"Do not worry too much about your difficulties in Mathematics, I can assure you that mine are still greater." Albert Einstein.

TROCKERS WORK

MATHEMATICAL INDUCTION & MATRICES

MATHEMATICAL INDUCTION

COMPILED BY TARAKINO N.P. (*Trockers*)

NOTES

Definition: It is a method of proof in which a statement is proved for one step in a process, and it is shown that if the statement holds for that step, it holds for the next.

STEPS

i. **Base Case/Initial Stage**
   Show it is true for $n = 1, n = 2$ etc and give a conclusion of this stage.
ii. **Assumption/Hypothesis Stage**
   Suppose/ assume the statement is true for $n = k$.
iii. **Thesis Stage**
   Stating the result for $n = k + 1$.
iv. **Proof Stage**
   Proving true for $n = k + 1$. Use the thesis stage and the hypothesis stage to get the required result. Give a conclusion of this stage.
v. **Conclusion Stage**
   State four conditions clearly: $n = 1, n = k, n = k + 1$ and $n \in \mathbb{Z}^+$

Areas to consider

- Divisibility
- Series/Summations
- Matrices
- Integration
- Differentiation

- Factorial Notation
- Composite Functions
- Binomial *etc*

DIVISIBILITY

Examples

1. **Prove that $4^n + 5$ is divisible by 3 for all $\in \mathbb{Z}^+$.**
2. **Use the principle of Mathematical Induction to verify that for n, any positive integer, $6^n - 1$ is divisible by 5.**

Suggested Solutions

1. Let P_n denotes the statement that $4^n + 5$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

   When $n = 1$: $P_1 = 4^1 + 5 = 9$, which is divisible by 3 or we can say 9 is a multiple of 3. $\therefore P_n$ is true for $n = 1$. *(Base Case/Initial Stage)*

   Assuming that P_n is true for $n = k$: **i.e.** $4^k + 5 = 3A \rightarrow 4^k = 3A - 5$. *(Assumption/Hypothesis Stage)*

   Then for $n = k + 1$: $P_{k+1} \rightarrow 4^{k+1} + 5 = 3B$. *(Thesis Stage)*
   Proof

$$4^{k+1} + 5 = 4^k.4^1 + 5$$

$$= 4^1.4^k + 5$$

$$= 4(3A - 5) + 5$$

$$= 4.3A - 20 + 5$$

$$= 4.3A - 15$$

$$= 3(4A - 5)$$

$$= 3B \sim B = 4A - 5;$$ which is true for $n = k + 1$.

$\therefore P_n$ is true for $n = k + 1$. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k + 1$ $\therefore P_n$ is also true for $n \in \mathbb{Z}^+$ *(Conclusion Stage)*

HINT

$6^n - 1$ is divisible by 5

$5^n - 1$ is divisible by 4

$4^n - 1$ is divisible by 3

$\vdots$

$k^n - 1$ is divisible by $(k - 1)$

SUMMATIONS/SERIES

Examples

1. **Prove by induction that: $\sum_{r=1}^{n} r = \frac{n}{2}(n + 1)$ for all $n \in \mathbb{Z}^+$.**
2. **Prove by induction that:**
   $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$ for all positive integral values of n.

Suggested Solutions

1. Let P_n denotes the statement that $\sum_{r=1}^{n} r = \frac{n}{2}(n + 1)$ for all $n \in \mathbb{Z}^+$.

   When $\mathrm{n} = 1$:
   $P_n\ LHS = 1$
   $P_n\ RHS = \frac{1}{2}(1 + 1) = \frac{1}{2}(2) = 1 \equiv P_n\ LHS$
   Since $P_n\ RHS = P_n\ LHS = 1 \therefore P_n$ is true for $\mathrm{n} = 1$. *(Base Case/Initial Stage)*

   Assuming that P_n is true for $\mathrm{n} = \mathrm{k}$: **i.e.** $\sum_{r=1}^{k} r = \frac{k}{2}(k + 1)$ *(Assumption/Hypothesis Stage)*

Then for n = k + 1: $P_{k+1} \rightarrow \sum_{r=1}^{k+1} r = \frac{k+1}{2}[(k+1)+1]$. *(Thesis Stage)*

Proof

Note:

a) $\boldsymbol{\sum_{r=1}^{k} r = 1 + 2 + 3 + \cdots + k}$

b) $\boldsymbol{\sum_{r=1}^{k+1} r = 1 + 2 + 3 + \cdots + k + (k+1) = \sum_{r=1}^{k} r + (k+1)}$

$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1)$

$= \frac{k}{2}(k+1) + (k+1)$

$= (k+1)\left(\frac{k}{2} + 1\right)$

$= (k+1)\left(\frac{k+2}{2}\right)$

$= (k+1)\left[\frac{(k+1)+1}{2}\right]$

$= \frac{k+1}{2}[(k+1)+1]$, which is true for n = k + 1.

$\therefore P_n$ is true for n = k + 1. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k+1$ $\therefore P_n$ is also true for $n \in \mathbb{Z}^+$ *(Conclusion Stage)*

2. Let P_n denotes the statement that:

   For all positive integral values of n, $1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$

   When n = 1:

   $P_n\ LHS = 3(1) - 2 = 3 - 2 = 1$

   $P_n\ RHS = \frac{1.[3(1)-1]}{2} = \frac{1(2)}{2} = \frac{2}{2} = 1 \equiv P_n\ LHS$

   Since $P_n\ RHS = P_n\ LHS = 1 \therefore P_n$ is true for n = 1. *(Base Case/Initial Stage)*

   Assuming that P_n is true for n = k: **i.e.** $1 + 4 + 7 + \cdots + (3k-2) = \frac{k(3k-1)}{2}$ *(Assumption/Hypothesis Stage)*

   Then for n = k + 1: $P_{k+1} \rightarrow 1 + 4 + 7 + \cdots + [3(k+1) - 2] = \frac{(k+1)[3(k+1)-1]}{2}$ *(Thesis Stage)*

