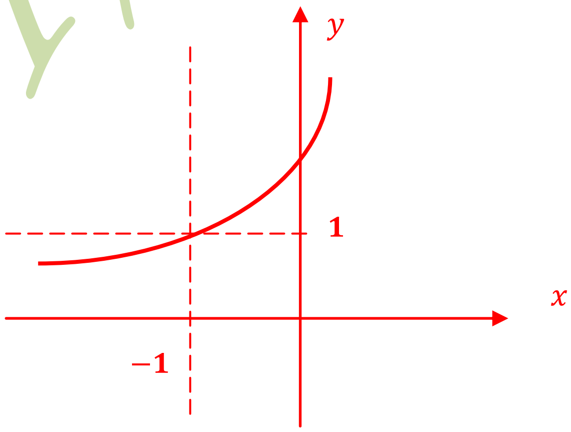
Translation y-axis moving 1 unit upwards

iii) $y = e^{x} - 2$



Translation along the $y - axis$ with translation vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

iv) $y = e^{x+1}$



Translation along the $x-axis$ with translation vector $\binom{-1}{0}$.

Question Seven

Sketch the graph of $f(x) = Cosx - 1$ for $0 \leq x \leq 2\pi$.

Hence on separate diagrams, sketch the graphs of the following functions and state the resultant transformation.

