Indices, Logarithms and Surds

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SYLLABUS (6042) REQUIREMENTS

- State the laws of indices
- Use laws of indices to solve problems
- Solve equations using laws of logarithms
- Simplify surds

INDICES

An index is a power (in plural form they are called indices/powers)

Rules of indices

[1] $a^x \times a^y = a^{x+y}$ (when multiplying indices of the same base add them)

[2] $a^x \div a^y = a^{x-y}$ (when dividing indices of the same base subtract them)

[3] $(a^x)^y = a^{xy}$ (when raising an index against another index, multiply them)

[4] $a^0 = 1$ (any number or letter raised to the power zero is 1)

[5] $a^{-x} = \frac{1}{a^x}$ (when raising a number to a negative index, take the reciprocal but with a positive index)

[6] $a^{\frac{1}{x}} = \sqrt[x]{a}$ (when raising a number to a fractional index, take the nth root of that number where the n^{th} value is the denominator)

[7] $a^{\frac{y}{x}} = \left(\sqrt[x]{a}\right)^y$ or $\sqrt[x]{a^y}$ (refer to rules #3 and #6 i.e. $a^{\frac{y}{x}} \equiv \left(a^{\frac{1}{x}}\right)^y$

DISGUISED QUADRATIC EQUATIONS

- They are also called pseudo/fake/hidden quadratic equations
- The word quadratic comes from the word quad which means square

Definition: These are equations which can be converted or transformed into quadratic equations using transformations

Steps

- Choose an appropriate substitution
- Solve the quadratic equations using the standard techniques
- Use the substitution again to transform back to the original variable

Worked Problems

Question 1

Find the value of

$$\frac{7^x + 7^{-x}}{2} = 1$$

Suggested Solution

$\dfrac{7^x + 7^{-x}}{2} = 1$

$7^x + 7^{-x} = 2$

$7^x + \dfrac{1}{7^x} = 2$

Let 7^x be c

$\Rightarrow c + \dfrac{1}{c} = 2$

$\Rightarrow c^2 - 2c + 1 = 0$

$\Rightarrow c^2 - c - c + 1 = 0$

$\Rightarrow c(c-1) - 1(c-1) = 0$

$\Rightarrow (c-1)(c-1) = 0$

$\Rightarrow c = 1$ twice

But $7^x = c$

$\Rightarrow 7^x = 1$

$\Rightarrow 7^x = 7^0$

$\therefore x = 0$

Question 2

Solve the equation $5^{2x+1} - 26(5^x) + 5 = 0$.

Suggested Solution

$5^{2x+1} - 26(5^x) + 5 = 0.$

$5^{2x}.5^1 - 26(5^x) + 5 = 0$

$5(5^x)^2 - 26(5^x) + 5 = 0$

Let $(5^x) = c$

$5c^2 - 26c + 5 = 0$

$5c^2 - 25c - 1c + 5 = 0$

$5c(c - 5) - 1(c - 5) = 0$

$(5c - 1)(c - 5) = 0\backslash$

$c = \frac{1}{5}$ or 5

But $(5^x) = c$

$\Rightarrow 5^x = \frac{1}{5}$ or $5^x = 5$

$\therefore x = -1 \text{ } or \text{ } 1$

Question 3

Solve the equation $3^{2x} - 12(3)^x + 27 = 0$

Suggested Solution

$(3^x)^2 - 12(3)^x + 27 = 0$

Let $y = 3^x$

$\Rightarrow y^2 - 12y + 27 = 0$

$\Rightarrow y^2 - 9y - 3y + 27 = 0$

$y(y - 9) - 3(y - 9) = 0$

$\Rightarrow (y - 9)(y - 3) = 0$

$\Rightarrow y = 3 \text{ } or \text{ } 9$

But $y = 3^x$

$\Rightarrow 3^x = 3$ or $3^x = 9$

$\Rightarrow 3^x = 3^1$ or $3^x = 3^2$

$\therefore x = 1$ or 2

Question 4

Solve the equation $4^x = 8^{2x+1}$

Suggested Solution

$4^x = 8^{2x+1}$

$(2^2)^x = (2^3)^{2x+1}$

$2^{2x} = 2^{6x+3}$

$\Rightarrow 2x = 6x + 3$

$4x = -3$

$\therefore x = -\frac{3}{4}$

Question 5

ZIMSEC NOVEMBER 2013 PAPER 1

Solve the equation $5^{x-1} + 5^{x-2} = 30$

Suggested Solution

$5^{x-1} + 5^{x-2} = 30$

$\frac{5^x}{5} + \frac{5^x}{25} = 30$

$\frac{5(5^x) + 5^x}{25} = 30$

$\frac{6(5^x)}{25} = 30$

$5^x = \frac{30 \times 25}{6}$

$5^x = 125$

$5^x = 5^3$

$\therefore x = 3$

Question 6

Solve the equation $9^x + 81 = 10(3^{x+1})$

Suggested Solution

$9^x + 81 = 10(3^{x+1})$

$(3^2)^x + 81 = 10(3^1 \times 3^x)$

$(3^x)^2 - 30(3^x) + 81 = 0$

Let $3^x = m$

$m^2 - 30m + 81 = 0$

$(m-3)(m-27) = 0$

$\Rightarrow m = 3 \; or \; 27$

But $3^x = m$

$\Rightarrow 3^x = 3 \; or \; 3^x = 27$

$3^x = 3 \; or \; 3^x = 3^3$

$\therefore x = 1 \; or \; 3$

Question 7

ZIMSEC NOVEMBER 2017 PAPER 1

Solve the equation $2^{1+2x} - 9(2^x) = -4$

Suggested Solution

$2^{1+2x} - 9(2^x) = -4$

$(2)^1 \times 2^{2x} + 9(2^x) = -4$

$2(2^x)^2 - 9(3^x) + 4 = 0$

Let $2^x = y$

$2y^2 - 9y + 4 = 0$

$(y-4)(2y-1) = 0$

$\Rightarrow y = 4 \; or \; \frac{1}{2}$

But $2^x = y$

$\Rightarrow 2^x = 4 \; or \; 2^x = \frac{1}{2}$

$2^x = 2^2 \; or \; 2^x = \; 2^{-1}$

$\therefore x = -1 \; or \; 2$

Question 8

ZIMSEC JUNE 2011 PAPER 1

Find the value of x which satisfies the equation $3(2^{2x+2}) - 2^{1-2x} = -5$.

