Integration

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05 APRIL 2020

SYLLABUS (6042) REQUIREMENTS

- Finding Integrals of polynomials, rational functions, exponentials and trigonometrical functions
- Integrating by recognition and by substitution
- Evaluating definite integrals
- Representing life phenomena using mathematical models involving integration and exploring their applications in life

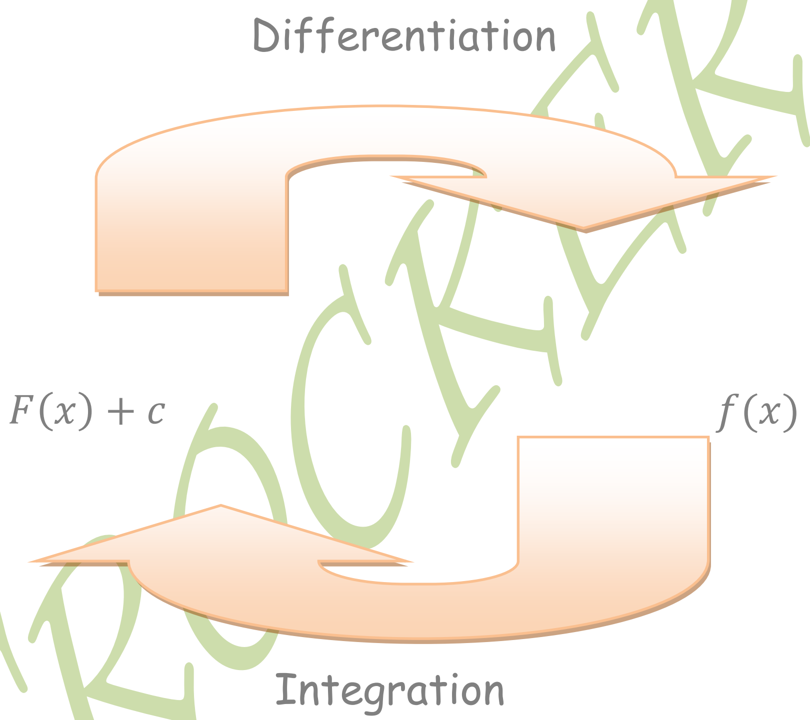
INTEGRATION

DEFINITION

It is the algebraic method or process of finding a function $[F(x)]$ given its derivative $[f(x)]$

It is also called anti-differentiation since it is an inverse process of differentiation.

In other words it is a process of finding the area under a curve for any given interval.



This can be written as:

$$\int f(x)\,dx = F(x) + c$$

Where

$\int$ is called the integration sign or integral sign

x is the variable of integration

$f(x)$ is called the integrand

$F(x)$ is called the anti derivative or primitive

dx is called the integrating agent

c is the arbitrary constant or constant of integration

Definite Integral

A definite integral is one which contains the upper and the lower limits and it does not contain any constant of integration i.e.

$$\int_a^b f(x)\,dx = [F(x)]_a^b$$

The upper limit is b and the lower limit is a.

When we integrate the integrand $[f(x)]$ we obtain the primitive $[F(x)]$ and we evaluate it as follow:

$[F(x)]_a^b = F(b) - F(a)$

Indefinite Integral

An indefinite integral is defined without upper and lower limits and it contains a constant of integration i.e.

$$\int f(x)\,dx = F(x) + c$$

Where c is the arbitrary constant.

Arbitrary Constant

From the definition of integration, we said it's a reverse process of differentiation but at times we might not get the exact function. Thus we have to introduce the constant such that when sufficient information is provided we can then get the required function. Let's consider the following illustration:

Question

Given that $y = x^2 + 3$, find its derivative.

Solution

$y = x^2 + 3$

$\frac{dy}{dx} = 2x.$

We now need to reverse the process:

Question

Integrate the function $y = 2x$

Solution

$$\int 2xdx = \frac{2x^{1+1}}{1+1}$$

$$= \frac{2x^2}{2}$$

$$= x^2$$

We have failed to go back to the original expression which was $y = x^2 + 3$, thus to circumvent the problem we should introduce an arbitrary constant.

So the solution becomes:

$$\int 2xdx = \frac{2x^{1+1}}{1+1} + c$$

$$= \frac{2x^2}{2} + c$$

$$= x^2 + c$$

NOTE

If $y = x^2 + 3$, then $\frac{dy}{dx} = 2x.$

If $y = x^2 - 25$, then $\frac{dy}{dx} = 2x.$

If $y = x^2$, then $\frac{dy}{dx} = 2x.$

The constant of integration expresses a sense of ambiguity since for a given derivative there can exist many integrands which may differ by a set of real numbers. Thus arbitrary constant, C, will represent this set of real numbers.

Integration of polynomials

When integrating polynomials we integrate each term separately

- If $y = x^n$ then $\int y dx = \frac{x^{n+1}}{n+1} + c$
- If $y = ax^n$ then $\int y dx = \frac{ax^{n+1}}{n+1} + c$
- If $y = (fx)^n$ then $\int y dx = \frac{[(f(x)]^{n+1}}{(n+1)\times f^{'}(x)} + c$

