TOPIC: ALGEBRA

SUBTOPIC: GROUP 1

OBJECTIVES

By the end of this subtopic learners should be able to;

- Define binary operation.
- Define closure, commutation, association and distribution of a binary operation.
- Identify the identity and inverse element of a binary operation.

Binary operation

- Binary means composed of two parts.
- Binary is a rule for combining two values to create a new value.
- Binary operation on a set is a calculation involving two elements of a set to produce another element of the set.
- A binary operation on set P is a function $*$ such that $r * q = y$ where r, q and y are elements of set P.
- The notion $(P;*)$ be used to indicate that set P has a binary operation.
- The symbol $*$ can be used to represent a law of binary composition.
- Basic binary operations are $\times; \div; +; -$.
- A binary operation on a finite set can be displayed o a **Cayley table or Latin square** which shows how operations are to be performed.

Example

A binary operation $*$ is defined on set $P = \{a_1; a_2; a_3; a_4\}$.

a) Construct the Cayley table to show all possible outcomes using a binary operation $*$.
b) Use the table to find $a_3 * a_2$.

Solution

a) $P = \{a_1; a_2; a_3; a_4\}$

$*$	a_1	a_2	a_3	a_4
a_1	a_4	a_3	a_2	a_1
a_2	a_3	a_1	a_4	a_2
a_3	a_2	a_4	a_1	a_3
a_4	a_1	a_2	a_4	a_2

To read the table;

- Read the first value from the left hand column and the second value from the top row.
- The answer is the intersection point.

b) $a_3 * a_2 = a_4$

Properties of binary operation

a) Closure

- A set P is closed under binary operation $*$ if , for every pair of elements r and q are elements of set P and $r * q$ is also an element of set P.
- For every ordered pair (r, q) for all $r, q \in P$ and the element $r * q$ is the element of set P, the operation $*$ is said to be closed binary operation on P.

Examples

1) Determine whether the set of natural numbers $(\mathbb{N})$ is closed under addition.

Solution

$\mathbb{N} = \{1,2,3..\}$

$(\mathbb{N};*)$ -------------- replace $*$ by addition sign.

$(\mathbb{N}; +)$

Using the notion $r * q = y \;\; \forall\, r, q, y \in \mathbb{N}$

Let $r = 1, q = 2$

Then $r * q = r + q$

$= 1 + 2$

$= 3$

$3 \in \mathbb{N}$

$\therefore$ The set of natural numbers is closed under addition.

2) Determine whether the set of positive integers is closed under division.

Solution

$\mathbb{Z}^+ = \{1; 2; 3; 4 \dots\}$

$(\mathbb{Z}^+;*)$--------------- replace $*$ by $\div$.

$(\mathbb{Z}^+;\div)$

Using $r * q = y \ \ \forall\, r, q, y \in \mathbb{Z}^+$

Let $r = 3, q = 5$

$r * q = r \div q$

$= 3 \div 5$

$= \frac{3}{5}$

$\frac{3}{5} \notin \mathbb{Z}^+$

$\therefore$The set of all positive numbers is not closed under division.

Modular Arithmetic

- Modular arithmetic is a set of remainders or residues obtained after the division of two positive integers.
- The set of integers modulo $\boldsymbol{n}$, often denoted as $\mathbb{Z}_n$ where n is a positive integer.
- Given that x and y are none zero integers, then
  $\frac{x}{n} = q + R$ where q is the quotient and R is a remainder.
  The focus is on the remainder after dividing x by n.
  $\therefore x \text{ mod } n = R$

In general:

If $\frac{x}{n} = q + R$, then

$x = nq + R$

$\therefore x \text{ mod } n = R$ for $0 \leq R < n$

Examples

1) Find the value of following.
   a) $3 \times 5 \text{ (mod 4)}$
   b) $30 + 4 \text{ (mod 8)}$
   c) $23 \times 4 \text{ (mod 11)}$

Solution

a) $3 \times 5 = 15$
   $15 = 4(3) + 3$
   $\therefore 15 \text{ mod } 4 = 3$

b) $30 + 4 = 34$
   $34 = 8(4) + 2$
   $\therefore 34 \text{ mod } 8 = 2$

c) $23 \times 4 = 92$
   $92 = 11(8) + 4$
   $\therefore 92 \bmod 11 = 4$

2) Evaluate $(2 + 5) \times 11 \pmod{17}$.

Solution

$(4 + 5) \times 11 = 9 \times 11$

$99 = 17(5) + 14$

$\therefore 99 \text{ mod } 17 = 14$

Example

1) Determine whether the given $*$ is closed on set R.
   a) $R = \{1; 2; 3; 4; 5\}$, addition (mod 6)
   b) $R = \{1; 2; 3; 4\}$, multiplication (mod 5)
   c) $R = \{1; 5; 6; 18; 21\}$, $a * b = ab, \forall\, a, b \in R$

Solution

a) $(R_6; +) = \{1; 2; 3; 4; 5\}$

$\therefore *$ is closed on set $\mathbb{Z}$.

b) $(R_5; \times) = \{1; 2; 3; 4\}$

$\therefore *$ is closed on set R

c) $R = \{1; 5; 6; 18; 21\}$

Using the system $a * b = ab$ to check for closure.