(i) $y = f\left(\frac{1}{2}x\right)$

(ii) $y = f\left(x - \frac{\pi}{2}\right)$

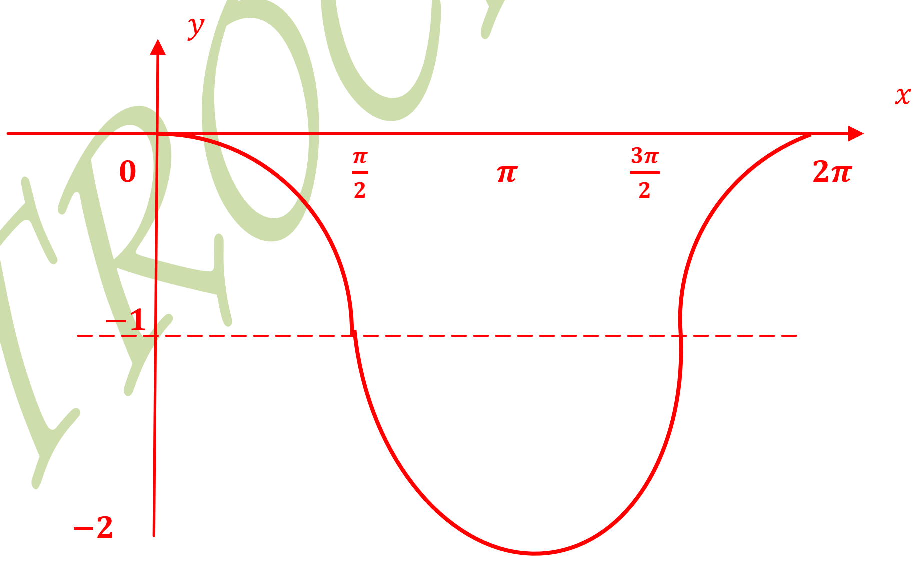
(iii) $y = 2f(x)$

(iv) $y = \left|f\left(x - \frac{\pi}{2}\right) + 1\right|$

Suggested solution

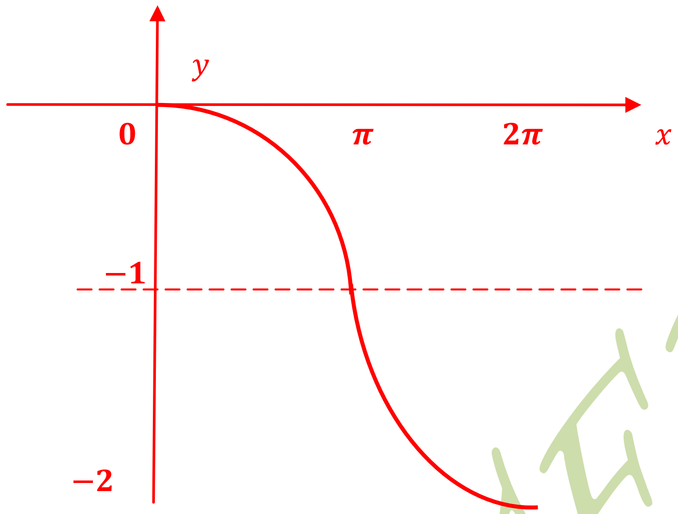
NB: $f(x)$ is a graph of $Cosx$ which has been translated 1 unit below the $y-axis$.

$y = f(x)$



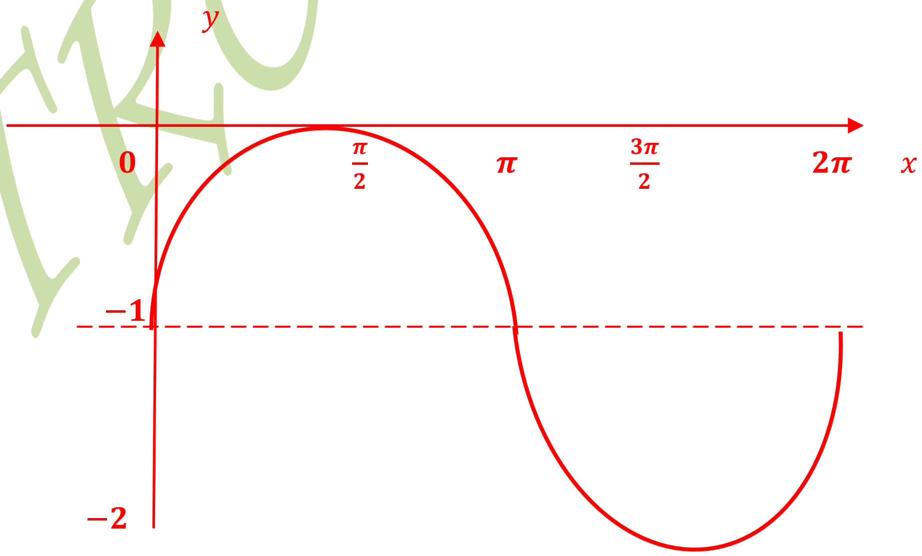
(i) $y = f\left(\frac{1}{2}x\right)$

Stretch along the $x - axis$ with factor $\frac{1}{2}$ or multiply every $x - coordinate$ by 2 or divide every $x - coordinate$ by $\frac{1}{2}$.



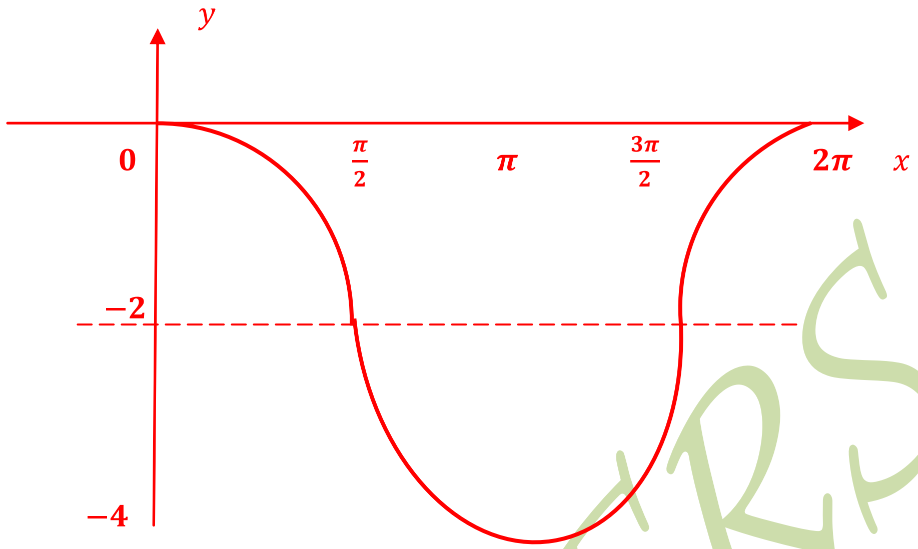
(ii) $y = f\left(x - \frac{\pi}{2}\right)$

Translation along the $x - axis$ with translation vector $\begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$ or moving $\frac{\pi}{2}$ $units$ rightwards.



