MATRICES

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SYLLABUS (6042) REQUIREMENTS

- carry out basic operations of matrices
- calculate determinant of a square matrix (up to 3 x 3)
- identify null matrix, identity matrix, singular and non-singular matrix
- find inverse of a 3 x 3 non- singular matrix
- apply the result $(AB)^{-1} = B^{-1}A^{-1}$ for non-singular matrices to solve problems
- solve simultaneous equations in 2 or 3 unknowns by reducing them to the matrix equation form $(A\boldsymbol{X} = \boldsymbol{b})$

MATRICES

NOTES

- A matrix is a rectangular array of numbers.
- Each entry in the matrix is called an element.
- Matrices are classified by the number of rows and columns that they have.
- Let A be the matrix with m rows and n columns then A called an $(m \times n)$ matrix said $m\ by\ n$
- $(m\ by\ n)$ is called the order of matrix A.
- If $(m = n)$ then the matrix is called a square matrix.
- The element of A can be denoted by a_{ij}.
- i is the row in which the element is found and j represent the column in which the element is found

Example

1. Given that $T = \begin{pmatrix} 1 & -6 & 3 \\ 2 & 5 & -1 \\ 0 & -10 & 4 \end{pmatrix}$
   a) State the order of T.
   b) List the elements a_{11} and a_{32}.

Suggested Solutions

1. a) Since T has 3 $rows$ and 3 $columns$ thus it is of order (3×3).

   b) $a_{11} = 1$ and $a_{32} = -10$.

MATRICES - ADDITION AND SUBTRACTION

NOTES

- Addition and subtraction of matrices is defined if and only if the matrices are of the same order.
- The sum/difference of matrices A and B is the matrix obtained by adding/subtracting the elements in corresponding positions of matrices A and B.
- If you multiply the matrix A by a scalar m then every element of A is multiplied by m.

Note: If A and B are matrices then the following results are true:

i. $A + B = B + A$

ii. $A - B \neq B - A$ but $A - B = -(B - A)$

Example

1. Given that $A = \begin{pmatrix} 5 & -2 & 4 \\ 3 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 6 & 2 \\ 0 & -2 & 4 \end{pmatrix}$

   i) Prove that $A + B = B + A$ and $A - B \neq B - A$

   ii) Find $4A - \frac{1}{2}B$

Suggested Solutions

1. i) $A + B = \begin{pmatrix} 5 & -2 & 4 \\ 3 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ -2 & 6 & 2 \\ 0 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 9 & -2 & 4 \\ 1 & 7 & 1 \\ 1 & -2 & 3 \end{pmatrix}$

$$B + A = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 6 & 2 \\ 0 & -2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & -2 & 4 \\ 3 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 9 & -2 & 4 \\ 1 & 7 & 1 \\ 1 & -2 & 3 \end{pmatrix}$$

Since $A + B = \begin{pmatrix} 9 & -2 & 4 \\ 1 & 7 & 1 \\ 1 & -2 & 3 \end{pmatrix}$ and $B + A = \begin{pmatrix} 9 & -2 & 4 \\ 1 & 7 & 1 \\ 1 & -2 & 3 \end{pmatrix} \therefore A + B = B + A$

Now $A - B = \begin{pmatrix} 5 & -2 & 4 \\ 3 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ -2 & 6 & 2 \\ 0 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 5 & -5 & -3 \\ 1 & 2 & -5 \end{pmatrix}$

$$B - A = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 6 & 2 \\ 0 & -2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & -2 & 4 \\ 3 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & -4 \\ -5 & 5 & 3 \\ -1 & -2 & 5 \end{pmatrix}$$

$= -\begin{pmatrix} 1 & -2 & 4 \\ 5 & -5 & -3 \\ 1 & 2 & -5 \end{pmatrix} \neq A - B$ but $-(A - B)$ (proven)

ii) $4A - \frac{1}{2}B = 4\begin{pmatrix} 5 & -2 & 4 \\ 3 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 \\ -2 & 6 & 2 \\ 0 & -2 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 20 & -8 & 16 \\ 12 & 4 & -4 \\ 4 & 0 & -4 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & -8 & 16 \\ 13 & 1 & -5 \\ 4 & 1 & -6 \end{pmatrix}$$

MATRICES - MULTIPLICATION

NOTES

- Let A and B be matrices. Then the multiplication of A and B is possible if and only if then number of $columns$ of A is equal to the number of $rows$ of B.
- If A is an $(m \times n)$ matrix and B is an $(x \times y)$ matrix, then the product AB exist if and only if $= x$.
- AB will be an $(m \times y)$ matrix.