Let $a = 6, b = 18$

$a * b = ab$

$6 * 18 = (6)(18)$

$= 108$

$108 \notin R$

$\therefore *$ is not closed on set R.

b) Commutativity

- A binary operation $*$ on set P is said to be commutative if;

  $\boldsymbol{r * q = q * r}, \quad \forall\, \boldsymbol{r, q \in P}.$

- The order of the pair does not affect the result.

Examples

1) Determine whether the multiplication is commutative on a set of integers.

Solution

$\mathbb{Z} = \{\ldots - 2; -1; 0; 1; 2 \ldots\}$

$(\mathbb{Z}; \times)$

For commutativity $r * q = q * r, \forall\, r, q \in \mathbb{Z}$

Let $r = -2, q = 1$

Considering LHS

$r * q = (-2) \times 1$

$\quad\quad = -2$

Considering RHS

$q * r = 1 \times (-2)$

$\quad\quad = -2$

LHS $=$ RHS

$\therefore$ Multiplication is commutative on a set of integers.

2) Which of the following sets are commutative on the given law of binary operation.
   a) Integers; addition
   b) Natural numbers ; division

Solution

a) $\mathbb{N} = \{\ldots -2; -1; 0; 1; 2; 3; 4; \ldots\}$

$(\mathbb{N};*)$ ---------------- substitute $+$ for $*$.

$(\mathbb{N};+)$

For commutativity $r * q = q * r\ \forall\, r, q \in \mathbb{N}$

Let $r = 2, q = 3$

LHS $= r * q$

$r * q = r + q$

$\quad\quad = 2 + 3$

$\quad\quad = 5$

RHS $= q * r$

$q * r = q + r$

$\quad\quad = 3 + 2$

$\quad\quad = 5$

LHS $=$ RHS

$\therefore$ Addition is commutative on a set of integers.

b) $\mathbb{N} = \{1; 2; 3 \ldots\}$

$(\mathbb{N};*)$ --------------------- replace $*$ by $\div$.

$(\mathbb{N};\div)$

For commutativity $r * q = q * r, \ \forall\, r, q \in \mathbb{Z}$

Let $r = 1, q = 2$

LHS $= r * q$

$r * q = r \div q$

$= (1) \div (2)$

$= \frac{1}{2}$

RHS $= q * r$

$q * r = q \div r$

$= (2) \div (1)$

$= 2$

LHS $\neq$ RHS

$\therefore$ Division is not commutative on a set of natural numbers.

3) Let $*$ be a binary operation on $\mathbb{N}$ is defined by $x * y = x + 2y - 1 \ \forall\, x, y \in \mathbb{N}$
   a) Prove that the operation is binary.
   b) Determine whether the operation is commutative.

Solution

a) $\mathbb{N} = \{1; 2; 3; 4 \ldots\}$

Condition for any binary operation: $r * q = y \ \forall\, r, q, y \in \mathbb{N}$

Using $x * y = x + 2y - 1$

Let $x = 2, y = 3$

$2 * 3 = 2 + 2(3) - 1$

$= 2 + 6 - 1$

$= 7$

$2 * 3 = 7$

$\therefore$ The operation is binary because 3,2 and 7 are elements of natural numbers.

b) For commutativity $r * q = q * r$

LHS $= r * q$

But $x * y = x + 2y - 1$

$r * q = r + 2(q) - 1$

$= r + 2q - 1$

LHS $= r + 2q - 1$

RHS $= q * r$

$q * r = q + 2(r) - 1$

$= q + 2r - 1$

LHS $\neq$ RHS

$\therefore$ The operation is not commutative.

4) Given that $R = \{1; 2; 3; 4\}$, multiplication mod 5.
   a) Draw the Caley table to show all possible outcomes.
   b) State whether the set R is commutative or not.

Solution

a) $(R_5;\times) = \{1; 2; 3; 4\}$

How to obtain elements of the table i.e.

3×4 mod 5

$12 = 5(2) + 2$

3×4 (mod 5) $= 2$

2×4 mod 5.

$8 = 5(1) + 3$

2×4 mod $5 = 3$

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

b) Use the table to check for commutative

For commutativity $r * q = q * r$.

LHS $= r * q$

Let $r = 2, q = 3$

From the table $2 * 3 = 1$

RHS $= q * r$

From the table $3 * 2 = 1$

LHS $=$ RHS

$\therefore$ The set R is commutative on multiplication mod 5.

c) **Associativity**

- The binary operation $*$ defined on set P is associative if;

  $\boldsymbol{r * (q * y) = (r * q) * y}$, for q, r and y are elements of set P.

- The position of the brackets does not matter.
- Multiplication and addition of all real numbers is associative but division is not associative.
- Both multiplication and addition of complex number are associative.

Examples

1) $*$ is defined on $\mathbb{Z}$ by $r * q = r - q$, determine whether the operation is associative or not.

Solution

For associativity $r * (q * y) = (r * q) * y, \forall\, r, q, y \in \mathbb{Z}$

LHS $= r * (q * y)$

Using the system $r * q = r - q$

$(q * y) = q - y$, then

$r * (q * y) = r * (q - y)$

$= r - (q - y)$

$= r - q + y$

RHS $= (r * q) * y$

$(r * q) = r - q$, then

$(r * q) * y = (r - q) * y$

$= (r - q) - y$

$= r - q - y$

LHS$\neq$ RHS

$\therefore$ The operation $(*)$ is not associative.

2) The operation $(*)$ is defined on $\mathbb{N}$ by $a * b = 2ab$, check whether the operation is associative or not.