Suggested Solution

$3(2^{2x+2}) - 2^{1-2x} = -5.$

$3(2^{2x}.2^2) - \frac{2^1}{2^{2x}} = -5$

$\Rightarrow 12(\,2^{2x}) - \frac{2^1}{2^{2x}} = -5$

Letting $p = 2^{2x}$ yields

$12p - \frac{2}{p} = -5$

$\Rightarrow 12p^2 + 5p - 2 = 0$

Solving for p yields

$p = \frac{1}{4} or - \frac{2}{3}$

Now equating

$p = 2^{2x} = \frac{1}{4} or - \frac{2}{3}$

$\Rightarrow 2^{2x} = \frac{1}{4} \quad or \quad 2^{2x} = -\frac{2}{3} \; (No\ solution)$

$\Rightarrow 2^{2x} = 2^{-2}$

$\therefore x = -1.$

Question 9

Solve the equation $4^x - 6^x = 9^x$

Suggested Solution

$4^x - 6^x = 9^x$

$\frac{4^x}{4^x} - \frac{6^x}{4^x} = \frac{9^x}{4^x}$

$1 - \left(\frac{6}{4}\right)^x = \left(\frac{9}{4}\right)^x$

$1 - \left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^{2x}$

Let $y = \left(\frac{3}{2}\right)^x$

$\Rightarrow y^2 = 1 - y$

$\Rightarrow y^2 + y = 1$

$\Rightarrow y^2 + y + \left(+\frac{1}{2}\right)^2 = 1 + \left(+\frac{1}{2}\right)^2$

$\Rightarrow \left(y + \frac{1}{2}\right)^2 = 1 + \frac{1}{4}$

$\Rightarrow y = -\frac{1}{2} \pm \sqrt{\frac{5}{4}}$

$\Rightarrow y = \frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$

But $y = \left(\frac{3}{2}\right)^x$

$\Rightarrow \left(\frac{3}{2}\right)^x = \frac{-1+\sqrt{5}}{2}$ or $\left(\frac{3}{2}\right)^x = \frac{-1-\sqrt{5}}{2}$

$\Rightarrow \left(\frac{3}{2}\right)^x = \frac{-1+\sqrt{5}}{2}$ or No solution

$\Rightarrow In\left(\frac{3}{2}\right)^x = In\left(\frac{-1+\sqrt{5}}{2}\right)$

$$\Rightarrow xIn\left(\frac{3}{2}\right) = In\left(\frac{-1+\sqrt{5}}{2}\right)$$

$$\Rightarrow \frac{xIn\left(\frac{3}{2}\right)}{In\left(\frac{3}{2}\right)} = \frac{In\left(\frac{-1+\sqrt{5}}{2}\right)}{In\left(\frac{3}{2}\right)}$$

$$\Rightarrow x = \frac{In\left(\frac{-1+\sqrt{5}}{2}\right)}{In\left(\frac{3}{2}\right)}$$

$\Rightarrow x = -1.18681439$

$\therefore x = -1.2$ (to 2 $s.f.$)

<u>Question 10</u>

Solve the equation $4^x - 207 = 49$

<u>Suggested Solution</u>

$4^x - 207 = 49$

$4^x = 207 + 49$

$4^x = 256$

$4^x = 4^4$

$\therefore x = 4$

PRACTICE QUESTIONS

Question 1

(i) Given that $= \sqrt{x}$, show that the equation $\sqrt{x} + \frac{10}{\sqrt{x}} = 7$ may be written as

$y^2 - 7y + 10 = 0.$

(ii) Hence, or otherwise solve the equation $\sqrt{x} + \frac{10}{\sqrt{x}} = 7.$

Answer (ii) $x = 4$ or 25

Question 2

Show that $\frac{6(2^{n+1}) - 4(2^{n-1})}{2^{n+1} - 2^n} = 10$

Question 3

Given that $\frac{2^x}{4^y} = 1$ and $(3^x)(\sqrt{3})^y = 9$, obtain two linear equations in x and y.

Hence determine the values of x and y.

Answer: $x = 2y$ and $2x + y = 4$; $x = \frac{8}{5}$ and $y = \frac{4}{5}$

Question 4

Solve the following equations:

a) $2(4^x) + 4^{-x} = 3$

Answer: $x = -\frac{1}{2}$ or 0

b) $27(9^x) + 9^{-x} = 12$

Answer: $x = -\frac{1}{2}$

c) $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 2\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$

Answer: $x = 3$

d) $x^{-\frac{1}{2}} + 2x^{-1} = 10$

Answer: $x = \frac{1}{4}$ or $\frac{4}{25}$

e) $x^{\frac{1}{3}} - 2x^{-\frac{1}{3}} = 1$

Answer: $x = -1$ or 8

f) $9^x + 81 = 10(3^{x+1})$

Answer: $x = 1$ or 3

g) $2(4^t) + 4^{-t} = 3$

Answer: $t = -\frac{1}{2}$ or 0

h) $5^{2x} - 12(5^x) + 35 = 0$

Answer: $x = 1$ or $\frac{In7}{In5}$

Question 5

Show that $\dfrac{40(5^{n-1}) - 25\,(5^{n-1})}{5^{n+1} - 2(5^n)} = 1$

Question 6

Solve the pair of simultaneous equations:

$$3^{3x} \times 9^y = 1$$

$$2^{2x} \times 4^y = \frac{1}{8}$$

Answer: $x = 3$ and $y = -\frac{9}{2}$

Question 7

Given that $y = 10^x$, show that

a) i) $y^2 = 100^x$ ii) $\frac{y}{10} = 10^{x-1}$,

b) Using the results from (a) and (b) write the equation

$100^x - 10\,001(10^{x-1}) + 100 = 0$ as an equation in y,

c) By first solving the equation in y, find the values of x which satisfy the given equation in x. **(C)**

Answer: b) $10y^2 - 10\,001y + 1000 = 0$

c) $x = -1$ or 3

Question 8

a) Solve the equation $(x^2 + 3x)^2 - 8(x^2 + 3x) = 20$

Answer: $x = -5, -2, -1$ or 2

b) Solve the equation $2^{2x} - 12(2^x) + 32 = 0$

Answer: $x = 2$ or 3

Question 9

Show that $\sqrt[n]{2 \times 4^m} = 2^{(2m+1)/n}$.