Note:

1. The product $AB \neq BA$.

Example

1. Given that $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$

   i) Prove that $AB \neq BA$

   ii) Find A^2

Suggested Solutions

1. i) $AB = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} (-1 \times 2) + (1 \times -2) + (0 \times 0) & (-1 \times 0) + (1 \times 0) + (0 \times -1) & (-1 \times 1) + (1 \times 2) + (0 \times 1) \\ (0 \times 2) + (2 \times -2) + (-2 \times 0) & (0 \times 0) + (2 \times 0) + (-2 \times -1) & (0 \times 1) + (2 \times 2) + (-2 \times 1) \\ (1 \times 2) + (0 \times -2) + (-1 \times 0) & (1 \times 0) + (0 \times 0) + (-1 \times -1) & (1 \times 1) + (0 \times 2) + (-1 \times 1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 - 2 + 0 & 0 + 0 + 0 & -1 + 2 + 0 \\ 0 - 4 + 0 & 0 + 0 + 2 & 0 + 4 - 2 \\ 2 + 0 + 0 & 0 + 0 + 1 & 1 + 0 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 & 1 \\ -4 & 2 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

$$and\ BA = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (2 \times -1) + (0 \times 0) + (1 \times 1) & (2 \times 1) + (0 \times 2) + (1 \times 0) & (2 \times 0) + (0 \times -2) + (1 \times -1) \\ (-2 \times -1) + (0 \times 0) + (2 \times 1) & (-2 \times 1) + (0 \times 2) + (2 \times 0) & (-2 \times 0) + (0 \times -2) + (2 \times -1) \\ (0 \times -1) + (-1 \times 0) + (1 \times 1) & (0 \times 1) + (-1 \times 2) + (1 \times 0) & (0 \times 0) + (-1 \times -2) + (1 \times -1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 0 + 1 & 2 + 0 + 0 & 0 + 0 - 1 \\ 2 + 0 + 2 & -2 + 0 + 0 & 0 + 0 - 2 \\ 0 + 0 + 1 & 0 - 2 + 0 & 0 + 2 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & -1 \\ 4 & -2 & -2 \\ 1 & -2 & 1 \end{pmatrix} \neq AB \text{ (proven).}$$

ii) $A^2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 1+0+0 & -1+2+0 & 0-2+0 \\ 0+0-2 & 0+4+0 & 0-4+2 \\ -1+0-1 & 1+0+0 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -2 \\ -2 & 4 & -2 \\ -2 & 1 & 1 \end{pmatrix}$$

SPECIAL MATRICES

NOTES

- An Identity Matrix is a special matrix in which if you pre-multiply or post-multiply any matrix A by it, A remains unchanged. i.e. $I_n A = AI_n = A$
- The identity matrix of an $(n \times n)$ matrixis represented by I_n where n is the number of rows or columns since it is a square matrix
- The general matrix of an identity matrix is given by: $I_n = \begin{pmatrix} 1 & 0 & \cdots & & 0 & 0 \\ 0 & 1 & \ddots & & \vdots & 0 \\ \vdots & 0 & & & 0 & \vdots \\ 0 & \vdots & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$
- For example: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix for a (2×2) matrix and

  $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix for a (3×3) matrix.
- The Zero Matrix is a matrix whose elements are all Zeroes.
- The Zero Matrix is represented by Z e.g. a (3×3) matrix is given by

  $Z_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- Adding Z to any matrix, say A leaves A unchanged and pre-multiplying or post-multiplying Z with A, leaves Z unchanged

- i.e. $Z + A = A + Z = A$ and $ZA = AZ = Z$

MATRICES - TRANSPOSING

NOTES

- The transpose of a matrix A is found by inter-changing rows and columns of matrix A.
- The transpose of matrix A is represented as A^T orA'
- If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$.
- The elements in the diagonal column does not change i.e. $a, e\ and\ i$