(iii)$y = 2f(x)$

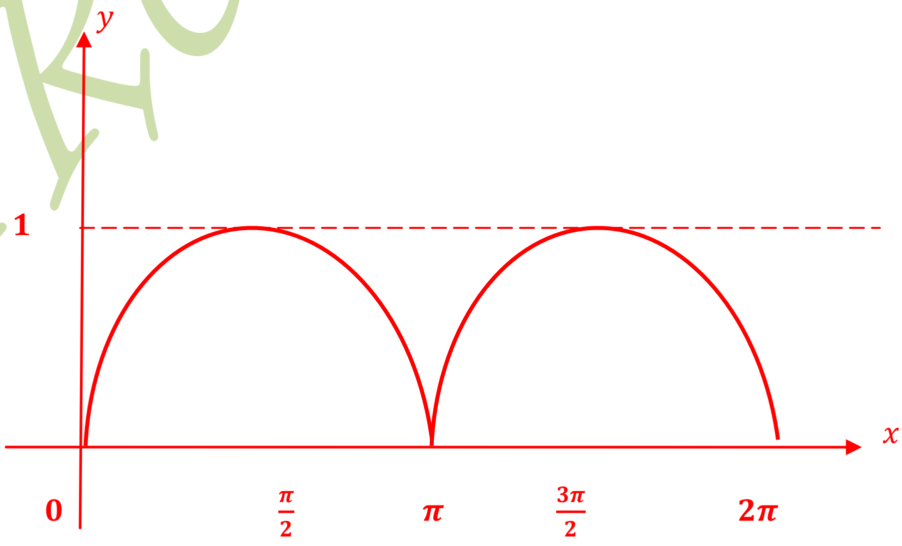
Stretch along the $y - axis$ with factor 2 or multiplying every $y - coordinate$ by 2.



(iv)$y = \left|f\left(x - \frac{\pi}{2}\right) + 1\right|$

(a) Translation along the $x - axis$ with translation vector $\begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$ or moving $\frac{\pi}{2}$ $units$ rightwards.

(b) Translation along the $y - axis$ with translation vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or moving 1 $unit$ upwards.



PAST EXAMINATION QUESTIONS

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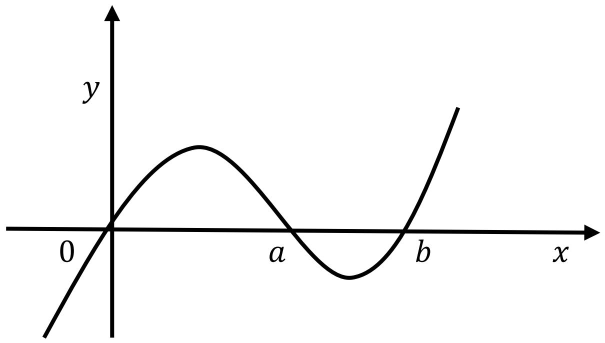


The diagram shows the graph of $y = f(x)$. The curve passes through the origin, and cuts the positive x −axis at the points where $x = a$ and $x = b$.