Hence, solve the simultaneous equations:

$$\sqrt[n]{2 \times 4^m} = 8 \text{ and } \frac{27^m}{9^{n+1}} = 81$$

Answer: $m = 4$ and $n = 3$

Question 10

Solve the equation

$$6x^2 - 25x + 14 = 0$$

Hence solve the equation

$$6(4x-3)^2 - 25(4x-3) + 14 = 0$$

Answer: $x = \frac{2}{3}$ or $\frac{7}{3}$ and $x = \frac{13}{8}$ or $\frac{11}{12}$

Question 11

Solve the following equations:

a) $x - 9\sqrt{x} + 20 = 0$

Answer: $x = 16$ or 25

b) $x^6 - 7x^3 = 8$

Answer: $x = -1$ or 2

c) $3\left(\frac{1}{3x-4}\right)^2 + \frac{5}{3x-4} + 2 = 0$

Answer: $x = 1$ or $\frac{5}{6}$

d) $(x^2 - x)^2 - 26(x^2 - x) + 120 = 0$

Answer: $x = -4, -2,$ 3 or 5

e) $\frac{2}{x^2} + \frac{7}{x} + 6 = 0$

Answer: $x = -\frac{2}{3}$ *or* $-\frac{1}{2}$

f) $3x^{-\frac{1}{3}} + 2x^{-\frac{2}{3}} + 1 = 0$

Answer: $x = -1$ *or* -8

g) $\frac{32}{2^{5x+10}} = 8^{x^2-1}$

Answer: $x = -1$ or $-\frac{2}{3}$

h) $2^{2x} + 4 \times 2^x - 5 = 0,$

Answer: $x = 0$

i) $x - 9\sqrt{x} + 20 = 0$

Answer: $x = 16 \text{ or } 2$

j) $\left(\frac{1}{x-2}\right)^2 + \frac{2}{x-2} + 1 = 0$

Answer: $x = 1$

k) $x^4 - 5.12x^2 + 4.08 = 0$

Answer: $x = \pm 2.03 \text{ or } \pm 0.993$

l) $x^{\frac{2}{3}} - 9x^{\frac{1}{3}} + 8 = 0$

Answer: $x = 1 \text{ or } 512$

m) $3^{2x} + 3^x = 2$

Answer: $x = 0$

n) $2x^4 - 11x^2 + 12 = 0$

Answer: $x = \pm 2 \text{ or } \pm\sqrt{\frac{3}{2}}$

o) $2^x + 4(2^{-x}) - 5 = 0,$

Answer: $t = 0 \text{ or } 2$

p) $12x^{-2} + 8(x^{-1}) + 1 = 0$

Answer: $x = -2 \text{ or } x = -6$

q) $6x^{-2} + 7x^{-1} - 5 = 0$

Answer: $x = -\frac{3}{5} \text{ or } 2$

r) $2x + 5\sqrt{x} + 3 = 0$

Answer: $x = 1 \text{ or } x = \frac{9}{4}$

s) $2x^{\frac{2}{3}} - 9\left(x^{\frac{1}{3}}\right) + 4 = 0$

Answer: $x = 8\ or\ x = 64$

t) $x + \sqrt{x} - 2 = 0$

Answer: $x = 1$

u) $x^4 - 5x^2 + 4 = 0$

Answer: $x = \pm 1\ or\ x = \pm 2$

v) $x^4 - 9x^2 + 14 = 0$

Answer: $x = \pm\sqrt{2}\ or\ x = \pm\sqrt{7}$

w) $x^6 - 9(x^3) + 8 = 0$

Answer: $x = 1\ or\ x = 1$

x) $(x+2)^2 - 3(x+2) - 4 = 0$

Answer: $x = -3\ or\ x = 2$

y) $x^4 + x^2 - 6 = 0$

Answer: $x = \pm\sqrt{2}$

z) $x^4 - 7x^2 + 12 = 0$

Answer: $x = \pm\sqrt{3}\ or\ x = \pm 2$

PAST EXAMINATION QUESTIONS

Question 1

ZIMSEC NOVEMBER 1996 PAPER 1

By using the substitution $y = x^{\frac{1}{3}}$, solve the equation

$$x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$$

Answer: $x = 8 \text{ or } 27$

Question 2

ZIMSEC NOVEMBER 1998 PAPER 1

Express 4^x in terms of y, where $y = 2^x$. **[1]**

Solve the equation

$$6(4^x) - 5(2^x) + 1 = 0,$$

expressing your answers for x in terms of logarithms where appropriate. **[5]**

Answer: $4^x = y^2$ and $x = -1 \text{ or } -\frac{In3}{In2}$

Question 3

ZIMSEC NOVEMBER 2003 PAPER 1

Solve the equation $27(9^x) + 9^{-x} = 12$. **[4]**

Answer: $x = -1 \text{ or } -\frac{1}{2}$

Question 4

ZIMSEC NOVEMBER 2004 PAPER 1

Given that the equation $\frac{x^2+4x}{3} + \frac{84}{x^2+4x} = 11$ is solved by using the substitution $y = x^2 + 4x$.

(i) reduce the equation to a quadratic in y, **[3]**

(ii) hence solve the original equation. **[3]**

Answer: $y^2 - 33y + 252 = 0$ and $x = -7, -6, 2$ or 3

Question 5

ZIMSEC NOVEMBER 2007 PAPER 1

Given that $2^x - 2^{-x} = 4$.

(i) Solve the equation for x.

(ii) Show that $|2^x + 2^{-x}| = 2\sqrt{5}$.