   Proof

   Note:

a) The summation of k terms $= \boldsymbol{1+4+7+\cdots+(3k-2)}$

b) The summation of (k+1) terms

$= \boldsymbol{1+4+7+\cdots+(3k-2)+\ [3(k+1)-2]}$

$= \boldsymbol{1+4+7+\cdots+(3k-2)+\ (3k-2)}$ ***(Bracket Expansion)***

$= \boldsymbol{\mathit{the\ summation\ of\ k\ terms}[1+4+7+\cdots+(3k-2)]\ +}$

$\boldsymbol{\mathit{the}\ (k+1)^{th}\ \mathit{term}[3k+1]}$

$$1+4+7+\cdots+[3(k+1)-2] = 1+4+7+\cdots+(3k+1)$$
$$= 1+4+7+\cdots+(3k-2)+(3k+1)$$
$$= \frac{k}{2}(3k-1)+(3k+1)$$
$$= \frac{k(3k-1)+2(3k+1)}{2}$$
$$= \frac{3k^2-k+6k+2}{2}$$
$$= \frac{3k^2+5k+2}{2}$$
$$= \frac{(k+1)(3k+2)}{2}$$ *(Factorisation)*
$$= \frac{(k+1)[(3k+3)-1]}{2}$$

$= \frac{(k+1)[3(k+1)-1]}{2}$ which is true for $\mathrm{n} = \mathrm{k}+1$.

$\therefore P_n$ is true for $\mathrm{n} = \mathrm{k}+1$. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k+1$ $\therefore P_n$ is also true for $n \in \mathbb{Z}^+$ *(Conclusion Stage)*

MATRICES

Examples

1. Prove by Mathematical Induction that if $\boldsymbol{Z = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}}$**, then** $\boldsymbol{Z^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}}$ **for all** $\boldsymbol{n \in \mathbb{Z}^+}$**.**

Suggested Solution

1. Let P_n denotes the statement that:

   If $Z = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, then $Z^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.

   When n = 1:

   $P_n\ LHS = Z^1 = Z = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

   $P_n\ RHS = \begin{pmatrix} 1+1 & -1 \\ 1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \equiv P_n\ LHS$

   Since $P_n\ RHS = P_n\ LHS = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \therefore P_n$ is true for n = 1. *(Base Case/Initial Stage)*

   Assuming that P_n is true for n = k: **i.e.** $Z^k = \begin{pmatrix} k+1 & -k \\ k & 1-k \end{pmatrix}$ *(Assumption/Hypothesis Stage)*

   Then for n = k + 1: $P_{k+1} \rightarrow Z^{k+1} = \begin{pmatrix} [(k+1)+1) & -(k+1) \\ k+1 & 1-(k+1) \end{pmatrix}$ *(Thesis Stage)*

   <u>Proof</u>

   Note:

   a) $\boldsymbol{Z^{k+1} = Z^k.Z^1}$

   $Z^{k+1} = Z^k.Z^1 = \begin{pmatrix} k+1 & -k \\ k & 1-k \end{pmatrix}.\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

   $= \begin{pmatrix} 2(k+1)-k & -1(k+1)+0 \\ 2k+1(1-k) & -k+0 \end{pmatrix}$

   $= \begin{pmatrix} 2k+2-k & -k-1 \\ 2k+1-k & -k \end{pmatrix}$

   $= \begin{pmatrix} k+2 & -(k+1) \\ k+1 & -k+1-1 \end{pmatrix}$

   $= \begin{pmatrix} k+2 & -(k+1) \\ k+1 & 1-(k+1) \end{pmatrix}$ which is true for n = k + 1.

$\therefore P_n$ is true for n = k + 1. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k+1\ \therefore P_n$ is also true for $n \in \mathbb{Z}^+$ *(Conclusion Stage)*

INTEGRATION

Example

1. Use Induction to prove that $\int x^n = \frac{x^{n+1}}{n+1} + c$ for $n \in \mathbb{Z}^+$.

Suggested Solution

1. Let P_n denotes the statement that: $\int x^n = \frac{x^{n+1}}{n+1} + c$ for $n \in \mathbb{Z}^+$.

   When n = 1:

   $P_n\ LHS = \int x^1 dx = \frac{x^2}{2} + c$

   $P_n\ RHS = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c \equiv P_n\ LHS$

   Since $P_n\ RHS = P_n\ LHS = \frac{x^2}{2} + c \therefore P_n$ is true for n = 1. *(Base Case/Initial Stage)*

   Assuming that P_n is true for n = k: **i.e.** $\int x^k = \frac{x^{k+1}}{k+1} + c$ *(Assumption/Hypothesis Stage)*

   Then for n = k + 1: $P_{k+1} \rightarrow \int x^{k+1} = \frac{x^{(k+1)+1}}{(k+1)+1} + c$

   (Thesis Stage)

   Proof

   Note:

a) $\int x^{k+1}dx \equiv \int x^k.x^1dx$. Then use the method of integration by parts. Choose u to be an expression which varnishes after performing 1 or more derivation.

$\int x^{k+1}dx = \int x^k.x^1dx$

$Let\ u = x^1 \rightarrow \frac{du}{dx} = 1\ and\ \frac{dv}{dx} = x^k \rightarrow v = \frac{x^{k+1}}{k+1}$

$Now\ use\ \int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}dx$

$\int x^{k+1}dx = x\frac{x^{k+1}}{k+1} - \int \frac{x^{k+1}}{k+1}(1)dx$

$= x.\frac{x^{k+1}}{k+1} - \frac{x^{(k+1)+1}}{(k+1)[(k+1)+1]} + c$

$= \frac{x^{(k+1)+1}}{k+1} - \frac{x^{(k+1)+1}}{(k+1)[(k+1)+1]} + c$ ***(Since $a^b.a^c = a^{b+c}$)***

$= \frac{x^{(k+1)+1}}{k+1}\left[1 - \frac{1}{[(k+1)+1]}\right] + c$

$= \frac{x^{(k+1)+1}}{k+1}\left[\frac{[(k+1)+1]}{[(k+1)+1]} - \frac{1}{[(k+1)+1]}\right] + c$

$= \frac{x^{(k+1)+1}}{k+1}\left[\frac{[(k+1)+1]-1}{[(k+1)+1]}\right] + c$

$= \frac{x^{(k+1)+1}}{k+1}\left[\frac{k+1}{[(k+1)+1]}\right] + c$

$= \frac{x^{(k+1)+1}}{(k+1)+1} + c$;which is true for $n = k + 1$.