Solution

For associativity $r * (q * y) = (r * q) * y, \forall\, r, q, y \in \mathbb{N}$

LHS $= r * (q * y)$

Using the system $a * b = 2ab$

$(q * y) = 2qy$, then

$r * (q * y) = r * (2qy)$

$= 2r(2qy)$

$= 4qry$

RHS $= (r * q) * y$

$r * q = 2rq$, then

$(r * q) * y = (2rq) * y$

$= 2(2rq)y$

$= 4rqy$

LHS$=$ RHS

$\therefore$ The operation $(*)$ is associative.

Sets

Union of sets

If A, B,and C are three subsets of the universal set, then

$A \cup (B \cup C) = (A \cup B) \cup C$

$\therefore$ Union of sets is associative on binary operation.

Intersection of sets

If A, B and C are three subsets of the universal set, then

$A \cap (B \cap C) = (A \cap B) \cap C$

$\therefore$ Intersection of sets is associative on binary operation.

Matrices

- Addition of matrices is associative.
- Multiplication of (compatible) matrices is also associative.
- Compatible matrices are matrices which can be multiplied.

Example

Given that A $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, B$= \begin{pmatrix} q & f \\ g & h \end{pmatrix}$ and C$= \begin{pmatrix} m & n \\ x & y \end{pmatrix}$, use matrix A, B and C to show that addition of matrices is associative.

Solution

For associativity $A * (B * C) = (A * B) * C$

LHS $= A * (B * C)$ -------------------------------- substitute $+$ for $*$.

$= A + (B + C)$

$$= A + \left(\begin{pmatrix} q & f \\ g & h \end{pmatrix} + \begin{pmatrix} m & n \\ x & y \end{pmatrix} \right)$$

$$= A + \begin{pmatrix} q+m & f+n \\ g+x & h+y \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} q+m & f+n \\ g+x & h+y \end{pmatrix}$$

$$= \begin{pmatrix} a+q+m & b+f+n \\ c+g+x & d+h+y \end{pmatrix}$$

RHS $= (A * B) * C$

$$= (A + B) + C$$

$$= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} q & f \\ g & h \end{pmatrix} \right) + C$$

$$= \begin{pmatrix} a+q & b+f \\ c+g & d+h \end{pmatrix} + \begin{pmatrix} m & n \\ x & y \end{pmatrix}$$

$$= \begin{pmatrix} a+q+m & b+f+n \\ c+g+x & d+h+y \end{pmatrix}$$

LHS $=$ RHS

∴ Addition of matrices is associative.

d) **Distribution**

- The binary operation $*$ defined on set P is distributive if $*$ both left and right is distributive.
- The Distributive Law says that multiplying a number by a group of numbers added together is the same as doing each multiplication separately, for example

  $2 \times (3 + 7) = 2 \times 3 + 2 \times 7$

  LHS $= 2 \times (3 + 7)$

  $= 20$

  RHS $= 2 \times 3 + 2 \times 7$

  $= 6 + 14$

$= 20$

LHS $=$ LHS

- Let $*$ and $+$ be binary operations on a set P. Then $*$ is said to be left distributive if;
  $\boldsymbol{a * (b + c) = (a * b) + (a * c)}$**,**$\forall\, \boldsymbol{a, b, c \in P.}$
- $*$ is said to be right distributive if;
  $\boldsymbol{(a + b) * c = (a * c) + (b * c), \forall a, b, c \in P.}$

Example

Show that standard multiplication $(\cdot)$ and standard addition $(+)$ on $\mathbb{R}$ is distributive.

Solution

Left distributive: $a * (b + c) = (a * b) + (a * c)$,$\forall\, a, b, c \in \mathbb{R}$

$= (a * b) + (a * c)$ ---------------- substitute $(\cdot)$ for $*$.

$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Right distribution: $(a + b) * c = (a * c) + (b * c)$

$= (a * c) + (b * c)$ ------------------- substitute $(\cdot)$ for $*$.

$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

Therefore standard multiplication and addition on R is distributive.

e) **Identity element (Neutral element)**

- An identity element (e) exist under operation $*$ defined on a set P if;
  $\boldsymbol{a * e = e * a = a}$**,** where a and e are elements of set P.
- The identity matrix of a 2 by 2 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- The later I can be used to denote the identity element.
- The identity element when multiplying real numbers is 1, for example
  $2 \times 1 = 1 \times 2 = 2$
- Zero (0) is the identity element when adding real numbers, for example
  $0 + 3 = 3 + 0 = 3$

Examples

Given that $* \, b = a + b - 1 \;\forall\, x, y \in \mathbb{Q}$.

Determine whether the system $(\mathbb{Q};*)$ possesses the identity element or not.

Solution

For an identity element to exist $a * e = e * a = a$

LHS $= a * e$

$a * e = a$

Using the system $a * b = a + b - 1$, then

$a * e = a$

$a + e - 1 = a$

$e - 1 = a - a$

$e = 1$

RHS $= e * a$

$e * a = a$

$e + a - 1 = a$

$e = 1$

LHS= RHS

∴The identity element exist and the identity element is 1.

f) **Inverse element**

- If $\boldsymbol{a}$ is an element of set P, then the element $\boldsymbol{a^{-1}}$ is the inverse element of a with respect to the binary operation such that;

  $\boldsymbol{a} * \boldsymbol{a^{-1}} = \boldsymbol{e}$ where $\boldsymbol{a}, \boldsymbol{a^{-1}}$ and $\boldsymbol{e}$ are elements of set P.