Examples

1. Given that $A = \begin{pmatrix} 2 & 5 & 11 \\ -8 & 0 & 7 \\ 3 & 10 & 9 \end{pmatrix}$; find A'.

Suggested Solution

1. $A' = \begin{pmatrix} 2 & -8 & 3 \\ 5 & 0 & 10 \\ 11 & 7 & 9 \end{pmatrix}$

MATRICES - DETERMINANTS

NOTES

- Determinants play a major role in finding the inverse of the matrix and also in solving a system of linear equations.

DETERMINANTS – 2X2 MATRICES

NOTES

- Suppose A is any (2×2) Matrix such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of A can be written as $|A|$ or $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Examples

1. Given that $A = \begin{pmatrix} 6 & -1 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$;find the determinant of A and the determinant of B.

Suggested Solutions

1. $\det(A) = ad - bc = 6(1) - (-1 \times 4) = 6 + 4 = 10.$
   $\det(B) = ad - bc = cos\theta(cos\theta) - (sin\theta \times -sin\theta)$
   $= cos^2\theta + sin^2\theta \equiv 1$

DETERMINANTS – 3X3 MATRICES

NOTES

- Suppose A is any (3×3) Matrix such that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then the determinant of A can be written as $|A|$ or $\det(A)$ or $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$
- Several techniques can be employed to find the determinants for (3×3) Matrices i.e.
  i) The Basket weave formula
  ii) The method of cofactors
  iii) Multiplying A by its Adjoint Matrix.

DETERMINANTS – The Basket Weave

NOTES

- We add the first and the second columns of A as follows:
- $\det(A) = \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}$
- $\det(A) =$ "$sum\ of\ products\ along\ the \searrow diagonals$" $-$ "$sum\ of\ products\ along \nearrow diagonals$"
- $= aei + bfg + cdh - gec - hfa - idb$

DETERMINANTS – The Method of Cofactors

NOTES

- This method can be applied to all $(n \times n)$ Matrices.
- Suppose A is any (3×3) Matrix such that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then to find the determinant you choose a row or column to work with; for instance we are going to choose the row containing the elements $(a \quad b \quad c)$.
- When choosing the row or column to work with, choose the one with 1 or more zeroes because it is easy to simplify.
- We multiply each element a_{ij} in the chosen row/column by the determinant of a (2×2) matrix which remains when the row and the column containing a_{ij} are deleted from A. This is illustrated as follows:
- $\det(A) = (-1)^{i+j}\, a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{i+j}\, b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (-1)^{i+j}\, c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- $'i'$ represent the row in which the element a_{ij} from the chosen row/column is found $'j'$ represent the column in which the element a_{ij} from the chosen row/column is found
- If $(i + j)$ is even then a_{ij} is even and if $(i + j)$ is odd then a_{ij} is odd. For example:
  i. $(-1)^{1+1}\, a = (-1)^2\, a = +a$
  ii. $(-1)^{2+1}\, b = (-1)^3\, b = -b$
  iii. $(-1)^{3+1}\, c = (-1)^4\, c = +c$
- $\therefore \det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

Examples

2. Given that $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}$; find the determinant of A using the both methods.

Suggested Solutions

a. <u>The Method of Cofactors</u>

1. $\det(A) = 3\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

$= 3(2-2) - 1(-4-2) + 1(2--1)$

$= 3(0) - 1(-6) + 1(3)$

$= 0 + 6 + 3$

$= 9$

b. <u>The Basket Weave Formula</u>

1. $\det(A) = \begin{vmatrix} 3 & 1 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & -1 \\ 1 & 1 & -2 & 1 & 1 \end{vmatrix}$

$= (3 \times -1 \times -2) + (1 \times 2 \times 1) + (1 \times 2 \times 1) - (1 \times -1 \times 1) - (1 \times 2 \times 3) - (-2 \times 2 \times 1)$