$\therefore P_n$ is true for $n = k + 1$. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k + 1$ $\therefore P_n$ is also true for $n \in \mathbb{Z}^+$ *(Conclusion Stage)*

DIFFERENTIATION

Examples

1. **Prove by Induction that $\frac{d(x^n)}{dx} = nx^{n-1}$.**
2. **Use Mathematical Induction to prove that if $y = xe^x$ then $\frac{d^n y}{dx^n} = (x+n)e^x$**

Suggested Solutions

1. Let P_n denotes the statement that $\frac{d(x^n)}{dx} = nx^{n-1}$ for all $n \in \mathbb{Z}^+$.

When n = 1:

$P_n\ LHS = \frac{d(x^1)}{dx} = 1$

$P_n\ RHS = 1.x^{1-1} = 1.x^0 = 1.1 = 1 \equiv P_n\ LHS$

Since $P_n\ RHS = P_n\ LHS = 1 \therefore P_n$ is true for n = 1. *(Base Case/Initial Stage)*

Assuming that P_n is true for n = k: **i.e.** $\frac{d(x^k)}{dx} = kx^{k-1}$ *(Assumption/Hypothesis Stage)*

Then for n = k + 1: $P_{k+1} \rightarrow \frac{d(x^{k+1})}{dx} = (k+1)x^{(k+1)-1}$. *(Thesis Stage)*

Proof

Note:

a) $\frac{d(x^{k+1})}{dx} \equiv \frac{d(x^k.x^1)}{dx}$. Then use the Product Rule.

$\frac{d(x^{k+1})}{dx} = \frac{d(x^k.x^1)}{dx}$

Let $v = x^1 \rightarrow \frac{dv}{dx} = 1$ *and* $u = x^k \rightarrow \frac{du}{dx} = kx^{k-1}$

Now use $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{dv}{dx}$.

$\frac{d(x^{k+1})}{dx} = x^k(1) + x^1(kx^{k-1})$

$= x^k(1) + kx^k$

$= x^k(k+1)$

$= (k+1)x^k$

$= (k+1)x^{k+1-1}$

$= (k+1)x^{(k+1)-1}$, which is true for n = k + 1.

$\therefore P_n$ is true for n = k + 1. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k + 1$ $\therefore P_n$ is also true for $n \in \mathbb{Z}^+$*(Conclusion Stage)*

2. Let P_n denotes the statement that if $y = xe^x$ then $\frac{d^n y}{dx^n} = (x+n)e^x$ for all $n \in \mathbb{Z}^+$.

When n = 1:
$P_n\ LHS = \frac{d^1 y}{dx^1} = \frac{dy}{dx} = xe^x + e^x(1) = xe^x + e^x = (x+1)e^x$
$P_n\ RHS = (x+1)e^x \equiv P_n\ LHS$
Since $P_n\ RHS = P_n\ LHS = 1 \therefore P_n$ is true for n = 1. *(Base Case/Initial Stage)*

Assuming that P_n is true for n = k: **i.e.** $\frac{d^k y}{dx^k} = (x+k)e^x$ *(Assumption/Hypothesis Stage)*

Then for n = k + 1: $P_{k+1} \rightarrow \frac{d^{k+1}y}{dx^{k+1}} = [x + (k+1)]e^x$ *(Thesis Stage)*
<u>Proof</u>

Note:

b) $\frac{d^{k+1}y}{dx^{k+1}} \equiv \frac{d^{k+1}y}{dx^{k+1}} \equiv \frac{d^k}{dx^k}\left[\frac{d(y)}{dx}\right] \equiv \frac{d}{dx}\left[\frac{d^k}{dx^k}\right]$

$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d^k}{dx^k}\left[\frac{d(y)}{dx}\right]$
$= \frac{d}{dx}\left[\frac{d^k}{dx^k}\right]$
$= \frac{d}{dx}[(x+k)e^x]$ *(Use the Product Rule)*
$= (x+k)e^x + e^x(1)$

$= e^x[x + (k+1)]$
$= [x + (k+1)]e^x$ which is true for n = k + 1.

$\therefore P_n$ is true for n = k + 1. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k + 1$ $\therefore P_n$ is also true for $n \in \mathbb{Z}^+$*(Conclusion Stage)*

Compiled by Nyasha Patrick Tarakino – BSc. (Hon's) Mathematics (M.S.U.)
+263772978155/+263717267175

Example

Prove by Induction that $\sum_{r=1}^{n} r(r)!$ =(n+1)!-1 for all $n \in \mathbb{Z}^+$.

Suggested Solution

2. Let P_n denotes the statement that: $\sum_{r=1}^{n} r(r)!$ = (n+1)!-1 for all $n \in \mathbb{Z}^+$

   When n = 1:

   $P_n\ LHS = 1(1)! = 1 \times 1 = 1$

   $P_n\ RHS = (1+1)! - 1 = 2! - 1 = 2 \times 1 - 1 = 2 - 1 = 1 \equiv P_n\ LHS$

   Since $P_n\ RHS = P_n\ LHS = 1 \therefore P_n$ is true for n = 1. *(Base Case/Initial Stage)*

   Assuming that P_n is true for n = k: **i.e.** $\sum_{r=1}^{k} r(r)!$ =(k+1)!-1 *(Assumption/Hypothesis Stage)*

   Then for n = k + 1: $P_{k+1} \rightarrow \sum_{r=1}^{k+1} r(r)!$ =[(k+1)+1]!-1 *(Thesis Stage)*