Answer: $x = 2.1$

Question 6

ZIMSEC JUNE 2009 PAPER 1

Solve the equation

$$3\sqrt{x} - 11 = 4x^{-\frac{1}{2}}$$ [4]

Answer: $x = \frac{1}{9}$ or 16

Question 7

ZIMSEC JUNE 2012 PAPER 1

Solve the equation $3(7^x) - 2(7^{-x}) = 5$. **[5]**

Answer: $x = \frac{In2}{In7}$ $(or\ 0.3562)$

Question 8

ZIMSEC JUNE 2013 PAPER 1

Solve the following simultaneous equations:

$$2^x + 3^y = 5$$

$$2^{x+2} - 3^{y+1} = 13$$

Answer: $x = 2$ and $y = 0$

Question 9

ZIMSEC JUNE 2014 PAPER 1

Solve the equation $\frac{27^{4x}}{3^4} = 9^{(x-1)}$. **[3]**

Answer: $x = \frac{1}{5}$

Question 10

ZIMSEC JUNE 2017 PAPER 1

Solve the equation $y^{\frac{2}{3}} - 5y^{\frac{1}{3}} + 6 = 0$. **[4]**

Answer: $y = 8$ or 27

Question 11

ZIMSEC JUNE 2019 PAPER 1

Show that the equation $2(2^{-x}) + 2^{x+2} = 10$ can be written as a quadratic equation in y where $y = 2^x$.

Hence or otherwise find the values of x which satisfy the equation.

Answer: $4y^2 - 10y + 2 = 0$ and $x = -2.2$ or 1.2

Question 12

ZIMSEC NOVEMEBER 2019 PAPER 1

Solve the equation $3^{x-1} + 3^{x-2} = 12$ **[3]**

Answer: $x = 3$

LOGARITHMS

Definition: The logarithm of any number is the power in which we raise the base to get the required number i.e.

$$log_a\, y = x \;\; \Rightarrow \; a^x = y$$

Rules of logarithms

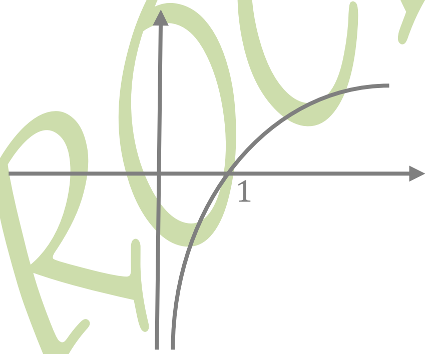
[1] $log_a\,(xy) = log_a x + log_a y$

[2] $log_a \left(\frac{x}{y}\right) = log_a x - log_a y$

[3] $log_a\,(x)^y = y(log_a x)$

Natural Logarithms (Inx)

Definition: These are logarithms which are in base e i.e. $log_e x = In_e x = Inx$



Rules of Natural logarithms

[1] $In\left(\frac{x}{y}\right) = In\,x - Iny$

[2] $In(xy) = Ina + In\,b$

[3] $In(x)^y = yIn(x)$

Properties of Natural logarithms

[1] $In1 = 0$

[2] $Ine = 1$

[3] $In(e)^x = x$

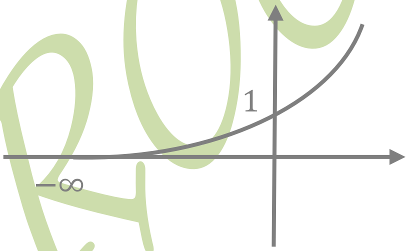
[4] $Iny = Inx \Rightarrow y = x$

NOTE

- *There are no logarithms for negative numbers*
- *Logarithms of numbers between* 0 *and* 1 *are negative*

Exponential Functions $\{e\}$

Definition: An exponential function (e) is the base of the natural logarithm (In) and it is just a rational number 2.718281828459045



Rules

The rules are the same as those for indices.

[1] $e^x \times e^y = e^{x+y}$ (when multiplying indices of the same base add them)

[2] $e^x \div e^y = e^{x-y}$ (when dividing indices of the same base subtract them)

[3] $(e^x)^y = e^{xy} = e^{yx}$ (when raising an index against another index, multiply them)

[4] $(e)^{\frac{1}{x}} = \sqrt[x]{e}$

[5] $(e)^{\frac{y}{x}} = \left(\sqrt[x]{e}\right)^y$

Properties of Exponential Function e

[1] $(e)^{Inx} = x$

[2] e^0

[3] $e^{-\infty} = 0$

This is not part of the objective but it is worth mentioning here:

$$Lim_{f(x)\to\infty}\left[\frac{1}{f(x)}\right] = 0 \quad i.e. \frac{1}{\infty} = 0$$

Worked Problems

Question 1

November 2004 ZIMSEC Paper 1

Solve $e^{In2x} + Ine^x = 6$

Suggested solution

$2x + x = 6$
$3x = 6$
$\therefore x = 2$

Question 2

November 2006 ZIMSEC Paper 1

Solve the following pair of simultaneous equations

$2logy = log2 + logx$
$2^y = 4^x$

Suggested solution

$2logy = log2 + logx$ (i)
$2^y = 4^x$ (ii)

From (i)

$logy^2 = \log(2x)$
$\therefore y^2 = 2x$ (iii)
From (ii)
$2^y = 2^{(2x)}$
$\therefore y = 2x$ (iv)

Substituting (iii) into (iv)

$y^2 = y$

$y^2 - y = 0$

$y(y-1) = 0$

$\therefore y = 0 \text{ or } 1$

$y = 2x$ (iv)

$\Longrightarrow 2x = 0 \;\; or \; 2x = 1$

$\therefore x = 0 \text{ and } \frac{1}{2}$

The solutions are $(0;0)$ and $\left(\frac{1}{2};1\right)$. But since we are dealing with Logarithms, the correct coordinates are $\left(\frac{1}{2};1\right)$. [NB: $\log(0)$ is not applicable]