WORKED EXAMPLES

Question One

Find $\int (x^2 + 2)dx$

Suggested Solution

$$\int (x^2 + 2)dx = \frac{x^3}{3} + 2x + c$$

Question Two

Evaluate

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\, dx$$

Suggested Solution

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\, dx = \int_0^4 (2x+1)^{-\frac{1}{2}} dx$$

$$= \left[\frac{(2x+1)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)\times 2}\right]_0^4$$

$$= \left[(2x+1)^{\frac{1}{2}}\right]_0^4$$

$$= \left[\sqrt{2x+1}\right]_0^4$$

$$= (\sqrt{9}) - (\sqrt{1})$$

$$= 3 - 1$$

$$= 2$$

Question Three

Find $\int e^{2In(1-3x)}dx$

Suggested Solution

$$\int e^{2In(1-3x)}dx = \int e^{In(1-3x)^2}dx$$

$$= \int (1-3x)^2 dx$$

$$= \frac{(1-3x)^{2+1}}{3\times -3} + c$$

$$= -\frac{(1-3x)^3}{9} + c$$

Question Four

Find $\int In\left[e^{(x^2+5x-1)}\right]dx$

Suggested Solution

$$\int In\left[e^{(x^2+5x-1)}\right]dx = \int x^2 + 5x - 1\, dx$$

$$= \frac{1}{3}x^3 + \frac{5}{2}x^2 - x + c$$

NOTE

- $e^{In[f(x)]} = f(x)$
- $In[e^{f(x)}] = f(x)$

Follow Up Questions

Question 1

Find $\int \frac{1}{\sqrt[3]{x}}\ dx$ **Answer** $\frac{3}{2}\sqrt[3]{x^2}$

Question 2

Find $\int 10 - 3x\ \ dx$ **Answer** $10x - \frac{3}{2}x^2 + c$

Question 3

Evaluate

$$\int_1^2 2x dx$$

Answer 3

Question 4

Evaluate

$$\int_{\frac{1}{2}}^{1} (x-2)dx$$

Answer $-\frac{5}{8}$

Question 5

Find $\int (x+2)^2\ \ dx$ **Answer** $\frac{1}{3}x^3 + x^2 + 4x + c$

Question 6

Find $\int \left(6x^7 + 4x^{\frac{2}{3}} + 8\right) dx$ **Answer** $\mathbf{\frac{3}{4}x^8 + \frac{12}{5}x^{\frac{5}{3}} + 8x + c}$

Integration of exponentials

- If $y = e^{f(x)}$ then $\int y dx = \frac{e^{f(x)}}{f^{'}(x)} + c$

WORKED EXAMPLES

Question One

Find $\int e^{(x^2+2)} dx$

Suggested Solution

$$\int e^{(x^2+2)} dx = \frac{e^{(x^2+2)}}{2x} + c$$

Question Two

Evaluate exactly

$$\int_2^5 \frac{1}{e^{1-3x}}\, dx$$

Suggested Solution

$$\int_2^5 \frac{1}{e^{1-3x}}\, dx = \int_2^5 e^{-1(1-3x)} dx$$

$$= \int_{2}^{5} e^{3x-1} dx$$

$$= \left[\frac{e^{3x-1}}{3}\right]_{2}^{5}$$

$$= \left(\frac{e^{14}}{3}\right) - \left(\frac{e^{5}}{3}\right)$$

$$= \frac{e^{14} - e^{5}}{3}$$

Question Three

Show that $\int e^{sin^2x} dx = \frac{e^{sin^2x}}{sin2x} + c$

Suggested Solution

$$\int e^{sin^2x} dx = \frac{e^{sin^2x}}{2sinxcosx} + c$$

$$= \frac{e^{sin^2x}}{sin2x} + c \ \ (as\ required)$$

Follow Up Questions

Question 1

Find $\int \frac{1}{e^{2-3x}} dx$ **Answer** $\frac{1}{3} e^{3x-2} + c$

Question 2

Evaluate

$$\int_{In2}^{In5} 2e^{2x} \ dx$$ **Answer 21**

Question 3

Evaluate

$$\int_{0}^{\frac{1}{7}In2} 14e^{7x} \, dx$$

Answer 2

Question 4

Evaluate

$$\int_{-2}^{2} e^{x+2} \, dx$$

Answer $e^4 - 1$

Question 5

Find $\int e^{2x+3} \, dx$

Answer $\frac{1}{2}e^{2x+3} + c$

Question 6

Find $\int (Ine^{2x} + 3e^{3x}) \, dx$

Answer $x^2 + e^{3x} + c$

Question 7

Evaluate

$$\int_{0}^{1} e^{2x} + x + 2 \, dx$$

Answer $\frac{e^2}{2} + 2$

Question 8

Evaluate

$$\int_{0}^{1} 3e^{1-2x} \, dx$$

Answer $\frac{3}{2}e - \frac{3}{2}e^{-1}$

Question 9

Show that

$$\int_{0}^{1} (e^x + 1)^2 \, dx = \frac{1}{2}e^2 + 2e - \frac{3}{2}$$

Question 10

Show that

$$\int_0^5 (2e^{2x})^2\ dx = e^{20} - 1$$

Integration of Trigonometric functions

- If $y = cosx$ then $\int ydx = sinx + c$
- If $y = sinx$ then $\int ydx = -\ cosx + c$
- If $y = sec^2x$ then $\int ydx = tanx + +c$

WORKED EXAMPLES

Question One

Find $\int cos2xdx$

Suggested Solution

$$\int cos2xdx = \frac{Sin2x}{2} + c$$

Question Two

Show that

$$\int_{\frac{\pi}{4}}^{\pi} tan^2 3x\, dx = \frac{1}{12}(4 - 9\pi)$$

Suggested Solution

$tan^2 3x \equiv sec^2 3x - 1$

$$\int_{\frac{\pi}{4}}^{\pi} tan^2 3x\, dx = \int_{\frac{\pi}{4}}^{\pi} (sec^2 3x - 1) dx$$

$$= \left[\frac{tan3x}{3} - x\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \left(\frac{tan3\pi}{3} - \pi\right) - \left(\frac{tan\frac{3\pi}{4}}{3} - \frac{\pi}{4}\right)$$

$$= (0 - \pi) - \left(-\frac{1}{3} - \frac{\pi}{4}\right)$$

$$= \frac{1}{3} - \frac{3\pi}{4}$$

$$= \frac{1}{12}(4 - 9\pi)$$

Question Three

Find $\int 3sin5xdx$

Suggested Solution

$$\int 3sin5xdx = -\frac{3}{5}cos5x + c$$

Follow Up Questions

Question 1

Find $\int 3Sinx - 4Sec^2 x\, dx$ **Answer $-3cosx - 4tanx + c$**

Question 2

Evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3Sin(2x) + 1 \; dx$$ **Answer π**