   <u>Proof</u>

   Note:

   a) $r! = r \times (r-1) \times (r-2) \times \dots \times (r-n);\quad r > n$

   b) $(r+1)! = (r+1) \times r \times (r-1) \times (r-2) \times \dots \times (r-n);\quad r > n$

      $\rightarrow (r+1)! = (r+1) \times r!$

      $\rightarrow (r+2)! = (r+2) \times (r+1)!$

$\sum_{r=1}^{k+1} r(r)! = \sum_{r=1}^{k} r(r)! + (k+1)[(k+1)!]$

$= [\ (k+1)! - 1] + \{(k+1)(k+1)!\}$

$= \{(k+1)(k+1)!\} + (k+1)! - 1]$

$= \{(k+1)[(k+1)!] + (k+1)!\} - 1]$

$= \{(k+1)!\,[k+1+1]\} - 1]$

$= \{(k+1)! \times (k+2)\} - 1]$

$= (k+2)! - 1$

$= [(k+1)+1]! - 1$; which is true for n = k + 1.

$\therefore P_n$ is true for $n = k + 1$. *(Proof Stage)*

Since P_n is true for $n = 1, n = k, n = k + 1$ $\therefore P_n$ is also true for $n \in \mathbb{Z}^+$ *(Conclusion Stage)*

COMPOSITE FUNCTIONS AND SEQUENCES

Examples

1. If $y = xe^x$, find:

i) $\frac{dy}{dx}$

ii) $\frac{d^2y}{dx^2}$

iii) $\frac{d^3y}{dx^3}$

iv) **Suggest a formula for** $\frac{d^ny}{dx^n}$

v) **Prove your result in (iv) by Mathematical Induction.**

2. Given that a sequence $U_1, U_2, U_3, \ldots$ is defined by $U_1 = 3$; $(n \geq 1)$ $U_{n+1} = 3 - \frac{2}{U_n}$.

Prove by induction that for $n \in \mathbb{Z}^+$, $U_n = \frac{2^{n+1} - 1}{2^n - 1}$

Suggested Solutions

.

1. i) $\frac{dy}{dx} = xe^x + e^x(1) = xe^x + e^x = (x+1)e^x$ (Use the Product Rule)

ii) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = [xe^x + e^x(1)] + e^x$

$= xe^x + e^x + e^x$

$= (x+2)e^x$

iii) $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = [xe^x + e^x(1)] + e^x + e^x$

$= xe^x + e^x + e^x + e^x$

$= (x+3)e^x$

iv) $\frac{d^ny}{dx^n} = (x+n)e^x$

v) The proof has been worked out on the differentiation section.

2. $U_n = \frac{2^{n+1} - 1}{2^n - 1}$ for all $n \in \mathbb{Z}^+$.

When $n = 1$:

$U_1 = \frac{2^{1+1}-1}{2^1-1} = \frac{2^2-1}{2^1-1} = \frac{4-1}{2-1} = 3 \equiv$ the given condition $U_1 = 3$

$\therefore U_n$ is true for $n = 1$. *(Base Case/Initial Stage)*

Assuming that U is true for $n = k$: **i.e.** $U_k = \frac{2^{k+1}-1}{2^k-1}$ *(Assumption/Hypothesis Stage)*

Then for $n = k+1$: $U_{k+1} = \frac{2^{(k+1)+1}-1}{2^{k+1}-1}$ *(Thesis Stage)*

Proof

$U_{n+1} = 3 - \frac{2}{U_n}$

$\rightarrow U_{n+1} = 3 - \frac{2}{\frac{2^{k+1}-1}{2^k-1}}$

$\rightarrow U_{n+1} = 3 - \frac{2(2^k-1)}{2^{k+1}-1}$

$\rightarrow U_{n+1} = \frac{3(2^{k+1}-1)-2(2^k-1)}{2^{k+1}-1}$

$\rightarrow U_{n+1} = \frac{3 \times 2^{k+1} - 3 - 2 \times 2^k - 2}{2^{k+1}-1}$

$\rightarrow U_{n+1} = \frac{3 \times 2^{k+1} - 3 - 2^{k+1} - 2}{2^{k+1}-1}$ ***(Since $a^b.a^c = a^{b+c}$)***

$\rightarrow U_{n+1} = \frac{(2 \times 2^{k+1}) - 1}{2^{k+1}-1}$

$\rightarrow U_{n+1} = \frac{2^{k+2}-1}{2^{k+1}-1}$ ***(Since $a^b.a^c = a^{b+c}$)***

$\rightarrow U_{n+1} = \frac{2^{(k+1)+1}-1}{2^{k+1}-1}$; which is true for $n = k+1$.

U is true for $n = k+1$. *(Proof Stage)*

Since U_n is true for $n = 1, n = k, n = k+1$ $\therefore U_n$ is also true for $n \in \mathbb{Z}^+$ *(Conclusion Stage)*

PRACTICE QUESTIONS

1. **Prove that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers of n.**
2. **Prove by Mathematical Induction that $n^5 - n$ is a multiple of 5 for all natural numbers n.**
3. **Prove that $5^{6n} + 2^{3n+1} - 1$ is a multiple of 7.**
4. **Prove that $6^n + 8^n$ is a multiple of 7, when n is odd.**

5. **Prove by Mathematical Induction that $n^3 - n$ is divisible by 6 for all natural numbers n.**
6. **Prove by induction that: $1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$ for any $n \geq 1$.**
7. **Prove by induction that:**

   $$\sum_{r=1}^{n} (r^3 - r) = \frac{1}{4}n(n-1)(n+1)(n+2) \text{ for all } n \in \mathbb{Z}^+$$

   Hence Evaluate

   $$\sum_{r=3}^{50} (r^3 - r) \ .$$
8. **Prove by induction that:**

   $$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)\,(2n+1) \text{ for all } n \in \mathbb{Z}^+$$
9. **Prove by induction that:**

   $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1) \text{ for all } n \in \mathbb{Z}^+$$
10. **Use Mathematical Induction to prove that:**

    $$\sum_{r=1}^{n} = \frac{1}{4}n^2\,(n+1)^2 \text{ for all } n \in \mathbb{Z}^+$$
11. **Use Mathematical Induction to prove that:**