Question 3

November 2016 ZIMSEC Paper 1

Solve exactly $2e^{2x} = 7e^x - 3$

Suggested solution

$2e^{2x} = 7e^x - 3$
$2e^{2x} - 7e^x + 3 = 0$

$2(e^x)^2 - 7e^x + 3 = 0$

Let $e^x = m$
$2m^2 - 7m + 3 = 0$

$(m-3)(2m-1)=0$

$\Rightarrow m = 3 \; or \; \frac{1}{2}$

But $e^x = m$

$\Rightarrow e^x = 3 \; or \; e^x = \frac{1}{2}$

$x = In(3) \; or \; x = In\left(\frac{1}{2}\right)$

$x = In(3) \; or \; x = In(2^{-1})$

$\therefore x = In3 \; or -In2 \; / \; In\left(\frac{1}{2}\right)$

Question 4

Solve $5^x = 8$

Suggested solution

$In(5^x) = \text{In}(8)$

$xIn(5) = \text{In}(8)$

$$x = \frac{\text{In}(8)}{In(5)}$$

$\therefore x = 1.292029674 = 1.29(\text{ to 3sf})$

Question 5

Solve $log_2(x+1) - log_2 x = log_2 7$

Suggested solution

$$log_2\left(\frac{x+1}{x}\right) = log_2 7$$

$$\left(\frac{x+1}{x}\right) = 7$$

$x + 1 = 7x$

$6x = 1$

$$\therefore x = \frac{1}{6}$$

Question 6

Solve $e^{3-x} = 2e^{-3x}$, giving your answer exactly in terms of logarithms

Suggested solution

$e^{3-x} = 2e^{-3x}$

$\frac{e^{3-x}}{e^{-3x}} = 2$

$\Rightarrow e^{3-x+3x} = 2$

$\Rightarrow e^{3+2x} = 2$

Taking natural logs to both sides

$In(e^{3+2x}) = In2$

$\Rightarrow 3 + 2x = In2$

$\therefore x = \frac{In2 - 3}{2}$

Question 7

Solve the following equation:

$6e^{2x} + 5e^{x} - 6 = 0$

Suggested solution

$6e^{2x} + 5e^{x} - 6 = 0$

$6(e^{x})^2 + 5e^{x} - 6 = 0$

Let $y = e^{x}$

$\Rightarrow 6y^2 + 5y - 6 = 0$

$\Rightarrow 6y^2 + 9y - 4y - 6 = 0$

$\Rightarrow 3y(2y + 3) - 2(2y + 3) = 0$

$\Rightarrow (2y + 3)(3y - 2) = 0$

$\Rightarrow y = -\frac{3}{2}\ or\ \frac{2}{3}$

But $y = e^{x}$

$\Rightarrow e^x = -\frac{3}{2}$ or $e^x = \frac{2}{3}$

$\therefore x = In\left(\frac{2}{3}\right)$ or equivalent

Question 8

Show that $Log_{10}(x+5) = 2 - Log_{10}x$ can be written as a quadratic equation in x.

Hence solve the equation.

Suggested solution

$Log_{10}(x+5) + Log_{10}x = 2$

$Log_{10}[\, x(x+5)] = 2$

$x(x+5) = 10^2$

$x^2 + 5x - 100 = 0$ (Shown)

Hence:

$x^2 + 5x - 100 = 0$

$x^2 + 5x + \left(\frac{5}{2}\right)^2 = 100 + \left(\frac{5}{2}\right)^2$

$\left(x + \frac{5}{2}\right)^2 = 100 + \frac{25}{4}$

$\left(x + \frac{5}{2}\right)^2 = \frac{425}{4}$

$x = -\frac{5}{2} \pm \sqrt{\frac{425}{4}}$

$x = -\frac{5}{2} + \sqrt{\frac{425}{4}}$ or $-\frac{5}{2} - \sqrt{\frac{425}{4}}$

$x = 7.80776406404$ and -12.80776406404

$\therefore x = 7.8$ (Since there are no Logarithms for negative numbers)

PAST EXAMINATION QUESTIONS

ZIMSEC QUESTIONS

Question 1

November 2018 ZIMSEC Paper 1

Solve the following simultaneous equations:

$$e^{2x+y} = 1$$

$$4x - 3y = 10$$

Answer: $x = 1$ and $y = -2$

Question 2

June 2018 ZIMSEC Paper 1

Solve the equations: $e^{2x} = 4e^{2-x}$ giving the answer in exact form.

Answer: $x = \frac{1}{3}(2 + In4)$

Question 3

June 2016 ZIMSEC Paper 1

Solve exactly the equations

(a) $\frac{1}{2}log_5(x-2) = 3log_5 2 - \frac{3}{2}log_5(x-2)$

(b) $2e^{2x} = 7e^x - 3$

Answer: $a)\ x = 2 + \sqrt{8}$ b) $x = In\left(\frac{1}{2}\right)$ or $x = In3$

Question 4

November 2014 ZIMSEC Paper 1

It is given that $e^{-3y}(1-4x) = (7+3x)$, show that $y = -\frac{1}{3}[In(7+3x) - In(1-4x)]$.

Question 5

June 2012 ZIMSEC Paper 1

Solve the following equation:

$log_{10}(2x^3 + 1) - 3log_{10}x = 1$

Answer: $x = \frac{1}{2}$

Question 6

November 2010 ZIMSEC Paper 1

If a and b are positive real numbers, $a \neq b$ and $log_a b + \frac{2}{log_a b} = 3$,express b in terms of a.

Answer: $b = a^2$

Question 7

June 2008 ZIMSEC Paper 1

Show that $e^{x+2Inx} = x^2e^x$.

Question 8

June 2007 ZIMSEC Paper 1

Find the exact value of the solution to the equation $2e^{2x} - 7e^x - 4 = 0$

Answer: $x = In4$

Question 9

November 2007 ZIMSEC Paper 1

Show that $In\left[\frac{n^2}{n^2-1}\right]. = 2In(n) - In(n+1) - In(n-1)$

Question 10

November 2006 ZIMSEC Paper 1

Solve the following simultaneous equations:

$$2logy = log2 + logx$$
$$2^y = 4^x$$

Answer: $(0,0)$ and $\left(\frac{1}{2}, 1\right)$

Question 11

June 2006 ZIMSEC Paper 1

a) Express $In\left[\frac{2x^2-3x-2}{\sqrt{1-x}}\right]$ in the form $In(ax+b) + In(cx+d) - gIn(1-x)$ where $a, b, c, d,$ and g are constants to be found.

b) Solve the equation $\frac{e^{3x}-e^{-3x}}{2} = 4$, giving your solution correct to 3 significant figures.