- The operation of an element $(\boldsymbol{a})$ and its inverse $(\boldsymbol{a^{-1}})$ gives an identity element$(\boldsymbol{e})$.
- a^{-1} is just an inverse element not $\frac{1}{a}$.
- When adding integers , the inverse of an integer $\boldsymbol{q}$ is $-\boldsymbol{q}$, for example;
  The additive inverse of $(+3)$ is (-3).
- When multiplying non zero rational numbers , the inverse of $\frac{\boldsymbol{x}}{\boldsymbol{y}}$ is $\frac{\boldsymbol{y}}{\boldsymbol{x}}$, for example

  The multiplicative inverse of $\frac{3}{7}$ is $\frac{7}{3}$.

Examples

1) The Cayley table below shows all possible outcomes of set $Z = \{1; 5; 7; 11\}$ under multiplication mod 12.

$\times$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

a) Find the identity element.
b) Find the inverse.

Solution

a) For identity element: $a * e = e * a = a$

$5 * 1 = 5$

$1 * 5 = 5$

$e = 1$

$\therefore$ The identity element is 1.

b) For inverse element: $a * a^{-1} = e$

$1 * 1 = 1$

$1 * 1 = e$

The inverse of 1 is 1 (self inverse).

$5 * 5 = 1$

$5 * 5 = e$

The inverse of 5 is 5 (self inverse).

$7 * 7 = 1$

$7 * 7 = e$

The inverse of 7 is 7 (self inverse).

$11 * 11 = 1$

$11 * 11 = e$, then

The inverse of 11 is 11 (self inverse).

TOPIC: ALGEBRA

SUBTOPIC: GROUPS 2

Objectives

By the end of the subtopic learners should be able to;

- Define a group.
- Determine whether the given structure is a group or not using basic properties of a group.
- Solve problems involving binary operations and properties of a group.

Groups

A **group** (G;$*$) is a non-empty set G with binary operation $*$ with the following basic properties(axioms);

i) Set G is **closed** under binary operation $*$ i.e. $r * q \in G$ for r and q are elements of set G.
ii) Binary operation $*$ is **associative** such that $r * (q * y) = (r * q) * y \; \forall \; r, q, y \in G$.
iii) There is an **identity** element in set G such that $e * a = a * e = a \; \forall \; a, e \in G$.
iv) Every element in set G has an **inverse** (a^{-1}) i.e. $a * a^{-1} = a^{-1} * a = e$ for a, a^{-1} and e are elements of set G.

Any given structure is said to be a group if the given structure satisfies all basic properties above.

Examples

1) Determine whether a set of integers under addition forms a group.

Solution

Checking for closure

For closure a$* \; b = c \; \forall \; a, b\,, c \in \mathbb{Z}$.

Let $a = 3, b = 5$

$3 + 5 = 8$

$8 \in \mathbb{Z}$

∴ Set of integers is closed under addition.

Checking for associative.

For associative $r * (q * y) = (r * q) * y\,, \forall \; r, q, y \in \mathbb{Z}$.

Let $r = 3, q = 5, y = -1$

LHS $= r * (q * y)$

$= r + (q + y)$

$= 3 * (5 + (-1))$

$= 3 + 4$

$= 7$

RHS $= (r * q) * y$

$(r + q) * y = (3 + 5) * (-1)$

$= 8 + (-1)$

$= 7$

LHS$=$ RHS

∴ Addition is associative on integers.

The identity element is zero.

If a is an element of set $\mathbb{Z}$, then the inverse of a is $(-a)$ i.e. the inverse of 2 is -2.

∴ Set of integers is a group under addition.

2) Determine whether each of the following sets forms a group.
   a) $\mathbb{R}$,the set of real numbers under the operation $a * b = 2a - b + 1$.
   b) P$=$ $\{0; 1; 2; 3\}$, under multiplication mod 4.
   c) $\mathbb{Q}$, set of non-zero rational numbers under the operation $*$ defined by $a * b = 3ab$.

Solution

a) $a * b = 2a - b + 1, \forall\, a, b \in \mathbb{R}$.

Checking for closure

Let $x = 15, y = 2$

$x * y = 2x - y + 1$

$15 * 2 = 2(15) - 2 + 1$

$= 29$

$29 \in \mathbb{R}$

∴ The set of real numbers is closed under binary operation $*$.

Checking for associative

For associative :$r * (q * y) = (r * q) * y$

LHS $= r * (q * y)$

Considering $q * y$.

Using the formula; $a * b = 2a - b + 1$.

By comparison $a = q, b = y$.

$q * y = 2q - y + 1$, then

$r * (q * y) = r * (2q - y + 1)$

By comparison $a = r, b = 2q - y + 1$.

$r * (q * y) = 2r - (2q - y + 1) + 1$

$= 2r - 2q + y - 1 + 1$

$= 2r - 2q + y$

RHS $= (r * q) * y$

Considering $r * q$.

By comparison $a = r, b = q$.

$r * q = 2r - q + 1$,then

$(r * q) * y = (2r - q + 1) * y$

By comparison $a = 2r - q + 1, b = y$.

$(r * q) * y = 2(2r - q + 1) - y + 1$

$= 4r - 2q + 2 - y + 1$

$= 4r - 2q - y + 3$

LHS $\neq$ RHS

$\therefore$ The binary operation $*$ defined on $\mathbb{R}$ is not associative.