(i) Write down, in terms of a and b, the solution set of the inequality $f(x) > 0$. **[2]**

(ii) Sketch the graph of $y = |f(x)|$. **[2]**

UCLES JUNE 1997 PAPER 1

It is given that

$$f(x) \equiv (x - \alpha)(x + \beta), \quad x \in \Re,$$

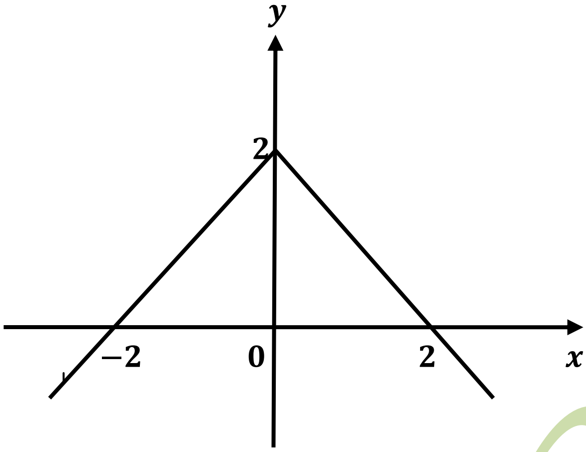
where α and β are positive constant. Sketch on separate diagrams the curves with the following equations, giving in each case the coordinates of the points at which the curve meets the x-axis.

(i) $y = f(x)$, **[2]**

(ii) $y = |f(x)|$, **[2]**

(iii) $y = f(x + 2\alpha)$. **[3]**

UCLES JUNE 1998 PAPER 1



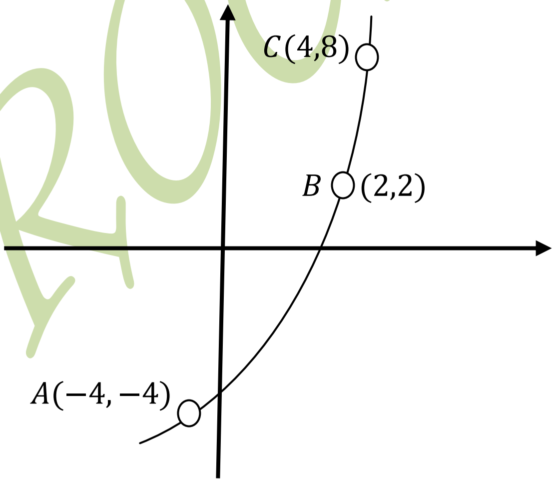
The diagram shows the graph of $y = f(x)$, where $f(x) = 2 - |x|$.

(i) Sketch the graph of $y = |f(x)|$. **[1]**

(ii) Hence solve the inequality $|f(x)| > 1$. **[3]**

UCLES NOVEMBER 2002 PAPER 1

The points $A(-4,-4)$, $B(2,2)$ and $C(4,8)$ are on the graph of $y = f(x)$ as shown in the diagram.



(i) Sketch the graph of $y = f\left(\frac{x}{2}\right)$, labelling clearly the points corresponding to A, B and C with their coordinates. **[2]**

(ii) Briefly explain why the point B is also on the graph of $y = f^{-1}(x)$. **[1]**

ZIMSEC NOVEMBER 2003 PAPER 1

Given that $(x) = \frac{x+4}{x+1}$, describe the geometrical transformations required to obtain the

graph of $f(x)$ from the graph $y = \frac{1}{x}$. **[4]**

ZIMSEC JUNE 2004 PAPER 1

(i) Sketch the graph of $f(x) = (x+2)(x-3)$. **[1]**

(ii) Sketch the graph of $y = |f(x)|$. **[1]**

(iii)By using the graphs of $y = |f(x)|$ and $y = |2x-2|$, or otherwise, solve the inequality $|2x-2| \geq |f(x)|$. **[4]**

ZIMSEC JUNE 2006 PAPER 1

(a) The graph of $y = x^2$ is first stretched in the positive x-direction by a scale factor of 4. It is then translated by 3 units in the negative x-direction.