Answer: a) $In(2x+1) + In(x-2) - \frac{1}{2}In(1-x)$ b) $x = 0.698$

Question 12

November 2004 ZIMSEC Paper 1

Solve the following equations $e^{In2x} + Ine^x = 6$

Answer: $x = 2$

Question 13

November 2000 ZIMSEC Paper 1

Given that $3^x = 4^y$, find the value of $\frac{y}{x}$ correct to 3 significant figures.

Answer: $\frac{In3}{In4}$

Question 14

November 1998 ZIMSEC Paper 1

Express 4^x in terms of y, where $y = 2^x$.

Solve the equation $6(4^x) - 5(2^x) + 1 = 0$, expressing your answers for x in terms of logarithms where appropriate.

Answer: $4^x = y^2$ and $x = -1$ or $-\frac{In3}{In2}$

CAMBRIDGE QUESTIONS

Question 1

a) Given that $2log_3(x-5) - log_3(2x-13) = 1$, show that $x^2 - 16x + 64 = 0$.

b) Hence, or otherwise, solve $2log_3(x-5) - log_3(2x-13) = 1$

Answer: $x = 8$

Question 2

Given that a and b are positive constants, solve the simultaneous equations:

$$a = 3b$$

$$log_3(a) + log_3(b) = 2$$

Give your answer as exact numbers.

Answer: $a = 2\sqrt{6}$ and $b = \frac{2}{3}\sqrt{6}$

Question 3

a) Write down the value of $log_6(36)$

b) Express $2log_a(3) + log_a 11$ as a single log to base a

Answer: a) 6

b) $log_a(99)$

Question 4

Solve:

a) $5^x = 8$

b) $log_2(x+1) - log_2 x = log_2 7$

c) $logy + log(y+1) = 2\,log(y+2)$,

d) $e^{2c} - 5 \times (e^c) + 4 = 0$.

Answer: a) $In8/In5$

b) $x = 1/6$

c) $y = -4/3$

d) $c = 0$ or $In4$

Question 5

Find the exact solutions to the equations:

a) $Inx + In3 = In6$.
b) $e^x + 3e^{-x} = 4$.
c) $log_2(2x+1) - log_2 x = 2$.

Answer: a) $x = 2$

b) $x = 0$ or $In3$

c) $x = 1/2$

Question 6

Given that $y = 3x^2$,

(a) Show that $log_3 y = 1 + 2log_3 x$.

(b) Hence, or otherwise, solve the equation $1 + 2log_3 x = log_3(28x - 9)$

Answer: b) $x = 1/3$ or 9

Question 7

Given that $log_a x = p$ and $log_a y = q$, express the following in terms of p and q.

(i) $log_a(xy)$

(ii) $log_a\left(\frac{a^2x^2}{y}\right)$

Answer: i) $p + q$

ii) $2 + p^2 - q$

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Question 8

Given that $y = 3 \times 10^{2x}$, show that $x = alog_{10}(by)$, where the values of the constants a and b are to be found.

Answer: $x = \frac{1}{2}log_{10}\left(\frac{1}{3}y\right)$

Question 9

Show that the equation $log_{10}(x + 5) = 2 - log_{10}x$ may be written as a quadratic equation in x.

Hence find the values of x, satisfying the equation $log_{10}(x + 5) = 2 - log_{10}x$.

Answer: $\frac{(-5+\sqrt{425})}{2}$

Question 10

Given that $\log_5 x = a$ and $\log_5 y = b$, find in terms of a and b,

(i) $log_5\left(\frac{x^2}{y}\right)$,

(ii) $log_5(25x\sqrt{y})$.

It is given that $log_5\left(\frac{x^2}{y}\right) = 1$ and that $log_5(25x\sqrt{y}) = 1$

(iii) Form simultaneous equations in a and b.

(iv) Show that $a = -0.25$ and find the value of b.

Hence, or otherwise,

(v) calculate , to 3 $decimal\ places$, the value of x and the value of y.

Answer: i) $2a - b$

ii) $2 + a + \frac{1}{2}b$

iii) $2a - b = 1$ and $a + \frac{1}{2}b = -1$

iv) $b = -3/2$

v) $x = 0.669$ and $y = 0.0894$

Question 11

Solve exactly for x the following equation $e^{2x} + 2e^{-2x} = 3$.

Answer: $x = 0$ or $\frac{In2}{2}$

Question 12

Without using calculator, show that

$$\frac{In\sqrt{64} + In\sqrt{27} - In\sqrt{125}}{In12 - In5} = \frac{3}{2}$$

Question 13

Solve the simultaneous equations:

$$3^{x-y} = 9^{y}$$
$$2^{x} = 6(2^{y}),$$

giving your answer in logarithmic form.

Answer: $x = \frac{3}{2}\left(\frac{In6}{In2}\right)$ and $y = \frac{1}{2}\left(\frac{In7}{In2}\right)$

Question 14

Without using calculator, show that

$$\frac{log_a\sqrt{64} + log_a\sqrt{27} - log_a\sqrt{125}}{log_a 12 - log_a 5} = \frac{3}{2},$$

where a is a positive constant.

Question 15

Solve $2e^{2x} - 3e^{x} + 1 = 0.$

Answer: $x = 0$ or $In\left(\frac{1}{2}\right)$

Question 16

a) Simplify $\frac{x^2+4x+3}{x^2+x}$

b) Find the value of x for which $log_2(x^2 + 4x + 3) - log_2(x^2 + x) = 4.$

Answer: a) $\frac{x+3}{x}$

b) $x = 1/5$

Question 17

Use logarithms to solve the equation $0.8^x = 0.05$, giving your answer to two significant figures.

Answer: $x = 13.4$

Question 18

Solve the equation $4^x = 2(3^x)$, giving your answer correct to 3 significant figures

Answer: $x = 2.41$

SURDS

Definition: A surd is a nasty root i.e. a root which is irrational and cannot be evaluated e.g. $\sqrt{3}$.