$\therefore$Set $\mathbb{R}$ is not a group.

b) P= $\{0; 1; 2; 3\}$, multiplication mod 4.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Use the table above to check for closure.

There is no new element in table.

∴ Set P is closed under addition mod 4.

Checking for associative

For associative: $r * (q * y) = (r * q) * y$

Let $r = 1, q = 2, y = 3$

LHS $= r * (q * y)$

Considering $(q * y)$

$q * y = 2 * 3$

$= 1$

$r * (q * y) = 1 * (2 * 3)$

$= 1 * 1$

$= 2$

RHS $= (r * q) * y$

Considering $r * q$

$r * q = 1 * 2$

$= 3$

$(r * q) * y = (1 * 2) * 3$

$= 3 * 3$

$= 2$

LHS =RHS

∴Set P is associative under addition mod 4.

The identity element is zero.

Inverses: Inverse of zero is zero

Inverse of 1 is 3.

Inverse of 2 is 2.

Inverse of 3 is 1.

∴ Every element has its own inverse.

∴Set P under addition mod 4 is a group.

c) $a * b = 3ab\ \forall a, b \in \mathbb{Q}$

Checking for closure

For closure $a * b = c\ \forall\ a, b, c \in \mathbb{Q}$

Let $a = 2, b = 7$

$a * b = 3ab$

$2 * 7 = 3(2)(7)$

$= 42$

$42 \in \mathbb{Q}$

∴Binary operation $*$ is closed under set $\mathbb{Q}$.

Checking for associative

For associative: $r * (q * y) = (r * q) * y$

LHS $= r * (q * y)$

Using the formula $a = 3ab$.

Considering $q * y$

By comparison $a = q, b = y$

$q * y = 3qy$

$r * (q * y) = r * (3qy)$

By comparison $a = r, b = 3qy$

$r * (3qy) = 3(r)(3qy)$

$= 9rqy$

RHS $= (r * q) * y$

Considering $r * q$

By comparison $a = r, b = q$

$r * q = 3rq$

$(r * q) * y = (3rq) * y$

By comparison $a = 3rq, b = y$.

$(3rq) * y = 3(3rq)(y)$

$= 9rqy$

LHS=RHS

∴ Set $\mathbb{Q}$ is associative on binary operation $*$.

Identity element

For identity element $a * e = e * a = a$.

LHS $= a * e$

$a * e = a$

By comparison $a = a, b = e$.

$a * e = a$

$3(a)(e) = a$

$e = \frac{1}{3}$

RHS $= e * a$

$e * a = a$

By comparison $a = e, b = a$.

$e * a = a$

$3(e)(a) = a$

$e = \frac{1}{3}$

The identity element is $\frac{1}{3}$.

∴ Identity element exist on binary operation defined on set $\mathbb{Q}$.

Inverse element

For inverse element: $a * a^{-1} = a^{-1} * a = e$.

LHS $= a * a^{-1}$

By comparison $a = a, b = a^{-1}$.

$a * a^{-1} = e$ ------------------------------ but $e = \frac{1}{3}$.

$3(a)(a^{-1}) = \frac{1}{3}$

$a^{-1} = \frac{1}{9a}$

RHS $= a^{-1} * a$

By comparison $a = a^{-1}, b = a$.

$a^{-1} * a = e$ ----------------- but $e = \frac{1}{3}$.

$3(a^{-1})(a) = \frac{1}{3}$

$a = \frac{1}{9a}$

LHS = RHS

The inverse element is $\frac{1}{9a}$.

∴The inverse element exist on binary operation defined on set $\mathbb{Q}$.

∴ Set $\mathbb{Q}$ is a group.

Abelian Group.

- A group (G;∗) which is commutative is called **Abelian group**.
- Check for commutative in order to determine whether the given structure is an Abelian group or not.
- All basic properties of a group are still valid.

Example

Given that R $= \{2; 4; 6; 8)$,multiplication.

a) Compile Cayley table to show all possible outcomes of set R.
b) Determine whether set R is a group or not
c) Show that set R is an Abelian group.

Solution

a) If the mod is not given, the mod is 10.

$\times$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

Leading diagonal which shows that the set R is commutative.

b) Check all basic properties of a group.

Checking for closure.

For closure a$* \, b = c, \forall a, b, c \in R$.

Let $a = 2, b = 6$

$a * b = 2 * 6$

$= 2$

$2 \in R$

∴ Set R is closed under multiplication mod 10.

Checking for associative

For associative $r * (q * y) = (r * q) * y$

Let $r = 2, q = 4, y = 8$

LHS $= r * (q * y)$

$= 2 * (4 * 8)$

$= 2 * 2$

$= 4$

RHS $= (r * q) * y$

$= (2 * 4) * 8$

$= 8 * 8$

$= 4$

RHS $=$ LHS

∴ Multiplication is associative on set R.

Identity element

The identity element is 6.

∴ The identity element exist on set R.

Inverse element:

Inverse of 2 is 8

Inverse of 4 is 4.

Inverse of 6 is 6.

Inverse of 8 is 2.

∴ Each element has its own inverse .

∴ Set R is a group.

c) Checking for commutative.

The group is symmetrical about the leading diagonal .

∴ The group is commutative.

∴ Set R is an Abelian group.

Exercise 1

1) Determine whether the following sets are closed under the given law of binary operation.
   a) H= $\{5; 7; 9; 12\}$; addition mod 13.
   b) Complex numbers; multiplication.
   c) Real numbers ; division.