Write the equation of the resulting graph. **[3]**

(b) Given that $y = f(x)$, where $f(x) = \frac{1}{x+1}, x > -1$

(i) find $y = f(x)$, **[3]**

(ii) state how the graph of $y = f(x)$ is related to the graph of $y = f(x)$, **[1]**

(iii)determine the range of $y = f(x)$, **[1]**

(iv)sketch the graph of $y = f(x) + 2$. **[2]**

ZIMSEC JUNE 2007 PAPER 1

Show $(x-3)$ is a factor of the function $f(x) = x^3 - 4x^2 - 3x + 18$. **[2]**

Factorise $f(x)$ completely. **[2]**

Hence sketch on separate diagrams, the graphs of

Write the equation of the resulting graph. **[3]**

(i) $y = f(x)$, showing clearly the coordinates of the points at which the graph meets the axes, **[1]**

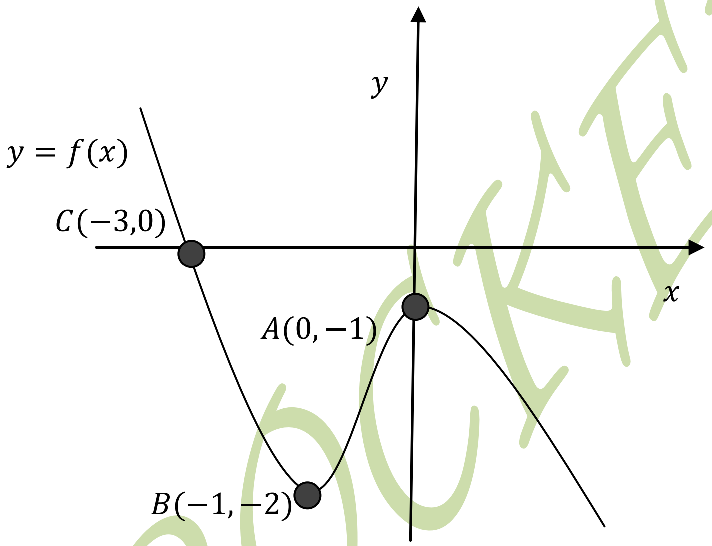
(ii) $y = f(x+3)$, showing clearly the coordinates of the points at which the graph meets the x- axis, **[1]**

(iii)$y = f(x) - 4$, showing clearly the coordinates of the point at which the graph meets the y- axis, **[1]**

State, without solving, the number of real roots of the equation $f(x) - 4 = 0$. **[1]**

ZIMSEC JUNE 2008 PAPER 1

The diagram shows the graph of $y = f(x)$.



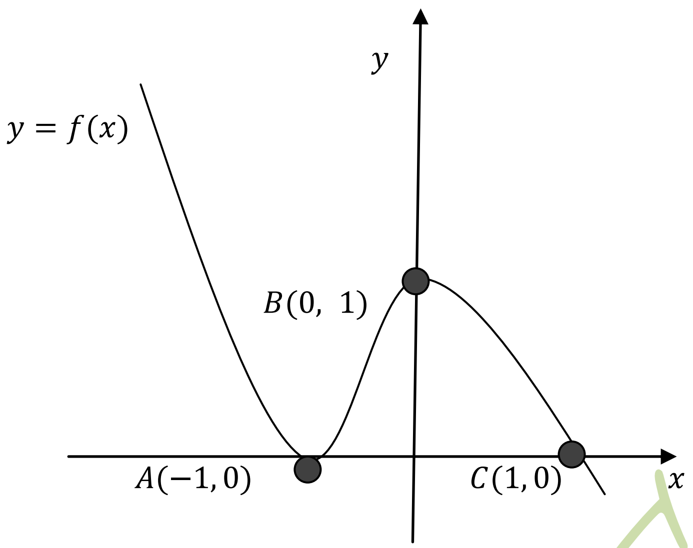
The points A, B and C have coordinates $(0,-1)$; $(-1,-2)$ and $(-3,0)$ respectively. Sketch on separate axes, the graphs of

(i) $y = f(x)$,

(ii) $y = f(x) + 2$.