Question 3

Evaluate

$$\int_{0}^{\frac{\pi}{2}} \frac{sin^2x}{1 + cosx} \; dx$$ **Answer $\frac{\pi}{2} - 1$**

Question 4

Find $\int (tan^2x + sin2x) \; dx$ **Answer $tanx - x - \frac{1}{2}cos2x + c$**

Question 5

Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2cos(2x) \; dx$$ **Answer -1**

Question 6

Show that

$$\int_{0}^{\frac{\pi}{3}} (sin2x + sec^2x) \; dx = \frac{3}{4} + \sqrt{3}$$

Question 7

Show that

$$12cos^2xsin^2x = \frac{3}{2}(1 - cos4x)$$

Hence show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 12cos^2xsin^2x \; dx = \frac{\pi}{18} + \frac{3\sqrt{3}}{16}$$

Question 8

a) Show that

$$\frac{d}{dx}(secx) = secxtanx \text{ and also } sec^2x + sec^2xtanx \equiv \frac{1}{1 - sinx}$$

Hence show that

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 - sinx}\, dx = \sqrt{2}$$

Integration by recognition

Integration by recognition means to integrate by recognizing the integrand as being the derivative of another function or simply that it relies on the fact that differentiating and then anti-differentiating a function gives you the original function, although with a constant. This implies that:

- If $y = \frac{f'(x)}{f(x)}$ then $\int ydx = In[f(x)] + c$

WORKED EXAMPLES

Question One

Find $\int \frac{1}{x+2} dx$

Suggested Solution

$$\int \frac{1}{x + 2} dx = In|x + 2| + c$$

Question Two

Find $\int \frac{3}{2x-1} dx$

Suggested Solution

$$\int \frac{3}{2x-1}dx = \frac{3}{2}\int \frac{2}{2x-1}dx$$

$$= \frac{3}{2}In|2x-1| + c$$

Question Three

Find $\int \frac{1}{8x+2}dx$

Suggested Solution

$$\int \frac{1}{8x+2}dx = \frac{1}{8}\int \frac{8}{8x+2}dx$$

$$= \frac{1}{8}In|8x+2| + c$$

Question Four

Show that $\int tanx\ dx = In|Secx| + c$

Suggested Solution

$$\int tanx\ dx = \int \frac{sinx}{cosx}dx$$

$$= \int \frac{sinx}{cosx}dx$$

$$= -1\int \frac{-sinx}{cosx}dx$$

$$= -In|\cos x| + c$$

$$= In|\cos x|^{-1} + c$$

$$= In\left|\frac{1}{\cos x}\right| + c$$

$$= In|secx| + c$$

Question Five

Evaluate exactly

$$\int_2^4 \frac{3}{6x-1}\,dx$$

Suggested Solution

$$\int_2^4 \frac{3}{6x-1}\,dx = \frac{1}{2}\int_2^4 \frac{6}{6x-1}\,dx$$

$$= \left[\frac{1}{2}In|6x-1|\right]_2^4$$

$$= \left[\frac{1}{2}In|23|\right] - \left[\frac{1}{2}In|11|\right]$$

$$= \frac{1}{2}In\left(\frac{23}{11}\right)$$

Follow Up Questions

Question 1

Find $\int \frac{17sinx}{cosx}dx$ **Answer** $\boldsymbol{-17In|cosx|+c}$

Question 2

Evaluate

$$\int_1^4 \frac{2x+4}{x^2+4x}\,dx$$

Answer $\boldsymbol{In\left|\frac{32}{5}\right|}$

Question 3

Evaluate

$$\int_1^5 \frac{14}{2x-1}\,dx$$

Answer $\boldsymbol{7In9}$

Question 4

Evaluate

$$\int_{2}^{5} \frac{1}{3x}\, dx$$

Answer $\frac{1}{3}In\left|\frac{5}{2}\right|$

Question 5

Find $\int \frac{5}{3-2x}\, dx$

Answer $-\frac{5}{2}In|3-2x|+c$

Question 6

Find $\int \frac{6x+5}{3x^2+5x+6}\, dx$

Answer $In|3x^2+5x+6|+c$

Question 7

Find $\int cotx\ dx$

Answer $In|sinx|c$

Question 8

Find $\int \frac{5}{(x-2)^2}+\frac{1}{x-3}\, dx$

Answer $\frac{5}{4}+In|3|+c$

Question 9

Show that

$$\int_{-1}^{3} \frac{1}{2x+3}\, dx = In3$$

Question 10

Show that

$$\int_{1}^{4} \left(\frac{1}{2x}+\frac{1}{x+1}\right) dx = In5$$

Question 11

Show that

$$\int_0^3 \frac{3}{x+5}\,dx = 3In\left(\frac{8}{5}\right)$$

Question 12

Show that

$$\int_0^3 \frac{1}{2x+1}\,dx = In\sqrt{7}$$

Integration of Rational functions

Rational functions are fractions which have both the numerator and the denominator which are polynomials or simply a function that is a ratio of two polynomials.

WORKED EXAMPLES

Question One

ZIMSEC NOVEMBER 2006 PAPER 1

Evaluate exactly

$$\int_1^4 \frac{2x+3}{x+1}\,dx$$

Suggested Solution

$$\int_1^4 \frac{2x+3}{x+1}\,dx$$

Aside

NB: use either long division or any suitable method to divide the rational function above

$$\frac{2x+3}{x+1} = 2 + \frac{1}{x+1}$$

Now:

$$\int_1^4 2 + \frac{1}{x+1}\,dx = [2x + In|x+1|]_1^4$$

$$= [2(4) + In|4+1|] - [2(1) + In|1+1|]$$

$$= [8 + In5] - [2 + In2]$$

$$= 6 + In\left(\frac{5}{2}\right)$$

<u>Question Two</u>

<u>ZIMSEC NOVEMBER 2019 PAPER 2</u>

Evaluate exactly

$$\int_{-2}^{0} \frac{1+2x}{1-2x}\,dx$$

<u>Suggested Solution</u>

$$\int_{-2}^{0} \frac{1+2x}{1-2x}\,dx$$

<u>Aside</u>

NB: use either long division or any suitable method to divide the rational function above