    $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2 \text{ for all } n \in \mathbb{Z}^+$$
12. **Prove by induction that for any $n \geq 1$;**

    $$\sum_{r=1}^{n} (3r-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1).$$
13. **Prove by induction that for any $n \geq 1$;**

    $$1 + 4 + 7 + \cdots + (3n-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$$
14. **Prove by induction that:**

    $\sum_{r=1}^{n}(2r-1) = n^2$ **for all positive integral values of n.**
15. **Prove by induction that:** $\displaystyle\sum_{r=1}^{n}(3r-2) = \frac{n}{2}(3n-1)$ **for all $n \in \mathbb{Z}^+$**
16. **Prove by Mathematical Induction that if $P = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$, then**

    $P^n = \begin{pmatrix} 2^n & (2^n - 1)a \\ 0 & 1 \end{pmatrix}$ **for all $n \in \mathbb{Z}^+$.**
17. **Prove by induction that** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.
18. **Prove by Mathematical Induction that if $Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, then**

    $Q^n = \begin{pmatrix} 1 & n & \frac{n}{2}(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ **for all $n \in \mathbb{Z}^+$.**

19. Prove by Mathematical Induction that $\begin{pmatrix}1 & a\\ 0 & a\end{pmatrix}^n = \begin{pmatrix}1 & \frac{a^{n+1}-1}{a-1}\\ 0 & a^n\end{pmatrix}$ **for all** $n \in \mathbb{Z}^+$.

20. Prove by Mathematical Induction that if $A = \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$, **then**

$A^n = \begin{pmatrix}1 & 3\\ 3n & 1\end{pmatrix}$ **for all** $n \in \mathbb{Z}^+$.

21. Prove by Mathematical Induction that if $X = \begin{pmatrix}3 & -1\\ 4 & -1\end{pmatrix}$, **then**

$X^n = \begin{pmatrix}2n+1 & -n\\ 4n & 1-2n\end{pmatrix}$ **for all** $n \in \mathbb{Z}^+$.

22. Prove by Mathematical Induction that if $B = \begin{pmatrix}1+cos\theta & sin\theta\\ sin\theta & 1-cos\theta\end{pmatrix}$, **then**

$B^{n+1} = 2^n \times B$ **for all** $n \in \mathbb{Z}^+$.

23. Use Mathematical Induction to prove that if $y = \frac{1}{2x+1}$ **then:**

$\frac{d^ny}{dx^n} = \frac{(-2)^n n!}{(2x+1)^{n+1}}$

24. Use Mathematical Induction to prove that if $y = cosx$ **then** $\frac{d^ny}{dx^n} = cos\left(x + \frac{n\pi}{2}\right)$

25. Prove by Induction that $\frac{d^n(x^m)}{dx^n} = \frac{m!}{(m-n)!}x^{m-n}; \quad m \geq n$.

26. If $\frac{dy}{dx} = 2xy - 1$, **show by Induction that** $\frac{d^{n+2}y}{dx^{n+2}} = 2(n+1)\frac{d^ny}{dx^n} + 2x\frac{d^{n+1}y}{dx^{n+1}}$.

27. Prove by Induction that if $y = sin\ ax$ **then** $\frac{d^ny}{dx^n} = a^n sin\left(x + \frac{n\pi}{2}\right)$

28. Given that a sequence $U_1, U_2, U_3, \ldots$ **is defined by** $U_1 = 5$; $(n \geq 1)$ $\quad U_{n+1} = 2U_n + 1$.
Prove by induction that for $n \in \mathbb{Z}^+$, $U_n = 3 \times 2^n - 1$

29. Given that $f(x) = \frac{x}{x+1}$, **find:**

i) $f.f(x$
ii) $f.[f.f(x)]$
iii) **Suggest a formula for** $f^n(x$
iv) **Prove the result in (iii) by Induction.**

MATRICES

NOTES

- A matrix is a rectangular array of numbers.
- Each entry in the matrix is called an element.
- Matrices are classified by the number of rows and columns that they have.
- Let A be the matrix with m rows and n columns then A called an $(m \times n)$ matrix said $m\ by\ n$
- $(m\ by\ n)$ is called the order of matrix A.
- If $(m = n)$ then the matrix is called a square matrix.
- The element of A can be denoted by a_{ij}.
- i is the row in which the element is found and j represent the column in which the element is found

Example

1. Given that $T = \begin{pmatrix} 1 & -6 & 3 \\ 2 & 5 & -1 \\ 0 & -10 & 4 \end{pmatrix}$
   a) State the order of T.
   b) List the elements a_{11} and a_{32}.

Suggested Solutions

1. a) Since T has 3 $rows$ and 3 $columns$ thus it is of order (3×3).

   b) $a_{11} = 1$ and $a_{32} = -10$.

MATRICES - ADDITION AND SUBTRACTION

NOTES

- Addition and subtraction of matrices is defined if and only if the matrices are of the same order.
- The sum/difference of matrices A and B is the matrix obtained by adding/subtracting the elements in corresponding positions of matrices A and B.
- If you multiply the matrix A by a scalar m then every element of A is multiplied by m.