Notes

- If we try to write them as decimals they would go on forever e.g. $\sqrt{3} = 1.4142213562$
- A number which cannot be expressed as a fraction of two integers is called an irrational number.

Rules

The following rules are used to evaluate surds:

[1] $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

[2] $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$

[3] $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 \Rightarrow \left(a^{\frac{1}{2}}\right)^2 = a$

Worked Problems

Question 1

Write $\sqrt{60}$ as the simplest possible surd.

Suggested solution

$\sqrt{60} = \sqrt{4 \times 15}$

$= \sqrt{4} \times \sqrt{15}$

$= 2\sqrt{15}$

Question 2

Express $8\sqrt{2}$ as a number under one square root.

Suggested solution

$8\sqrt{2} = \sqrt{8^2 \times 2}$

$= \sqrt{64 \times 2}$

$= \sqrt{128}$

Question 3

Simplify $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$

Suggested solution

$\sqrt{8} + \sqrt{18} - 2\sqrt{2} = \sqrt{4 \times 2} + \sqrt{9 \times 2} - 2\sqrt{2}$

$= 2\sqrt{2} + 3\sqrt{2} - 2\sqrt{2}$

$= 3\sqrt{2}$

Question 4

Simplify $(2\sqrt{5} - 7)^2$

Suggested solution

$(2\sqrt{5} - 7)^2 = (2\sqrt{5} - 7)(2\sqrt{5} - 7)$

$= 4\sqrt{5}.\sqrt{5} - 14\sqrt{5} - 14\sqrt{5} + 49$
$= 20 - 28\sqrt{5} + 49$
$= 69 - 28\sqrt{5}$

Alternatively

NB: Now using $(a + b)^2 = a^2 + 2ab + b^2$

$\Rightarrow (2\sqrt{5} - 7)^2 = (2\sqrt{5})^2 + 2(2\sqrt{5})(-7) + (-7)^2$

$= 20 - 28\sqrt{5} + 49$
$= 69 - 28\sqrt{5}$

Question 5

If $x = (\sqrt{5} + 2)$, express $x^2 - 3x + 4$ in the form $a + b\sqrt{c}$ where a, b and c are constants and real numbers.

Suggested solution

$x^2 - 3x + 4 = (\sqrt{5} + 2)^2 - 3(\sqrt{5} + 2) + 4$

$= (\sqrt{5})^2 + 2(\sqrt{5})(2) + (2)^2 - 3\sqrt{5} - 6 + 4$

$= 5 + 4(\sqrt{5}) + 4 - 3\sqrt{5} - 2$

$= 7 + \sqrt{5}$

RATIONALISATION

Definition: Getting rid of all surds off the bottom of the fraction

Method

Multiply the top and bottom of the fraction by the expression on the bottom but with changed sign (conjugate)

For example the conjugate of $(11 + 2\sqrt{7})$ is $(11 - 2\sqrt{7})$.

Multiplying a complex by its conjugate produces the difference from two squares i.e.

- $(a + b)(a - b) = a^2 - b^2$
- $(11 + 2\sqrt{7})(11 - 2\sqrt{7}) = 11^2 - (2\sqrt{7})^2 = 121 - 28 = 93.$

Worked Problems

Question 1

Rationalise the denominator of $\frac{5}{2-\sqrt{3}}$

Suggested Solution

$$\frac{5}{2-\sqrt{3}}$$

$$= \frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$

$$= \frac{10+5\sqrt{3}}{2^2-(\sqrt{3})^2}$$

$$= 10 + 5\sqrt{3}$$

Question 2

November 2002 ZIMSEC Paper 1

Express $\frac{2}{3-\sqrt{7}} - \sqrt{63}$ in the form $a + b\sqrt{c}$ where a, b and c are integers.

Suggested Solution

$$\frac{2}{3-\sqrt{7}} - \sqrt{63}$$

$$= \frac{2(3+\sqrt{7})}{(3-\sqrt{7})(3+\sqrt{7})} - \sqrt{7 \times 9}$$

$$= \frac{6+2\sqrt{7}}{3^2-(\sqrt{7})^2} - 3\sqrt{7}$$

$$= \frac{6+2\sqrt{7}}{9-7} - 3\sqrt{7}$$

$$= \frac{6+2\sqrt{7}}{2} - 3\sqrt{7}$$

$$= 3 + \sqrt{7} - 3\sqrt{7}$$

$$= 3 - 2\sqrt{7}$$

Question 3

November 2014 ZIMSEC Paper 1

Express $\frac{6}{3+\sqrt{5}+\sqrt{14}}$ in the form $a + b\sqrt{c} + d\sqrt{e}$ where a, b, c, d and e are real numbers.

Suggested Solution

$$\frac{6}{3+\sqrt{5}+\sqrt{14}}$$

$$= \frac{6\left[(3+\sqrt{5}) - \sqrt{14}\right]}{\left[(3+\sqrt{5}) + \sqrt{14}\right]\left[(3+\sqrt{5}) - \sqrt{14}\right]}$$

$$= \frac{6\left[(3+\sqrt{5}) - \sqrt{14}\right]}{\left[(3+\sqrt{5})^2 + (\sqrt{14})^2\right]}$$

$$= \frac{6[(3+\sqrt{5}) - \sqrt{14}]}{[9+6\sqrt{5}+5-14]}$$

$$= \frac{6[(3+\sqrt{5}) - \sqrt{14}]}{6\sqrt{5}}$$

$$= \frac{3+\sqrt{5} - \sqrt{14}}{\sqrt{5}}$$

$$= \frac{(3+\sqrt{5} - \sqrt{14}) \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}}$$

$$= \frac{(3\sqrt{5}+\sqrt{5}\sqrt{5} - \sqrt{5}\sqrt{14})}{\sqrt{5} \times \sqrt{5}}$$

$$= \frac{(3\sqrt{5}+5 - \sqrt{14 \times 5})}{5}$$

$$= \frac{(3\sqrt{5}+5 - \sqrt{70})}{5}$$

$$= \left(1 + \frac{3}{5}\sqrt{5} - \frac{1}{5}\sqrt{70}\right)$$

PRACTICE PROBLEMS

Question 1

Express $\frac{6}{3+\sqrt{2}+\sqrt{11}}$ in the form $a + b\sqrt{c} + d\sqrt{e}$, where a, b, c, d and e are real numbers.