$$\frac{1+2x}{1-2x} = \frac{2x+1}{-2x+1} = -1 + \frac{2}{-2x+1}$$

Now:

$$\int_1^4 -1 + \frac{2}{-2x+1}\,dx = [-x - In|1-2x|]_{-2}^{0}$$

$$= [0 + In|1|] - [-1(-2) - In|1+4|]$$

$= 0 - [2 - In5]$

$= In5 - 2$

Follow Up Questions

Question 1

Find $\int \frac{x+2}{x-1} dx$ **Answer** $\boldsymbol{x + 3In|x - 1|}$

Question 2

Evaluate

$$\int_1^4 \frac{2x + 4}{2x} dx$$

Answer $\boldsymbol{3 + In|16|}$

Question 3

Evaluate

$$\int_1^5 \frac{1 - x}{x + 1} dx$$

Answer $\boldsymbol{-4 + In9}$

Question 4

Evaluate

$$\int_1^3 \frac{2x}{2x - 1} dx$$

Answer $\boldsymbol{\frac{1}{2}In|5|}$

Question 5

Find $\int \frac{x+1}{-x} dx$ **Answer** $\boldsymbol{-x - In|x| + c}$

Question 6

Find $\int \frac{6x+5}{3x+2} dx$ **Answer** $\boldsymbol{2 + \frac{1}{3}In|3x + 2| + c}$

Question 7

Show that

$$\int_{-1}^{3} \frac{4x^2+8x}{2x+3}\, dx = 12 - 3In3$$

Integration by Partial fractions

We only integrate a rational function when it is proper i.e. the degree of the numerator if less that the degree of the denominator and also when the integrand is spilt into partial fractions. After that we use traditional integration techniques to evaluate given integrals.

WORKED EXAMPLES

Question One

ZIMSEC JUNE 2019 PAPER 1

The function $f(x) = \frac{x^3-x-2}{(x-1)(x^2+1)}$

(a) Express $f(x)$ in the form $A + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$ where, A, B, C and D are constants.
(b) Hence

$$\int_{2}^{3} f(x)\; dx$$

Suggested Solution

a) Let $f(x) \equiv A + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$

$$\equiv \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)}{x^2+1}$$

$\Rightarrow A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1) = x^3 - x - 2$

Comparing coefficients of x^3: $A = 1$

When $x = 1$: $2B = -1$

$\Rightarrow B = -1$

Comparing coefficients of x^2: $-A + B + C = 0$

$-1 - 1 + C = 0$

$\Rightarrow C = 2$

Comparing coefficients of x: $A - C + D = -1 = 0$

$1 - 2 + D = -1$

$\Rightarrow D = 0$

$\therefore f(x) \equiv 1 - \frac{1}{x-1} + \frac{2x}{x^2+1}$

b) Hence

$$\int_2^3 f(x)\ dx = \int_2^3 1 - \frac{1}{x-1} + \frac{2x}{x^2+1}\ dx$$

$$= [x - In|x-1| + In|x^2+1|]_2^3$$

$$= x + In\left|\frac{x^2+1}{x-1}\right|_2^3$$

$$= \left[3 + In\left|\frac{9+1}{3-1}\right|\right] - \left[2 + In\left|\frac{4+1}{2-1}\right|\right]$$

$$= [3 + In|5|] - [2 + In|5|]$$

$$= 1$$

Question Two

ZIMSEC NOVEMBER 2019 PAPER 1

(a) Express $f(x) = \frac{4x+5}{(x+1)(2x+1)^2}$ in partial fractions.

(b) Hence find the exact value of

$$\int_0^2 f(x)\ dx$$

<u>Suggested Solution</u>

a) Let $f(x) \equiv \frac{A}{x+1} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2}$

$$\equiv \frac{A(2x+1)^2 + B(x+1)(2x+1) + C(x+1)}{(x+1)(2x+1)^2}$$

$\Rightarrow A(2x+1)^2 + B(x+1)(2x+1) + C(x+1) = 4x+5$

When $x = -1$: $A(-2+1)^2 = -4+5$

$\Rightarrow A = 1$

Comparing coefficients of x^2: $4A + 2B = 0$

$4 + 2B = 0$

$\Rightarrow B = -2$

Comparing coefficients of x: $4A + 3B + C = 4$

$4 - 6 + C = 4$

$\Rightarrow C = 6$

$\therefore f(x) \equiv \frac{1}{x+1} - \frac{2}{2x+1} + \frac{6}{(2x+1)^2}$

b) Hence

$$\int_0^2 f(x)\ dx = \int_0^2 \frac{1}{x+1} - \frac{2}{2x+1} + 6(2x+1)^{-2}\ dx$$

$$= [In|x+1| - In|2x+1| - \frac{6}{2}(2x+1)^{-1}]_0^2$$

$$= \left[In\left|\frac{x+1}{2x+1}\right| - \frac{3}{2x+1}\right]_0^2$$

$$= \left[In\left|\frac{2+1}{4+1}\right| - \frac{3}{4+1}\right] - \left[In\left|\frac{0+1}{0+1}\right| - \frac{3}{0+1}\right]$$

$$= In\left|\frac{3}{5}\right| - \frac{3}{5} + 3$$

$$= In\left(\frac{3}{5}\right) + \frac{12}{5}$$

Follow Up Questions

Question 1

(a) Resolve $f(x) = \frac{x^2+x-11}{(x-3)(x-2)^2}$ in partial fractions.