Solution

a) H= $\{5; 7; 9; 12\}$, addition mod 13.
   $(H_{13}; +)$
   For closure $a * b = c\ \forall a, b, c \in H$.
   Let $a = 5, b = 12$.
   $a + b = 5 + 12$
   $\quad\quad = 17$
   $17 \text{ mod } 13 = 4$
   $4 \notin$H
   ∴ Set H is not closed under addition mod 13.
b) Any number is a complex number , then complex numbers are closed under multiplication.
c) Real numbers is closed under division.

2) Given that $H = \{2; 3; 4; 6\}$, $a * b = 2a + b - 4\ \forall\ a, b \in H$, check whether $*$ is closed to set H or not.

Solution

$H = \{2; 3; 4; 6\}, a * b = 2a + b - 4\ \forall a, b \in H$

Using $a * b = 2a + b - 4$ to check for closure.

Let $a = 3, b = 6$

$3 * 6 = 2(3) + 6 - 4$

$\quad\quad = 8$

$8 \notin H$

∴ $*$ is not closed on set H.

3) A binary operation $(*)$ is defined on a set $G = \{1; 2; 3\}, a * b = 2(a - b)\ \forall\ a, b \in G$. Show that the binary operation $(*)$ is not closed on set G.

Solution

$G = \{1; 2; 3\}, a * b = 2(a - b)\ \forall a, b \in G$

Using $a * b = 2(a - b)$ to check for closure.

Let $a = 2, b = 3$

$2 * 3 = 2(2 - 3)$

$= -2$

$-2 \notin G$

$\therefore$ Binary operation $(*)$ is not closed on set G.

4) $*$ is defined on a set of natural numbers by $q * r = q^{r-1}, \forall\, q, r \in \mathbb{N}$, evaluate
   a) $3 * 2$
   b) $10 * 1$
   c) $2 * (5 * 3)$

Solution

a) $q * r = q^{r-1} \forall\, q, r \in \mathbb{N}$
   $3 * 2 = q * r$
   $q = 3, r = 2$
   $3 * 2 = 3^{2-1}$
   $= 3$

b) $q * r = q^{r-1}$
   $10 * 1 = q * r$
   $q = 10, r = 1$
   $10 * 1 = 10^{1-1}$
   $= 1$

c) $q * r = q^{r-1} \forall$
   $5 * (2 * 3)$
   Consider $(2 * 3)$
   $q = 2, r = 3$
   $5 * (2 * 3) = 5 * (2^{3-1})$
   $= 5 * 4$
   For $5 * 4, q = 5, r = 4$
   $5 * 4 = q^{r-1}$
   $= 5^{4-1}$
   $= 125$

5) Find the value of the following,
   a) $3 \times 7 \pmod 4$.

b) $4 \times 2 + 21 \pmod{11}$.
c) $6 + 5(100 \div 4) \pmod{15}$

Solution

a) $3 \times 7 \pmod 4$

$21 = 4(5) + 1$

$21 \text{ mod } 4 = 1$

b) $4 \times 2 + 21 \pmod{11}$

$29 = 11(2) + 7$

$29 \text{ mod } 11 = 7$

c) $3 + 5(100 \div 4)\ (\text{ mod } 15)$

$128 = 15(8) + 8$

$128 \text{ mod } 15 = 8$

6) It is given that $a * b = a + (b + 1), \forall\ a, b \in \mathbb{R}$. Show that $*$ is commutative.

Solution

For commutative $r * q = q * r$

Using the system $a * b = a + (b + 1)$

LHS $= r * q$

$r * q = r + (q + 1)$

$= r + q + 1$

RHS $= q * r$

$= q + (r + 1)$

$= q + r + 1$

LHS $=$ RHS

$\therefore *$ is commutative on $\mathbb{R}$.

7) The Cayley table shows results of set $G = \{m; n; p; q\}$ under the binary operation $*$.

$*$	m	n	p	q
m	n	m	q	p
n	m	n	p	q
p	q	p	m	n
q	p	q	n	m

Use the table above to simplify the following

a) $n * m$
b) $m * p$
c) $n * (m * q)$
d) $(m * p) * m$
e) $(m * n) * (p * q)$

Solution

a) $n * m = m$
b) $m * p = q$
c) $n * (m * q) = n * p$
$= p$

d) $(m * p) * m = q * m$
$= p$

e) $(m * n) * (p * q) = m * n$
$= m$

8) A binary operation $(*)$ is defined on $\mathbb{R}$, $a * b = \sqrt{a^2 + b} + ab \ \forall \ a, b \ \in \mathbb{R}$.
Check whether the binary operation $(*)$ is commutative or not.

Solution

For commutative : $r * q = q * r$

LHS $= r * q$

Using the system $a * b = \sqrt{a^2 + b} + ab$

$r * q = \sqrt{r^2 + q} + rq$

RHS $= q * r$

$q * r = \sqrt{q^2 + r} + qr$

LHS $\neq$ RHS

∴ Binary operation $(*)$ is not commutative.

9) Show that the set $B = \{1; 3; 5; 7\}$ is commutative under multiplication $(\text{mod } 8)$.

Solution

Using the Cayley table to check for commutative.

$\{1; 3; 5; 7\}$, multiplication $(\text{mod } 8)$.

$\times$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Leading diagonal.

∴Set B is commutative because it is symmetric to the leading diagonal.