Note: If A and B are matrices then the following results are true:

i. $A + B = B + A$
ii. $A - B \neq B - A$ but $A - B = -(B - A)$

Example

1. Given that $A=\begin{pmatrix}5&-2&4\\3&1&-1\\1&0&-1\end{pmatrix}$ and $B=\begin{pmatrix}4&0&0\\-2&6&2\\0&-2&4\end{pmatrix}$
   i) Prove that $A+B=B+A$ and $A-B\neq B-A$
   ii) Find $4A-\frac{1}{2}B$

Suggested Solutions

1. i) $A+B=\begin{pmatrix}5&-2&4\\3&1&-1\\1&0&-1\end{pmatrix}+\begin{pmatrix}4&0&0\\-2&6&2\\0&-2&4\end{pmatrix}=\begin{pmatrix}9&-2&4\\1&7&1\\1&-2&3\end{pmatrix}$

$B+A=\begin{pmatrix}4&0&0\\-2&6&2\\0&-2&4\end{pmatrix}+\begin{pmatrix}5&-2&4\\3&1&-1\\1&0&-1\end{pmatrix}=\begin{pmatrix}9&-2&4\\1&7&1\\1&-2&3\end{pmatrix}$

Since $A+B=\begin{pmatrix}9&-2&4\\1&7&1\\1&-2&3\end{pmatrix}$ and $B+A=\begin{pmatrix}9&-2&4\\1&7&1\\1&-2&3\end{pmatrix}$ $\therefore A+B=B+A$

Now $A-B=\begin{pmatrix}5&-2&4\\3&1&-1\\1&0&-1\end{pmatrix}-\begin{pmatrix}4&0&0\\-2&6&2\\0&-2&4\end{pmatrix}=\begin{pmatrix}1&-2&4\\5&-5&-3\\1&2&-5\end{pmatrix}$

$B-A=\begin{pmatrix}4&0&0\\-2&6&2\\0&-2&4\end{pmatrix}-\begin{pmatrix}5&-2&4\\3&1&-1\\1&0&-1\end{pmatrix}$

$=\begin{pmatrix}-1&2&-4\\-5&5&3\\-1&-2&5\end{pmatrix}$

$=-\begin{pmatrix}1&-2&4\\5&-5&-3\\1&2&-5\end{pmatrix}\neq A-B$ but $-(A-B)$ (proven)

ii) $4A-\frac{1}{2}B=4\begin{pmatrix}5&-2&4\\3&1&-1\\1&0&-1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}4&0&0\\-2&6&2\\0&-2&4\end{pmatrix}$

$=\begin{pmatrix}20&-8&16\\12&4&-4\\4&0&-4\end{pmatrix}-\begin{pmatrix}2&0&0\\-1&3&1\\0&-1&2\end{pmatrix}$

$$= \begin{pmatrix} 18 & -8 & 16 \\ 13 & 1 & -5 \\ 4 & 1 & -6 \end{pmatrix}$$

MATRICES - MULTIPLICATION

NOTES

- Let A and B be matrices. Then the multiplication of A and B is possible if and only if then number of $columns$ of A is equal to the number of $rows$ of B.
- If A is an $(m \times n)$ matrix and B is an $(x \times y)$ matrix, then the product AB exist if and only if $= x$.
- AB will be an $(m \times y)$ matrix.

Note:

1. The product $AB \neq BA$.

Example

1. Given that $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$

   i) Prove that $AB \neq BA$

   ii) Find A^2

Suggested Solutions

1. i) $AB = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} (-1\times 2)+(1\times -2)+(0\times 0) & (-1\times 0)+(1\times 0)+(0\times -1) & (-1\times 1)+(1\times 2)+(0\times 1) \\ (0\times 2)+(2\times -2)+(-2\times 0) & (0\times 0)+(2\times 0)+(-2\times -1) & (0\times 1)+(2\times 2)+(-2\times 1) \\ (1\times 2)+(0\times -2)+(-1\times 0) & (1\times 0)+(0\times 0)+(-1\times -1) & (1\times 1)+(0\times 2)+(-1\times 1) \end{pmatrix}$$

$$= \begin{pmatrix} -2-2+0 & 0+0+0 & -1+2+0 \\ 0-4+0 & 0+0+2 & 0+4-2 \\ 2+0+0 & 0+0+1 & 1+0-1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 & 1 \\ -4 & 2 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

$and\ BA = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} (-2\times -1)+(0\times 0)+(1\times 1) & (-2\times 1)+(0\times 2)+(1\times 0) & (-2\times 0)+(0\times -2)+(1\times -1) \\ (-2\times -1)+(0\times 0)+(2\times 1) & (-2\times 1)+(0\times 2)+(2\times 0) & (-2\times 0)+(0\times -2)+(2\times -1) \\ (0\times -1)+(-1\times 0)+(1\times 1) & (0\times 1)+(-1\times 2)+(1\times 0) & (0\times 0)+(-1\times -2)+(1\times -1) \end{pmatrix}$$

$$= \begin{pmatrix} 2+0+1 & -2+0+0 & 0+0-1 \\ 2+0+2 & -2+0+0 & 0+0-2 \\ 0+0+1 & 0-2+0 & 0+2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & -1 \\ 4 & -2 & -2 \\ 1 & -2 & 1 \end{pmatrix} \neq AB \text{ (proven).}$$

ii) $A^2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 1+0+0 & -1+2+0 & 0-2+0 \\ 0+0-2 & 0+4+0 & 0-4+2 \\ -1+0-1 & 1+0+0 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -2 \\ -2 & 4 & -2 \\ -2 & 1 & -1 \end{pmatrix}$$

SPECIAL MATRICES

NOTES

- An Identity Matrix is a special matrix in which if you pre-multiply or post-multiply any matrix A by it, A remains unchanged. i.e. $I_nA = AI_n - A$
- The identity matrix of an $(n \times n)$ is represented by I_n where n is the number of rows or columns since it is a square matrix
- The general matrix of an identity matrix is given by: $I_n = \begin{pmatrix} 1 & 0 & & \dots & & 0 & 0 \\ 0 & 1 & & \ddots & & \vdots & 0 \\ \vdots & 0 & & & & 0 & \vdots \\ 0 & \vdots & \dots & & 0 & 1 & 0 \\ 0 & 0 & \dots & & \dots & 0 & 1 \end{pmatrix}$
- For example: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix for a (2×2) matrix and $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix for a (3×3) matrix.
- The Zero Matrix is a matrix whose elements are all Zeroes.
- The Zero Matrix is represented by Z e.g. a (3×3) matrix is given by $I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- Adding Z to any matrix, say A leaves A unchanged and pre-multiplying or post-multiplying Z with A, leaves Z unchanged
- i.e. $Z + A = A + Z = A$ and $ZA = AZ = Z$

MATRICES - TRANSPOSING

NOTES

- The transpose of a matrix A is found by inter-changing rows and columns of matrix A. i.e.
- The transpose of matrix A is represented as A^T orA'
- If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$.
- The elements in the diagonal column does not change i.e. a, e and i

Examples

1. Given that $A = \begin{pmatrix} 2 & 5 & 11 \\ -8 & 0 & 7 \\ 3 & 10 & 9 \end{pmatrix}$; find A'.

Suggested Solution

1. $A' = \begin{pmatrix} 2 & -8 & 3 \\ 5 & 0 & 10 \\ 11 & 7 & 9 \end{pmatrix}$

MATRICES - DETERMINANTS

NOTES

- Determinants play a major role in finding the inverse of the matrix and also in solving a system of linear equations.