(b) Hence evaluate

$$\int_4^6 f(x)\ dx$$

Answer a) $\frac{\mathbf{1}}{\boldsymbol{x-3}} + \frac{\mathbf{5}}{\boldsymbol{(x-2)^2}}$ **b)** $\frac{\mathbf{5}}{\mathbf{4}} + \boldsymbol{In|3|}$

Question 2

(a) Express $f(x) = \frac{(3x^2+1)^2}{x^2}$ in the form $9x^2 + 6 + \frac{1}{x^2}$

(b) Hence evaluate

$$\int_1^2 f(x)\ dx$$

Answer a) $\boldsymbol{Ax^2 + B + \frac{C}{x^2}}$ **b)** $\mathbf{27.5}$

Question 3

(a) Express $f(x) = \frac{13-x}{x^2+4x-5}$ in partial fractions.

(b) Hence find

$$\int f(x)\ dx$$

Answer a) $\frac{\mathbf{2}}{\boldsymbol{x-1}} - \frac{\mathbf{3}}{\boldsymbol{x+5}}$ **b)** $\boldsymbol{2In|x-1| - 3In|x+5| + c}$

Question 4

Evaluate

$$\int_{2}^{3} \frac{1}{x^2 - 2x + 1}\, dx$$

Answer $\frac{1}{2}$

Question 5

Find

$$\int \frac{3x - 7}{x^2 - 2x - 15}\, dx$$

Answer $\frac{1}{4} In|x - 3| + \frac{11}{4} In|x + 5| + c$

Question 6

Show that

$$\int_{4}^{5} \frac{x^2 + 9x + 9}{x^2 + x - 6}\, dx \approx 3.754$$

Question 7

Show that

$$\int_{2}^{3} \frac{1}{x^2 - 1}\, dx = \frac{1}{2} In\left|\frac{3}{2}\right|$$

Question 8

Find

$$\int \frac{x}{(x-1)(x-2)(x-3)}\, dx$$

Answer $\frac{1}{2} In|x - 1| - 2In|x - 2| + \frac{3}{2} In|x - 3| + c$

Integration by substitution

It is a process of transforming one integral to another integral which is easy to compute. In simpler terms, a difficult integral is transformed to an easier integral through substitution. It is also called the u substitution or the reverse chain rule. We should make the substitution for a function whose derivative also occurs in the integrand.

Steps for indefinite integrals

- Choose a new variable e.g. u
- Determine the value of dx
- Make the substitution
- Integrate the resulting integral
- Return to the initial variable

Steps for definite integrals

- Choose a new variable e.g. u
- Determine the value of dx
- Change the limits
- Make the substitution
- Integrate the resulting integral

ALGEBRAIC SUBSTITUTION

WORKED EXAMPLES

Question One

Find $\int e^{(2x+2)}dx$

Suggested Solution

$$\int e^{(2x+2)}dx$$

Let $u = 2x + 2$

$$\frac{du}{dx} = 2 \Rightarrow \frac{du}{2} = dx$$

Now

$$\int e^{(2x+2)}dx = \int e^u \frac{du}{2} = \int \frac{e^u}{2} du$$

$$= \frac{e^u}{2} + c$$

But $u = 2x + 2$

$$\therefore \int e^{(2x+2)}dx = \frac{e^{(2x+2)}}{2} + c$$

Question Two

Evaluate

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\, dx$$

Suggested Solution

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\, dx = \int_0^4 (2x+1)^{-\frac{1}{2}}dx$$

Let $u = 2x + 1$

$$\frac{du}{dx} = 2 \Rightarrow \frac{du}{2} = dx$$

$u = 2x + 1$

x	**4**	**0**
u	9	1

Now

$$\int_0^4 (2x+1)^{-\frac{1}{2}}dx = \int_1^9 (u)^{-\frac{1}{2}}\frac{du}{2} = \frac{1}{2}\int_1^9 (u)^{-\frac{1}{2}}du$$

$$= \frac{1}{2}\left[\frac{(u)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^9$$

$$= \left[(u)^{\frac{1}{2}}\right]_1^9$$

$$= [\sqrt{u}]_1^9$$

$$= (\sqrt{9}) - (\sqrt{1})$$

$$= 3 - 1$$

$$= 2$$

Question Three

Find $\int cos2xdx$

Suggested Solution

$$\int cos2xdx$$

Let $u = 2x$

$$\frac{du}{dx} = 2 \Rightarrow \frac{du}{2} = dx$$

Now

$$\int cos2xdx = \int cosu\frac{du}{2} = \frac{1}{2}\int cosu\, du$$

$$= \frac{1}{2}sin\, u + c$$

But $u = 2x + 2$

$$\therefore \int cos2xdx = \frac{Sin2x}{2}$$

TRIGONOMETRIC SUBSTITUTION

WORKED EXAMPLES

Question One

ZIMSEC JUNE 2012 PAPER 2

By using the substitution $u = sinx$ show that

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{cosx}{3 + cos^2x}\, dx = \frac{1}{4} In\left(\frac{1}{3}\right)$$

Suggested Solution

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{cosx}{3 + cos^2x}\, dx$$

Let $u = sinx$

$$\frac{du}{dx} = cosx \Rightarrow \frac{du}{cosx} = dx$$

$u = sinx$

x	$\frac{3}{2}\pi$	π
u	-1	0

Now:

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{cosx}{3 + cos^2x}\, dx = \int_{0}^{-1} \frac{cosx}{3 + cos^2x}\left(\frac{du}{cosx}\right) = \int_{0}^{-1} \frac{1}{3 + cos^2x}\, du$$

But $cos^2x = 1 - sin^2x$

$$\Rightarrow \int_{0}^{-1} \frac{1}{3 + cos^2x}\, du = \int_{0}^{-1} \frac{1}{3 + 1 - sin^2x}\, du$$

$$= \int_{0}^{-1} \frac{1}{4 - sin^2x}\, du$$

$$= \int_0^{-1} \frac{1}{4-u^2}\, du$$

$$= \int_0^{-1} \frac{1}{(2-u)(2+u)}\, du$$

Aside

Let

$$\frac{1}{(2-u)(2+u)} \equiv \frac{A}{(2-u)} + \frac{B}{(2+u)}$$

$$\equiv \frac{A(2+u)+B(2-u)}{(2-u)(2+u)}$$

$$\Rightarrow A(2+u)+B(2-u) = 1$$

When $u = 1: 4A = 1$

$$\Rightarrow A = \frac{1}{4}$$

When $u = -1: 4B = 1$

$$\Rightarrow B = \frac{1}{4}$$

$$\Rightarrow \frac{1}{(2-u)(2+u)} \equiv \frac{\frac{1}{4}}{(2-u)} + \frac{\frac{1}{4}}{(2+u)}$$