$= (6) + (2) + (2) - (-1) - (6) - (-4)$

$= 10 + 1 - 6 + 4$

$= 5 + 4$

$= 9$

INVERSE OF MATRICES

2×2 MATRICES

NOTES

- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; where $ad - bc$ is the determinant.
- $AA^{-1} = A^{-1}A = I$ where I is the identity matrix.
- If $ad = bc$ then $\frac{1}{ad-bc} = \frac{1}{0}$, which is not defined and in this case, A^{-1} does not exist and the matrix A is described as singular or non-invertible.
- If A^{-1} exist, then the matrix A is described as being non-singular or invertible.
- If we have two matrices A and B then $(AB)^{-1} = B^{-1}A^{-1}$

Examples

1. Find the inverse of $A = \begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix}$

Suggested Solutions

1. $\text{Det}(A) = ad - bc = 3(1) - 2(-4) = 3 + 8 = 11$

   Now $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 1 & -(-4) \\ -2 & 3 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$.

3×3 MATRICES

NOTES

- For any $(n \times n)$ matrix the Adjoint and determinant are defined and satisfy: $A \times adj(A) = adj(A) \times A = \det(A) \times I_n$
- If $\det(A) \neq 0$ then A is said to be invertible and A^{-1} exist i.e. $A^{-1} = \frac{1}{\det(A)} \times adj(A)$.
- If $\det(A) = 0$ then A is said to be non-invertible and A^{-1} does not exist.

Examples

1. Find the inverse of $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}$.

Suggested Solutions

1. We begin by finding the Minors of matrix A, followed by the Cofactors of matrix A then the Adjoint and finally A^{-1}.

i. **The Matrix of Minors**

   - For each of every element of matrix A, a_{ij}, we define the minor, M_{ij} of a_{ij}, to be the determinant of the 2×2 matrix which remains when the row and the column containing a_{ij} is deleted from A.
   - $'i'$ represent the row in which the element a_{ij} is found and $'j'$ represent the column in which the element a_{ij} is found

- $M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$

$M_{11} = \begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} = (-1 \times 2) - (1 \times 4) = -2 - 4 = -6$

$M_{12} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = (3 \times 2) - (1 \times 4) = 6 - 4 = 2$

$M_{13} = \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = (3 \times 1) - (-1 \times 1) = 3 + 1 = 4$

$M_{21} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = (1 \times 2) - (1 \times 0) = 2 - 0 = 2$

$M_{22} = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = (2 \times 2) - (1 \times 0) = 4 - 0 = 4$

$M_{23} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2 \times 1) - (1 \times 1) = 2 - 1 = 1$

$M_{31} = \begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} = (1 \times 4) - (0 \times -1) = 4 - 0 = 4$

$M_{32} = \begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} = (2 \times 4) - (0 \times 3) = 8 - 0 = 8$

$M_{33} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = (2 \times -1) - (1 \times 3) = -2 - 3 = -5$

$\therefore M = \begin{pmatrix} -6 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 8 & -5 \end{pmatrix}$

ii. **The Matrix of Cofactors**

- We define the cofactors of A as follows:

a. $C_{ij} = M_{ij}$ if $i + j$ is even.

b. $C_{ij} = -M_{ij}$ if $i + j$ is odd.

- Hence the following pattern of signs is produced; $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.
- Now C is the matrix of cofactors of matrix A and it differs from matrix M by the above pattern of signs

$$C=\begin{pmatrix}+(-6) & -(2) & +(4)\\ -(2) & +(4) & -(1)\\ +(4) & -(8) & +(-5)\end{pmatrix}=\begin{pmatrix}-6 & -2 & 4\\ -2 & 4 & -1\\ 4 & -8 & -5\end{pmatrix}$$

iii. **The Adjoint**

- The Adjoint of A is C' or C^T, which is the transpose of the matrix of cofactors of A.