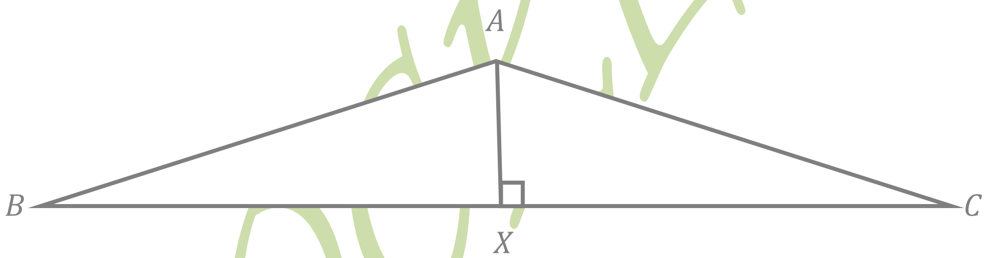
Answer: $2 + 3\sqrt{2} - \sqrt{22}$

Question 2

Express $(3\sqrt{3} + \sqrt{2})^2$ in the form $a + b\sqrt{c}$.

Answer: $29 + 6\sqrt{6}$

Question 3



The area of triangle ABC is $9\ cm^2$. The length of perpendicular height AX is $(3 + \sqrt{2}\ )cm$ and X is the midpoint of BC. Find the length of the BX, expressing your answer in the form $a\ +\ b\sqrt{c}$, where a, b and c are integers

Answer: $\frac{27}{7} + \frac{9}{7}\sqrt{2}$

Question 4

Rationalise the denominators of the following surds:

a) $\dfrac{2 + 3\sqrt{5}}{3 - \sqrt{5}}$

Answer: $\frac{21}{4} + \frac{11}{4}\sqrt{5}$

b) $\dfrac{5}{4-\sqrt{6}}$

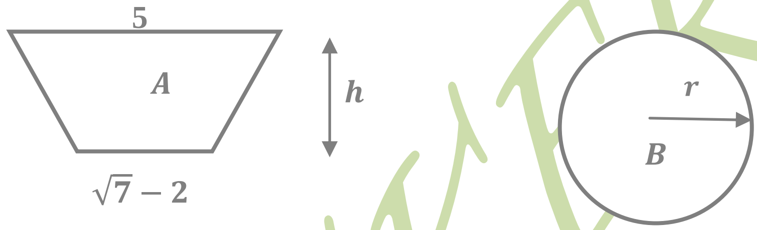
Answer: $2+\frac{1}{2}\sqrt{6}$

<u>Question 5</u>

If $x = 2-\sqrt{3}$, then express $2x^2 - 4x^{-1} + 8$ in the form $a + b\sqrt{c}$.

Answer: $14 - 12\sqrt{3}$

<u>Question 6</u>



$\left[use\ \pi = \frac{22}{7}\right]$

The diagrams above show a trapezium **A**; with parallel sides 5 cm, $(\sqrt{7}-2)cm$ and its perpendicular distance is $h\ cm$ and a circle **B** with radius $r\ cm$. Given that $r = (1+2\sqrt{7})cm$ and also that the area of circle **B** $= 44 \times$ area of trapezium **A**, express h in the form $a + b\sqrt{c}$.

Answer: $\frac{59}{14} - \frac{17}{14}\sqrt{7}$

<u>Question 7</u>

a) Simplify $6\sqrt{5} - \sqrt{125} + \sqrt{45}$

Answer: $4\sqrt{5}$

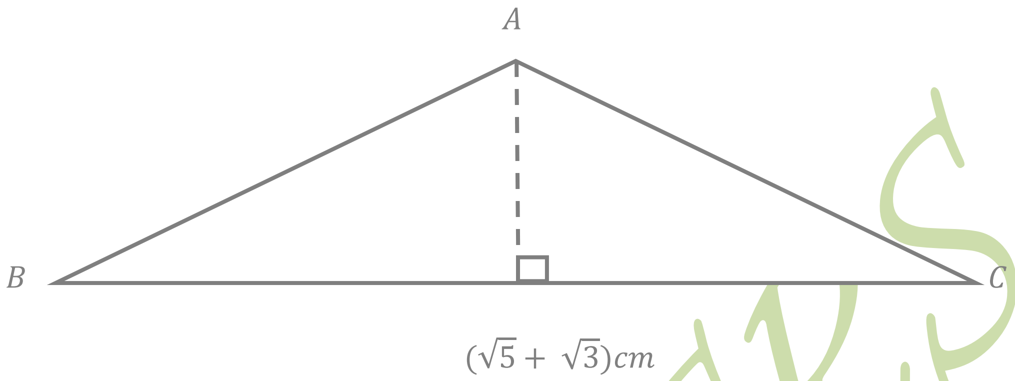
b) Express $\sqrt{48}$ as a simple surd.

Answer: $4\sqrt{3}$

c) Express $3\sqrt{7}$ in the form $\sqrt{k}$ where k is an integer.

Answer: $\sqrt{63}$

Question 8



The area of triangle ABC is 10 cm^2. The length of the base BC is $(\sqrt{5}+\sqrt{3})cm$. Find the perpendicular height from A to BC, expressing your answer in the form $a\sqrt{b} + c\sqrt{d}$, where a, b, c and b are integers

Answer: $10\sqrt{5} - 10\sqrt{3}$

Question 9

Rationalise the denominators of the following surds:

a) $\frac{1}{\sqrt{7}-2}$

Answer: $\frac{1}{3}\left(\sqrt{7}+2\right)$

b) $\frac{\sqrt{2}}{\sqrt{11}-\sqrt{3}}$.

Answer: $\frac{1}{8}\left(\sqrt{22}+\sqrt{6}\right)$

Question 10

If $x = 1 - \sqrt{2}$, then express $2x^2$ in the form $a + b\sqrt{c}$.

Answer: $6 - 4\sqrt{2}$

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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