Now

$$\int_0^{-1} \frac{1}{(2-u)(2+u)}\, du = \frac{1}{4}\int_0^{-1} \frac{1}{2-u} + \frac{1}{2+u}\, du$$

$$= \frac{1}{4}\left[-In(2-u) + In(2+u)\right]_0^{-1}$$

$$= \frac{1}{4}\left[In\left(\frac{2+u}{2-u}\right)\right]_0^{-1}$$

$$= \frac{1}{4}\left[In\left(\frac{2-1}{2+1}\right)\right] - \frac{1}{4}\left[In\left(\frac{2+0}{2-0}\right)\right]$$

$$= \frac{1}{4}\left[In\left(\frac{1}{3}\right)\right] - \frac{1}{4}\left[In(1)\right]$$

$$= \frac{1}{4} In\left(\frac{1}{3}\right) \quad (as\ required)$$

Question Two

ZIMSEC NOVEMBER 2018 PAPER 2

a) Use the substitution $x = Cos^2\theta$ to show that

$$\int \sqrt{\frac{x}{1-x}}\,dx = -\int 2Cos^2\theta\, d\theta$$

b) Hence evaluate

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx,$$

correct to 3 decimal places.

Suggested Solution

a) $\int \sqrt{\frac{x}{1-x}}\,dx$

Let $x = Cos^2\theta$

$$\frac{dx}{d\theta} = -2Cos\theta Sin\theta$$

$\Rightarrow dx = -2Cos\theta Sin\theta d\theta$

Now:

$$\int \sqrt{\frac{x}{1-x}}\,dx = -\int \sqrt{\frac{Cos^2\theta}{1-Cos^2\theta}}(2Cos\theta Sin\theta)d\theta$$

Use $Cos^2\theta + Sin^2\theta = 1$

$$= -\int \sqrt{\frac{Cos^2\theta}{Sin^2\theta}}(2Cos\theta Sin\theta)d\theta$$

$$= -\int \frac{Cos\theta}{Sin\theta}(2Cos\theta Sin\theta)d\theta$$

$$= -\int Cos\theta(2Cos\theta)d\theta$$

$$\therefore \int \sqrt{\frac{x}{1-x}}dx = -\int 2Cos^2\theta\, d\theta \quad (as\ required)$$

b) Now

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}dx$$

$\theta = cos^{-1}(\sqrt{x})$

x	$\frac{1}{2}$	0
θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$

$$-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2Cos^2\theta d\theta$$

Use $Cos2\theta = 2Cos^2\theta - 1$

$$-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\left(\frac{Cos2\theta + 1}{2}\right)d\theta = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} Cos2\theta + 1d\theta$$

$$= \left[-\frac{Sin2\theta}{2} - \theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$= \left[-\frac{Sin2\left(\frac{\pi}{4}\right)}{2} - \frac{\pi}{4}\right] - \left[-\frac{Sin2\left(\frac{\pi}{2}\right)}{2} - \frac{\pi}{2}\right]$$

$$= \left[-\frac{Sin\left(\frac{\pi}{2}\right)}{2} - \frac{\pi}{4}\right] - \left[-\frac{Sin(\pi)}{2} - \frac{\pi}{2}\right]$$

$$= \left[-\frac{1}{2} - \frac{\pi}{4}\right] - \left[0 - \frac{\pi}{2}\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$= 0.2853981634$$

$$= 0.285(\text{to } 3d.p.$$

Question Three

ZIMSEC JUNE 2019 PAPER 2

Find the exact value of

$$\int_0^{\frac{\pi}{4}} [1 + Sin^2(2x)]dx$$

Suggested Solution

$$\int_0^{\frac{\pi}{4}} [1 + Sin^2(2x)]dx$$

Use $Cos4x = 1 - 2Sin^2 2x$

$$\int_0^{\frac{\pi}{4}} \left(\frac{1 - Cos4x}{2}\right) + 1\, dx = \left[\left(\frac{1}{2}x - \frac{Sin4x}{8}\right) + x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\frac{3}{2}x - \frac{Sin4x}{8}\right)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\left(\frac{\pi}{4}\right) - \frac{Sin\pi}{8}\right] - \left[\frac{3}{2}(0) - \frac{Sin0}{8}\right]$$

$$= \frac{3\pi}{8}$$

Question Four

ZIMSEC NOVEMBER 2019 PAPER 2

a) Prove that

$$Sin3\theta \equiv 3Sin\theta - 4Sin^3\theta$$

b) Hence, or otherwise evaluate

$$\int_0^{\frac{\pi}{3}} Sin^3\theta d\theta$$

Suggested Solution

a) $Sin3\theta \equiv 3Sin\theta - 4Sin^3\theta$

LHS:

$$Sin3\theta \equiv Sin(2\theta + \theta)$$

$$\equiv Sin2\theta Cos\theta + Cos\theta Sin\theta$$

$$\equiv 2Sin\theta Cos\theta Cos\theta + Sin\theta(1 - 2Sin^2\theta)$$

$$\equiv 2Sin\theta Cos^2\theta + (Sin\theta - 2Sin^3\theta)$$

$$\equiv 2Sin\theta(1 - Sin^2\theta) + (Sin\theta - 2Sin^3\theta)$$

$$\equiv (2Sin\theta - 2Sin^3\theta) + (Sin\theta - 2Sin^3\theta)$$

$$\equiv 3Sin\theta - 4Sin^3\theta \quad (as\ required)$$

b) HENCE

$$\int_0^{\frac{\pi}{3}} Sin^3\theta d\theta = \int_0^{\frac{\pi}{3}} \frac{(3Sin\theta - Sin3\theta)}{4} d\theta$$