$$Adj(A)=\begin{pmatrix}-6 & -2 & 4\\ -2 & 4 & -8\\ 4 & -1 & -5\end{pmatrix}$$

iv. **The Inverse**

- We now compute $A\times Adj(A)$

$$A\times Adj(A)=\begin{pmatrix}2 & 1 & 0\\ 3 & -1 & 4\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}-6 & -2 & 4\\ -2 & 4 & -8\\ 4 & -1 & -5\end{pmatrix}$$

$$=\begin{pmatrix}(2\times -6)+(1\times -2)+(0\times 4) & (2\times -2)+(1\times 4)+(0\times -1) & (2\times 4)+(1\times -8)+(0\times -5)\\ (3\times -6)+(-1\times -2)+(4\times 4) & (3\times -2)+(-1\times 4)+(4\times -1) & (3\times 4)+(-1\times -8)+(4\times -5)\\ (1\times -6)+(1\times -2)+(2\times 4) & (1\times -2)+(1\times 4)+(2\times -1) & (1\times 4)+(1\times -8)+(2\times -5)\end{pmatrix}$$

$$=\begin{pmatrix}-12-2+0 & -4+4+0 & 8-8+0\\ -18+4+16 & -6-4-4 & 12+8-20\\ -6-2+8 & -2+4-2 & 4-8-10\end{pmatrix}$$

$$=\begin{pmatrix}-14 & 0 & 0\\ 0 & -14 & 0\\ 0 & 0 & -14\end{pmatrix}$$

- Now $A\times adj(A)=adj(A)\times A=\det(A)\times I_n$

Hence $\begin{pmatrix}-14 & 0 & 0\\ 0 & -14 & 0\\ 0 & 0 & -14\end{pmatrix}=-14\times\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=-14\times I_n.$

$\therefore \det(A)=-14$

- N.B. This is the other method of computing the determinant of any $(n\times n)$ matrix.

Now $A^{-1}=\frac{1}{\det(A)}\times Adj(A)=-\frac{1}{14}\begin{pmatrix}-6 & -2 & 4\\ -2 & 4 & -8\\ 4 & -1 & -5\end{pmatrix}$.

MATRICES -Solving Systems of Linear Equations

2×2 MATRICES

NOTES

- In general if:

$ax + by = e$

$cx + dy = f$

- Then the equations can be re-written as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

- Now we find the inverse of a 2×2 matrix and pre-multiply both sides of the equations by it, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$$

$$\rightarrow I_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$$

Examples

1. Use the matrix method to solve the following system of linear equations.
   $2x - y = 1$

$x + 3y = 4$

Suggested Solutions

1. The equations can be written in matrix form as follows:

$$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Now let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

$\text{Det}(A) = ad - bc = 2(3) - 1(-1) = 6 + 1 = 7$

Now $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -(-1) \\ -1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.

Pre-multiplying both sides by the inverse yields

$$\frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\rightarrow I\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{bmatrix} (3 \times 1) + (1 \times 4) \\ (-1 \times 1) + (2 \times 4) \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore x = 1$ and $y = 1$.

3×3 MATRICES

NOTES

- In general if:

$ax + by + cz = m$

$dx + ey + fz = n$

$gx + hy + iz = o$

- Then the equations can be re-written as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

- Now we find the inverse of a 3×3 matrix and pre-multiply both sides of the equations by it, i.e.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}\begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

$$\rightarrow I_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}\begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}\begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

Example

1. Solve the following system of linear equations.

   $2x + y \qquad = 1$

   $3x - y + 4z = 2$

   $x + y + 2z \quad = -1$

Suggested Solutions

2. The equations can be written in matrix form as follows:

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Now let $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}$ then $A^{-1} = -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}$

(see the whole solution under the inverse for 3X3 matrices)

Pre-multiplying both sides by this inverse:

$$\to -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\to I_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -6 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{bmatrix} (-6\times 1)+(-2\times 2)+(4\times -1) \\ (-2\times 1)+(4\times 2)+(-8\times -1) \\ (4\times 1)+(-1\times 2)+(-5\times -1) \end{bmatrix}$$

$$\to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -6-4-4 \\ -2+8+8 \\ 4-2+5 \end{pmatrix}$$

$$\to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -14 \\ 14 \\ 7 \end{pmatrix}$$

$$\to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} \left(\frac{-14}{-14}\right) \\ \left(\frac{14}{-14}\right) \\ \left(\frac{7}{-14}\right) \end{bmatrix}$$

$$\to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -\frac{1}{2} \end{pmatrix}$$

$\therefore x = 1, y = -1$ and $z = -\frac{1}{2}$.

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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