DETERMINANTS – 2X2 MATRICES

NOTES

- Suppose A is any (2×2) Matrix such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of A can be written as $|A|$ or $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Examples

1. Given that $A = \begin{pmatrix} 6 & -1 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$;find the determinant of A and the determinant of B.

Suggested Solutions

1. $\det(A) = ad - bc = 6(1) - (-1 \times 4) = 6 + 4 = 10.$
   $\det(B) = ad - bc = cos\theta(cos\theta) - (sin\theta \times -sin\theta)$
   $= cos^2\theta + sin^2\theta \equiv 1$

DETERMINANTS – 3X3 MATRICES

NOTES

- Suppose A is any (3×3) Matrix such that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then the determinant of A can be written as $|A|$ or $\det(A)$ or $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$
- Several techniques can be employed to find the determinants for (3×3) Matrices i.e.
  i) The Basket weave formula
  ii) The method of cofactors
  iii) Multiplying A by its Adjoint Matrix.

DETERMINANTS – The Basket Weave Formula

NOTES

- We add the first and the second columns of A as follows:
- $\det(A) = \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}$

- $\det(A) =$ "*sum of products along the* ↗ *diagonals*" $-$ "*sum of products along* ↘ *diagonals*"
- $= aei + bfg + cdh - gec - hfa - idb$

DETERMINANTS – The Method of Cofactors

NOTES

- This method can be applied to all $(n \times n)$ Matrices.
- Suppose A is any (3×3) Matrix such that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then to find the determinant you choose a row or column to work with; for instance we are going to choose the row containing the elements $(a \quad b \quad c)$.
- When choosing the row or column to work with, choose the one with 1 or more zeroes because it is easy to simplify.
- We multiply each element a_{ij} in the chosen row/column by the determinant of a (2×2) matrix which remains when the row and the column containing a_{ij} are deleted from A. This is illustrated as follows:
- $\det(A) = (-1)^{i+j}\, a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{i+j}\, b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (-1)^{i+j}\, c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- $'i'$ represent the row in which the element a_{ij} from the chosen row/column is found $'j'$ represent the column in which the element a_{ij} from the chosen row/column is found
- If $(i + j)$ is even then a_{ij} is even and if $(i + j)$ is odd then a_{ij} is odd. For example:
  i. $(-1)^{1+1}\, a = (-1)^2\, a = +a$
  ii. $(-1)^{2+1}\, b = (-1)^3\, b = -b$
  iii. $(-1)^{3+1}\, c = (-1)^4\, c = +c$
- $\therefore \det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

Examples

2. Given that $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}$; find the determinant of A using the both methods.

Suggested Solutions

a. <u>The Method of Cofactors</u>

1. $\det(A) = 3\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

$= 3(2-2) - 1(-4-2) + 1(2--1)$

$= 3(0) - 1(-6) + 1(3)$

$= 0 + 6 + 3$

$= 9$

b. <u>The Basket Weave Formula</u>

1. $\det(A) = \begin{vmatrix} 3 & 1 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & -1 \\ 1 & 1 & -2 & 1 & 1 \end{vmatrix}$

$= (3\times -1 \times -2) + (1 \times 2 \times 1) + (1 \times 2 \times 1) - (1 \times -1 \times 1) - (1 \times 2 \times 3) - (-2 \times 2 \times 1)$

$= (6) + (2) + (2) - (-1) - (6) - (-4)$

$= 10 + 1 - 6 + 4$

$= 5 + 4$

$= 9$

INVERSE OF MATRICES

2x2 MATRICES

NOTES

- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; where $ad - bc$ is the determinant.
- $AA^{-1} = A^{-1}A = I$ where I is the identity matrix.
- If $ad = bc$ then $\frac{1}{ad-bc} = \frac{1}{0}$, which is not defined and in this case, A^{-1} does not exist and the matrix A is described as singular or non-invertible.
- If A^{-1} exist, then the matrix A is described as being non-singular or invertible.
- If we have two matrices A and B then $(AB)^{-1} = B^{-1}A^{-1}$

Examples

1. Find the inverse of $A = \begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix}$

Suggested Solutions

1. $\text{Det}(A) = ad - bc = 3(1) - 2(-4) = 3 + 8 = 11$

   Now $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 1 & -(-4) \\ -2 & 3 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$.

3x3 MATRICES

NOTES

- For any $(n \times n)$ matrix the Adjoint and determinant are defined and satisfy:
  $A \times adj(A) = adj(A) \times A = \det(A) \times I_n$
- If $\det(A) \neq 0$ then A is said to be invertible and A^{-1} exist i.e.
  $A^{-1} = \frac{1}{\det(A)} \times adj(A)$.
- If $\det(A) = 0$ then A is said to be non-invertible and A^{-1} does not exist.