$$= \left[\frac{\left(-3Cos\theta + \frac{Cos3\theta}{3}\right)}{4}\right]_0^{\frac{\pi}{3}}$$

$$= \left[\frac{\left(-3Cos\frac{\pi}{3} + \frac{Cos\pi}{3}\right)}{4}\right] - \left[\frac{\left(-3Cos0 + \frac{Cos0}{3}\right)}{4}\right]$$

$$= \left[\frac{\left(-\frac{3}{2} - \frac{1}{3}\right)}{4}\right] - \left[\frac{\left(-3 + \frac{1}{3}\right)}{4}\right]$$

$$= \left[\frac{\left(-\frac{11}{6}\right)}{4}\right] - \left[\frac{\left(-\frac{8}{3}\right)}{4}\right]$$

$$= -\frac{11}{24} + \frac{2}{3}$$

$$= \frac{5}{24}$$

OR OTHERWISE

$$\int_0^{\frac{\pi}{3}} Sin^3\theta d\theta = \int_0^{\frac{\pi}{3}} Sin\theta Sin^2\theta\, d\theta$$

Use $Sin^2\theta + Sin^2\theta = 1$

Now:

$$\int_0^{\frac{\pi}{3}} Sin\theta Sin^2\theta\, d\theta = \int_0^{\frac{\pi}{3}} Sin\theta(1 - Cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} (Sin\theta - Cos^2\theta Sin\theta)\, d\theta$$

NB

$$\frac{d(Cos^3\theta)}{d\theta} = -3Cos^2\theta Sin\theta$$

Now:

$$\int_0^{\frac{\pi}{3}} (Sin\theta - Cos^2\theta Sin\theta)\, d\theta = \int_0^{\frac{\pi}{3}} \left(Sin\theta + \frac{-3Cos^2\theta Sin\theta}{3}\right) d\theta$$

$$= \left[-Cos\theta + \frac{Cos^3\theta}{3}\right]\begin{matrix}\frac{\pi}{3}\\0\end{matrix}$$

$$= \left[-Cos\left(\frac{\pi}{3}\right) + \frac{\left(Cos\frac{\pi}{3}\right)^3}{3}\right] - \left[-Cos0 + \frac{(Cos0)^3}{3}\right]$$

$$= \left[-\frac{1}{2} + \frac{1}{24}\right] - \left[-1 + \frac{1}{3}\right]$$

$$= -\frac{1}{2} + \frac{1}{24} + 1 - \frac{1}{3}$$

$$= \frac{5}{24}$$

Follow Up Questions

Question 1

By using the substitution $u = e^{-\frac{x}{2}}$ find

$$\int_0^2 \frac{\sqrt{1 + e^{-\frac{x}{2}}}}{e^{\frac{x}{2}}}\, dx,$$

to 3 decimal places. **Answer 1.638**

Question 2

Use the substitution $u = 10cosx - 1$ to show that

$$\int_0^{\frac{\pi}{3}} \sqrt{10cosx - 1}\; sinxdx = \frac{19}{15}$$

Question 3

By using the substitution $u = x^5 + 1$ show that

$$\int_{-1}^{1} 5x^4 \sqrt{x^5 + 1}\ dx = \frac{4\sqrt{2}}{3}$$

Question 4

Use the substitution $u = cosx$ show that

$$\int sinxcos^2x\ dx = -\frac{1}{3}cos^3x + c$$

Question 5

By using the substitution $u = x^2 + 3$ show that

$$\int_{0}^{1} \frac{x}{(x^2 + 3)^3}\ dx = \frac{7}{576}$$

Question 6

Find

$$\int tan^5x\ dx$$

Question 7

By using the substitution $u = sinx$ show that

$$\int_{\pi}^{\frac{3}{2}\pi} sin^2x\ cosx\ dx = \frac{1}{3}$$

Question 8

Show that

$$\int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} cos^2 4x\ dx = \pi$$

Question 9

Use the substitution $u = x^3 + 1$ show that

$$\int 3x^2(x^3+1)^5 dx = \frac{(x^3+1)^6}{6} + c$$

Question 10

By using the substitution $u = sinx + 1$ show that

$$\int_{\pi}^{\frac{3}{2}\pi} cosx\sqrt{sinx+1}\ dx = -\frac{2}{3}$$

PAST EXAMINATION QUESTIONS

ZIMSEC NOVEMBER 2018 PAPER 2

a) Express $\frac{2x^2-7}{(x-3)(2x+5)}$ in the form $A + \frac{B}{x-3} + \frac{C}{2x+5}$. **[5]**

b) Hence evaluate

$$\int_4^5 \frac{2x^2 - 7}{(x-3)(2x+5)} dx$$

correct to 3 decimal places. **[4]**

ZIMSEC NOVEMBER 2017 PAPER 1

Find the exact value of

$$\int_1^2 \frac{3}{x^2(3-x)} dx$$ [**8**]

ZIMSEC NOVEMBER 2016 PAPER 1

(i) Express $\frac{x-3}{x^2-1}$ in partial fractions. **[2]**

(ii) Hence from (i) find

$$\int_{-1}^{1} \frac{x-3}{x^2-1} dx,$$

giving your answer in terms of a single logarithm. **[4]**

ZIMSEC JUNE 2016 PAPER 2

By using the substitution $u = sinx$, show that

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{cosx}{3 + cos^2 x}\, dx = \frac{1}{4} In\left(\frac{1}{3}\right).$$ [**8**]

ZIMSEC NOVEMBER 2015 PAPER 1

Evaluate exactly

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} cosec^2 x\, dx$$ [**4**]