10) $*$ is defined on $\mathbb{N}$ by $a * b = a + b + |ab|$, show that binary operation $(*)$ defined on $\mathbb{N}$ is commutative.

Solution

For commutative: $r * q = q * r$

$\text{LHS} = r * q$

Using the system $a * b = a + b + |ab|$

$r * q = r + q + |rq|$

$\text{RHS} = q * r$

$q * r = q + r + |qr|$

$\text{LHS} = \text{RHS}$

$\therefore$Binary operation $(*)$ is commutative on $\mathbb{N}$.

11) Compute the Cayley table of the following sets under the given law of binary operation.
   a) $\{0; 1; 2; 3; 4; 5; 6\}$, addition (mod 7).
   b) $H = \{1; -1; i; -i)$, multiplication.

Solution

a) $\{0; 1; 2; 4; 5; 6)$, addition (mod 7)

$+$	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

b) H $= \{1; -1; i; -i\}$

$\times$	1	-1	-i	i
1	1	-1	-i	i
-1	-1	1	i	-i
-i	-i	i	-1	-1
i	i	-1	1	-1

Exercise 2

1) The binary operation $(*)$ on $\mathbb{N}$ is given by $a * b = 2ab - 1, \forall\, a, b \in \mathbb{N}$.
   a) Evaluate
      i) $2 * 3$

ii) $4 * 1$
iii) $(3 * 1) * 5$

b) Solve the following equations
i) $7 * y = 55$
ii) $(2 * y) * 4 = 23$

Solution

a) Using the system $a * b = 2ab - 1$

i) $2 * 3 = 2(2))(3) - 1$
$= 11$

ii) $4 * 1 = 2(4)(1) - 1$
$= 7$

iii) $(3 * 1) * 5$---------- firstly simply elements inside brackets.
$(3 * 1) = 2(3)(1) - 1$
$= 5$
Then $(3 * 1) * 5 = 5 * 5$
$= 2(5)(5) - 1$
$= 49$

b) Using the system $a * b = 2ab - 1$ to solve;

i) $7 * y = 55$

By comparison $a = 7, b = y$, then

$2(7)(y) - 1 = 55$

$14y - 1 = 55$

$14y = 56$

$y = 4$

ii) $(y * 2) + 4 = 23$

Considering $(y * 2)$,

By comparison $a = y, b = 2$, then

$y * 2 = 2(2)(y) - 1$

$= 4y - 1$

$(y * 2) * 4 = 23$

$(4y - 1) * 4 = 23$

By comparison $a = 4y - 1, b = 4$

$2(4y-1)(4)-1=23$

$8(4y-1)=24$

$4y-1=3$

$4y=4$

$y=1$

2) Binary operation $*$ is defined on $\mathbb{Q}$ by $a*b=2(a+b)+1, \forall a,b \in \mathbb{Q}$.

Check whether binary operation$(*)$ defined on $\mathbb{Q}$ is associative or not.

Solution

$a*b=2(a+b)+1, \forall\, a,b \in \mathbb{Q}$

For associativity: $r*(q*y)=(r*q)*y$

LHS $=r*(q*y)$

$=r*(2(q+y)+1)$

$=r*(2q+2y+1)$

$=2(r+2q+2y+1)+1$

$=2r+4q+4y+2+1$

$=2r+4q+4y+3$

RHS $=(r*q)*y$

Firstly simplify $(r*q)$

By comparison $a=r, b=q$.

$(r*q)=2(r+q)+1,$

$=2r+2q+1$ then

$(r*q)*y=(2r+2q+1)*y$

By comparison $a=2r+2q+1, b=y$.

$(r*q)*y=2(2r+2q+1+y)+1$

$=4r+4q+2+2y+1$

$=4r+4q+2y+3$

LHS $\neq$ RHS

$\therefore$ Binary operation $(*)$ defined on $\mathbb{Q}$ is not associative.

3) Compute the Cayley table for the set $\{3; 6; 9; 12\}$ under multiplication $(\text{mod } 15)$.

Use the table to find,

a) The identity element.
b) The inverse of each element.

Solution

Mod 15

To obtain all elements of the set use the formula $y \text{ mod } n = R, \forall\, 0 \leq R < n$.

For example

3×6 (mod 15)

$18 = 15(1) + 3$

Then 3×6 (mod15) $= 3$.

9×9 (mod 15)

$81 = 15(5) + 6$

Then 9×9 (mod 15) $= 6$

$\times$	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	9	3

a) For identity: $e * a = a * e = a$.

The identity element is 6.

b) For inverse $a * a^{-1} = a^{-1} * a = e$.

Inverse of 3 is 12

Inverse of 6 is 6.

Inverse of 9 is 9.

Inverse of 12 is 3.

4) Results of Set $H = \{A; B; C; D\}$, under the binary operation $*$ is shown in the table below.

$*$	A	B	C	D
A	B	A	D	C
B	A	B	C	D
C	D	C	A	B
D	C	D	B	A

Use the table above to find

a) The identity element.
b) The inverse of each element.
c) A∗B
d) A∗ (B∗C)

Solution

a) For identity element $e * a = a * e = a$.

From the table , the operation of B and any element does not change the result i.e.

$B * A = A$

$A * B = A$

$C * B = C$

$D * B = D$

∴ The identity element is B.

b) For inverse element $a * a^{-1} = e$

The identity element is B.

$A * A = B$

$\therefore$ Inverse of A is A.

$B * B = B$

$\therefore$ Inverse of B is B.