Show clearly in each case the coordinates of the points corresponding to A, B and C. **[4]**

ZIMSEC JUNE 2011 PAPER 1



The diagram shows the graph of $y = f(x)$. Sketch on separate diagrams, showing the images of A, B and C, the graphs of

(i) $y = f(2x)$,

(ii) $y = f(-x) + 3$,

(iii) $y = 2f(x + 1)$. **[6]**

ZIMSEC NOVEMBER 2011 PAPER 1

Describe the transformations needed to transform the graph of $y = sinx$ onto the graph of $y = 6sin(2x) - \pi$. **[3]**

ZIMSEC JUNE 2015 PAPER 1

Given that $f(x) = -x^3 + 2x^2 + 3x - 6$.

(i) factorise $f(x)$ completely

(ii) sketch the curve $y = f(x)$ showing all the intersections with the axes. **[6]**

[**You need not find the turning points**]

Hence or otherwise, write down the solution of the inequality $f(x) > 0$. **[2]**

ZIMSEC JUNE 2016 PAPER 1

The function f is defined by

$$f(x) = (x-2)(x+3), x \in \Re.$$

Sketch on separate diagrams, showing clearly all the intercepts and turning points, the graphs of:

(i) $y = f(x)$. **[4]**

(ii) $y = |f(x)|$ **[2]**

(iii)$y = f(2x)$. **[2]**

(iv)$y = f(x-1)$. **[2]**

(v) $y = -2f(x)$. **[2]**

ZIMSEC NOVEMBER 2016 PAPER 1

(i) Given that $f(x) = 2 - |x+3|$, sketch the graph of $y = |f(x)|$ **[2]**

(ii) Hence from (**i**) solve the inequality $|f(x)| > 1$. **[2]**

ZIMSEC JUNE 2017 PAPER 1

State the geometrical transformations which map the graph of $y = x^2$ onto the graph

of $y = x^2 - 8x + 10$. **[3]**

ZIMSEC NOVEMEBER 2017 PAPER 2

Sketch, on the same axis, the graphs of $f(x) = cosx$ and $f(x) = cos(x - 60°)$ for

$0° \leq x \leq 360°$ showing clearly the intercepts with the axes. **[3]**

Hence state the geometrical relationship between the graph $f(x) = cosx$ and the graph of

$f(x) = cosx$. **[2]**

ZIMSEC NOVEMBER 2018 PAPER 2

The function f is defined by

$$f(x) = 1 + Cosx \qquad\qquad 0 \leq x < 2\pi.$$

Sketch on separate diagrams the graphs of the following functions showing clearly all intercepts and turning points.

(i) $y = f(x)$. **[2]**

(ii) $y = |f(x) - 3|$ **[3]**

(iii) $y = f\left(\frac{x}{2}\right)$. **[2]**

(iv) $y = f\left(x - \frac{\pi}{2}\right)$. **[2]**

(v) $y = -2f(x)$. **[3]**

ZIMSEC NOVEMBER 2019 PAPER 1

Given that $h(\theta) = ksin\theta; k < 0$.

a) Sketch the graph of $y = h(\theta)$ for $0 \leq \theta \leq 2\pi$. **[3]**

b) Describe the transformations which maps $f(x) = sinx$ onto $y = 6sin(x - \pi)$. **[3]**

ZIMSEC NOVEMBER 2019 PAPER 2

(a) Express $g(x) = \frac{2x-1}{x-3}$ in the form $g(x) = a + \frac{b}{x-3}$. **[2]**

(b) Hence state the correct sequence of transformation which transform the graph of $y = \frac{1}{x}$ onto the graph $y = g(x)$. **[4]**

(c) Sketch the graph of $y = g(x)$ showing clearly any asymptotes and intercepts with axes. **[4]**

ASANTE SANA

********THERE IS A LIGHT AT THE END OF EVERY TUNNEL ********

CONTRIBUTIONS ARE WELCOME; FEEL FREE TO CONTACT ME SO THAT WE CAN IMPROVE THE DOCUMENT TOGETHER.

ENJOY

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