ZIMSEC NOVEMBER 2014 PAPER 1

By using the substitution $U = sinx$, or otherwise, find the exact value of

$$\int_{0}^{\frac{\pi}{2}} \frac{cosx}{3 + cos^2 x}\, dx$$ [**5**]

ZIMSEC NOVEMBER 2013 PAPER 1

By using the substitution $x = tan\theta$, find

$$\int_{0}^{1} \frac{4}{1 + x^2}\, dx$$ [**4**]

ZIMSEC JUNE 2013 PAPER 2

Show that

$$\int_{1}^{2} \frac{x^2 + 6x + 7}{(x + 2)(x + 3)}\, dx = In\left(\frac{75e}{64}\right)$$ [**5**]

ZIMSEC NOVEMBER 2012 PAPER 2

Find the exact value of

$$\int_2^3 \frac{3}{2x^2 - x - 1}\,dx$$

giving your answer as a single logarithm. **[6]**

ZIMSEC NOVEMBER 2011 PAPER 1

a) Express the function $\frac{3}{x^2(x+3)}$ in partial fractions. **[3]**

b) Hence evaluate the integral of

$$\int_2^4 \frac{3}{x^2(x+3)}\,dx,$$

leaving your answer in exact form. **[4]**

ZIMSEC JUNE 2011 PAPER 1

(i) Use the substitution $= \sqrt{x}$, or otherwise, find the exact value of

$$\int_0^4 \frac{1}{\sqrt{x}+1}\,dx$$ [7]

(ii) Using integration by parts, show that

$$\int_0^1 \frac{In\left(\sqrt{x}+1\right)}{\sqrt{x}}\,dx = \left[2\sqrt{x} + In\left(\sqrt{x}+1\right)\right]_0^1 - \int_0^1 \frac{dx}{\sqrt{x}+1}.$$

Hence find the exact value of the integral. **[5]**

ZIMSEC NOVEMBER 2009 PAPER 1

Show that $\frac{d}{dx}\left(\frac{x}{\sqrt{4-x^2}}\right) = \frac{4}{(4-x^2)^{\frac{3}{2}}}$

Hence evaluate

$$\int_0^1 \frac{8}{(4-x^2)^{\frac{3}{2}}}dx. \qquad [4]$$

ZIMSEC NOVEMBER 2008 PAPER 1

Express $\frac{1+x}{(2+x)(3x+5)}$ in partial fractions. **[2]**

(i) By putting $x = \sqrt{3}$, or otherwise, obtain an expression for

$$\frac{1+\sqrt{3}}{(2+\sqrt{3})(3\sqrt{3}+5)}$$

in the form $a + b\sqrt{3}$, where a and b are integers. **[3]**

(ii) Show that

$$\int_{-1}^{1} \frac{1+x}{(2+x)(3x+5)}dx = In3 - \frac{4}{3}In2. \qquad [4]$$

ZIMSEC NOVEMBER 2008 PAPER 2

Show that

$$\int_{\frac{\pi}{15}}^{\frac{\pi}{5}} \frac{cos5x}{3-4sin5x}dx = \frac{1}{20}In\left(\frac{3-2\sqrt{3}}{3}\right). \qquad [5]$$

ZIMSEC JUNE 2008 PAPER 1

a) Express $\frac{11x-1}{(x+1)^2(x-3)}$ in the form $\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{C}{x-3}$. **[4]**

b) Hence evaluate

$$\int \frac{11x-1}{(x+1)^2(x-3)}dx$$ [3]

ZIMSEC JUNE 2007 PAPER 1

Use the substitution $u=2x+3$ to find

$$\int_0^1 \frac{4x}{(2x+3)^3}dx$$ [**5**]

ZIMSEC NOVEMBER 2005 PAPER 1

Find the exact value of

$$\int_0^1 tan^2x\,dx$$ [**4**]

ZIMSEC NOVEMBER 2004 PAPER 1

Express $\frac{1}{x(1000-x)}$ in partial fractions and hence show that

$$\int \frac{1}{x(1000-x)}dx=\frac{1}{1000}In\left(\frac{x}{1000-x}\right)+c.$$ [**6**]

ZIMSEC JUNE 2004 PAPER 1

Express $\frac{6-x}{(2x+1)(x^2+3)}$ in the form $\frac{A}{2x+3}+\frac{Bx+C}{x^2+3}$ where A, B and C are numerical constants to be found. **[3]**

Hence show that

$$\int_0^1 \frac{6-x}{(2x+1)(x^2+3)}dx = In\left(\frac{3\sqrt{2}}{2}\right)$$ [4]

UCLES NOVEMBER 2001 PAPER 1

The value of

$$\int_0^{\frac{\pi}{2}} \frac{sin2x}{sinx+2}dx$$

is denoted by I. Use the substitution $u = sinx$ to show that

$$I = \int_0^1 2 - \frac{4}{u+2}du$$ [5]

Hence find the exact value of I. **[3]**

UCLES NOVEMBER 2000 PAPER 1

By using the substitution $x = u^2 - 1$, or otherwise, find

$$\int_0^{\frac{\pi}{2}} \frac{x}{\sqrt{1+x}}dx$$

giving your answer in terms of x. **[5]**

UCLES PAPER 1 JUNE 1998

a) Express $\frac{1}{(x+3)(4-x2x+5)}$ in partial fractions. **[2]**

Hence find

$$\int_0^2 \frac{1}{(x+3)(4-x)}dx$$

giving the answer in terms of a single logarithm. **[4]**

b) (i) Find $\int cos^2x\, dx$ **[2]**

(ii)By using the substitution $u = sinx$, or otherwise, find $\int cos^3x\, dx$. **[4]**

UCLES PAPER 1 NOVEMBER 1997

The value of

$$\int_1^4 \frac{1}{x(1+\sqrt{x})}dx$$

is denoted by I. Use the substitution $u = \sqrt{x}$ to show that

$$I = \int_0^1 \frac{4}{u(1+u)}du$$ [**3**]

Hence, by using partial fractions, show that $I = In\left(\frac{16}{9}\right)$. **[5]**

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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