$C * D = B$

$\therefore$ Inverse of C is D.

$D * C = B$

$\therefore$Inverse of D is C.

c) $A * B = A$

d) A $*$ (D$*$C) =A$*$B
=A

5) Determine whether non-zero real numbers are associative under the operation $*$ defined by $a * b = 2ab + 1$ or not.

Solution

For associativity; $r * (q * y) = (r * q) * y$.

Using the formula $a * b = 2ab + 1$.

LHS $= r * (q * y)$

Considering $(q * y)$.

By comparison $q = a, y = b$, then

$$
\begin{aligned}
q * y &= 2(q)(y) + 1 \\
&= 2qy + 1
\end{aligned}
$$

$r * (q * y) = r * (2qy + 1)$

By comparison $a = r, b = 2qy + 1$.

$$
\begin{aligned}
r * (2qy + 1) &= 2(r)(2qy + 1) + 1 \\
&= 4rqy + 2r + 1
\end{aligned}
$$

RHS $= (r * q) * y$

Considering $(r * q)$.

Using the formula $a * b = 2ab + 1$.

By comparison $a = r, b = q$.

$r * q = 2(r)(q) + 1$

$= 2rq + 1$

$(r * q) * y = (2rq + 1) * y$

By comparison $a = 2rq + 1, b = y$

$(2rq + 1) * y = 2(2rq + 1)(y) + 1$

$= 4rqy + 2y + 1$

LHS $\neq$ RHS

$\therefore$ Non zero numbers are not associative under binary operation $*$.

6) Given that $a * b = a + 2b,\ \forall\, a, b \in \mathbb{Q}$. Determine whether that $(\mathbb{Q};*)$ possess an identity element.

Solution

$a * b = a + 2b, \forall\, a, b \in \mathbb{Q}$

For identity element $a * e = e * a = a$

LHS $= a * e = a$

$a + 2e = a$

$2e = 0$

$e = 0$

$e + 2a = a$

$e = a - 2a$

$e = -a$

LHS$\neq$ RHS

$\therefore$ $(\mathbb{Q};*)$ does not possess an identity element.

7) $*$is defined on $\mathbb{R}$ by $a * b = 2a + b - 1\ \ \forall\, a, b \in \mathbb{R}$.If a^{-1} denotes the inverse element and e denotes the identity element of the operation $(*)$, show that $a^{-1} = 2(\frac{e+1}{2} - a)$.

Solution

For inverse element: $a * a^{-1} = e$.

Using the formula $a * b = 2a + b - 1$.

By comparison $a = a, b = a^{-1}$.

$a * a^{-1} = 2a + a^{-1} - 1 = e$

$2a + a^{-1} - 1 = e$

$a^{-1} = e + 1 - 2a$

$a^{-1} = 2\left(\frac{e+1}{2} - 2a\right)$ (Shown).

8) Prove that non zero integers are associative under the operation $*$ defined by $a * b = a + b$.

Solution

For associative; $r * (q * y) = (r * q) * y$.

LHS $= r * (q * y)$.

Considering $(q * y)$.

Using the formula $a * b = a + b$.

By comparison $a = q, b = y$.

$q * y = q + y$

$r * (q * y) = r * (q + y)$

By comparison $= r, b = q + y$.

$r * (q + y) = r + q + y$

RHS $= (r * q) * y$

Considering $(r * q)$.

Using the formula $a * b = a + b$.

By comparison $a = r, b = q$.

$r * q = r + q$

$(r * q) * y = (r + q) * y$

By comparison $a = r + q, b = y$.

$(r * q) * y = r + q + y$

LHS = RHS

$\therefore$ Non zero integers are associative under the operation $*$.

9) Given that $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Compile a Cayley table for I and H under multiplication.

Solution

$II = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$II = 1$

$IH = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0-1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$IH = H$

$\times$	I	H
I	I	H
H	H	I

10) Use the table below to answer the following questions.

	J	M	N	P	Q	R
J	J	M	N	P	Q	R
M	M	N	J	R	P	Q
N	N	J	M	Q	R	P
P	P	Q	R	J	M	N
Q	Q	R	P	N	J	M
R	R	P	Q	M	N	J

a) Solve the following equations.
   i) $\text{P}y = M$
   ii) $\text{Q}x = \text{N}$
   iii) $y\text{N} = \text{R}$
b) Simplify the following
   i) QR
   ii) NM
   iii) $M(QN)$
   iv) $(RQ)(PN)$
c) Write down the identity element.
d) Find all elements which are self-inverse.

Solution

a) Using the table to solve;

i) $\text{P}y = \text{M}$

From the table $\text{PQ} = M$.

$\therefore y = \text{Q}$

ii) $\text{Q}x = \text{N}$

From the table $\text{QP} = \text{N}$.

$\therefore x = \text{P}$

iii) $y\text{N} = \text{R}$

From the table PN $=$ R.

$\therefore y =$ P

b) Using the table to simplify;

i) QR $=$ M

ii) NM $=$ J

iii) M(QN)---------------------------------- Firstly simplify brackets.

M(QR) $=$ MM

$=$ N

iv) (RQ)(PN)--------------------------------Simplify brackets at first.

(RQ)(PN) $=$NR

$=$P

c) For identity element : $a * e = e * a = a$.

Using the table to check.

JM$=$ MJ$=$M

JN$=$NJ$=$N

$\therefore$The identity element is J.

d) Self inverse elements are J, P, Q and R.