Examples

1. Find the inverse of $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}$.

Suggested Solutions

1. We begin by finding the Minors of matrix A, followed by the Cofactors of matrix A then the Adjoint and finally A^{-1}.

i. The Matrix of Minors

- For each of every element of matrix A, a_{ij}, we define the minor, M_{ij} of a_{ij}, to be the determinant of the 2×2 matrix which remains when the row and the column containing a_{ij} is deleted from A.
- $'i'$ represent the row in which the element a_{ij} is found and $'j'$ represent the column in which the element a_{ij} is found
- $M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$

$$M_{11} = \begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} = (-1 \times 2) - (1 \times 4) = -2 - 4 = -6$$

$$M_{12} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = (3 \times 2) - (1 \times 4) = 6 - 4 = 2$$

$M_{13} = \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = (3 \times 1) - (-1 \times 1) = 3 + 1 = 4$

$M_{21} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = (1 \times 2) - (1 \times 0) = 2 - 0 = 2$

$M_{22} = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = (2 \times 2) - (1 \times 0) = 4 - 0 = 4$

$M_{23} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2 \times 1) - (1 \times 1) = 2 - 1 = 1$

$M_{31} = \begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} = (1 \times 4) - (0 \times -1) = 4 - 0 = 4$

$M_{32} = \begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} = (2 \times 4) - (0 \times 3) = 8 - 0 = 8$

$M_{33} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = (2 \times -1) - (1 \times 3) = -2 - 3 = -5$

$\therefore M = \begin{pmatrix} -6 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 8 & -5 \end{pmatrix}$

ii. The Matrix of Minors

- We define the cofactors of A as follows:

a. $C_{ij} = M_{ij}$ if $i + j$ is even.

b. $C_{ij} = -M_{ij}$ if $i + j$ is odd.

- Hence the following pattern of signs is produced; $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.
- Now C is the matrix of cofactors of matrix A and it differs from matrix M by the above pattern of signs

$C = \begin{pmatrix} +(-6) & -(2) & +(4) \\ -(2) & +(4) & -(1) \\ +(4) & -(8) & +(-5) \end{pmatrix} = \begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -1 \\ 4 & -8 & -5 \end{pmatrix}$

iii. The Adjoint

- The Adjoint of A is C' or C^T, which is the transpose of the matrix of cofactors of A.

$Adj(A) = \begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}$

iv. The Inverse

- We now compute $A \times Adj(A)$

$$A \times Adj(A) = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} (2\times -6)+(1\times -2)+(0\times 4) & (2\times -2)+(1\times 4)+(0\times -1) & (2\times 4)+(1\times -8)+(0\times -5) \\ (3\times -6)+(-1\times -2)+(4\times 4) & (3\times -2)+(-1\times 4)+(4\times -1) & (3\times 4)+(-1\times -8)+(4\times -5) \\ (1\times -6)+(1\times -2)+(2\times 4) & (1\times -2)+(1\times 4)+(2\times -1) & (1\times 4)+(1\times -8)+(2\times -5) \end{pmatrix}$$

$$= \begin{pmatrix} -12-2+0 & -4+4+0 & 8-8+0 \\ -18+4+16 & -6-4-4 & 12+8-20 \\ -6-2+8 & -2+4-2 & 4-8-10 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{pmatrix}$$

- Now $A \times adj(A) = adj(A) \times A = \det(A) \times I_n$

Hence $\begin{pmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{pmatrix} = -\frac{1}{14} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{1}{14} \times I_n$.

$\therefore \det(A) = -14$

- N.B. This is the other method of computing the determinant of any $(n \times n)$ matrix.

Now $A^{-1} = \frac{1}{\det(A)} \times Adj(A) = -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}$.

MATRICES -Solving Systems of Linear Equations

2x2 MATRICES

NOTES

- In general if:

$ax + by = e$

$cx + dy = f$

- Then the equations can be re-written as follows:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

- Now we find the inverse of a 2×2 matrix and pre-multiply both sides of the equations by it, i.e.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$

$\rightarrow I_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$

$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$

Examples

1. Use the matrix method to solve the following system of linear equations.
   $2x - y = 1$
   $x + 3y = 4$

Suggested Solutions

1. The equations can be written in matrix form as follows:
   $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

   Now let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

   Det$(A) = ad - bc = 2(3) - 1(-1) = 6 + 1 = 7$

Now $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -(-1) \\ -1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.

Pre-multiplying both sides by the inverse yields

$\frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\rightarrow I\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{bmatrix} (3\times 1) + (1\times 4) \\ (-1\times 1) + (2\times 4) \end{bmatrix}$

$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 \\ 7 \end{pmatrix}$

$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore x = 1$ and $y = 1$.

3x3 MATRICES

NOTES

- In general if:

$ax + by + cz = m$

$dx + ey + fz = n$

$gx + hy + iz = o$

- Then the equations can be re-written as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

- Now we find the inverse of a 3×3 matrix and pre-multiply both sides of the equations by it, i.e.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} \begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

$$\rightarrow I_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} \begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} \begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

Examples

1. Solve the following system of linear equations.

$$2x + y = 1$$
$$3x - y + 4z = 2$$
$$x + y + 2z = -1$$

Suggested Solutions

2. The equations can be written in matrix form as follows:

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Now let $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}$ then $A^{-1} = -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}$

(see the whole solution under the inverse for 3X3 matrices)

Pre-multiplying both sides by this inverse:

$$\rightarrow -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\rightarrow I_3\begin{pmatrix}x\\y\\z\end{pmatrix}=-\frac{1}{14}\begin{pmatrix}-6&-2&4\\-2&4&-8\\4&-1&-5\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=-\frac{1}{14}\begin{bmatrix}(-6\times 1)+(-2\times 2)+(4\times -1)\\(-2\times 1)+(4\times 2)+(-8\times -1)\\(4\times 1)+(-1\times 2)+(-5\times -1)\end{bmatrix}$$

$$\rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=-\frac{1}{14}\begin{pmatrix}-6-4-4\\-2+8+8\\4-2+5\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=-\frac{1}{14}\begin{pmatrix}-14\\14\\7\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{bmatrix}\left(\frac{-14}{-14}\right)\\\left(\frac{14}{-14}\right)\\\left(\frac{7}{-14}\right)\end{bmatrix}$$

$$\rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-1\\-\frac{1}{2}\end{pmatrix}$$

$\therefore x=1, y=-1$ and $z=-\frac{1}{2}$.

PROVERBS 16V3