Numerical Methods

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SYLLABUS (6042) REQUIREMENTS

- Approximate the root of an equation by graphical means or sign change
- derive a converging iterative formula for solving a given equation
- Solve the equation using iterative procedure
- derive the Newton – Raphson formula
- Solve equations using the Newton – Raphson method
- Recognise cases where the iterative method may fail to converge to the required root
- Solve problems involving the use of iterative procedures in root finding
- Derive the trapezium rule
- Estimate the area under a curve using the trapezium rule
- Explain how to set bounds for the area under the curve using rectangle or trapezia
- Solve problems involving the trapezium rule

Approximating the Root of an Equation

- Most of the equations we have looked at so far since grade one have had exact **solutions** (or **roots**) that could be found using a series of algebraic manipulations.
- However, this is not true of all equations. For example: $x^5 + 5x - 3 = 0$, $e^x = x - 2$ and $\sin x = \ln x + 1$ cannot be solved exactly by any obvious algebraic method.
- Thus these roots can be estimated either by graphical considerations, sign change or iterations

GRAPHICAL MEANS AND SIGN CHANGE

SIGN CHANGE

The change-of-sign rule states that: If two values of x, a_o and b_o, can be found such that $f(a_o)$ and $f(b_o)$ are of different sign then $f(x) = 0$ must have at least one root in the interval $[a_o, b_o]$.

Notes

- A change of sign method searches systematically for a change of sign in the value of a $f(x)$ in order to find a root α such that $f(\alpha) = 0$.
- Having established that a change of sign occurs between $x = a_o$ and $x = b_o$, then a root α lies in the interval $a_o < x < b_o$, provided that the function is continuous (**it has no breaks in it)** over the same interval.

Graphical Illustration

Suppose the function is $f(x)$ and it is continuous over the same interval $a_o < x < b_o$.

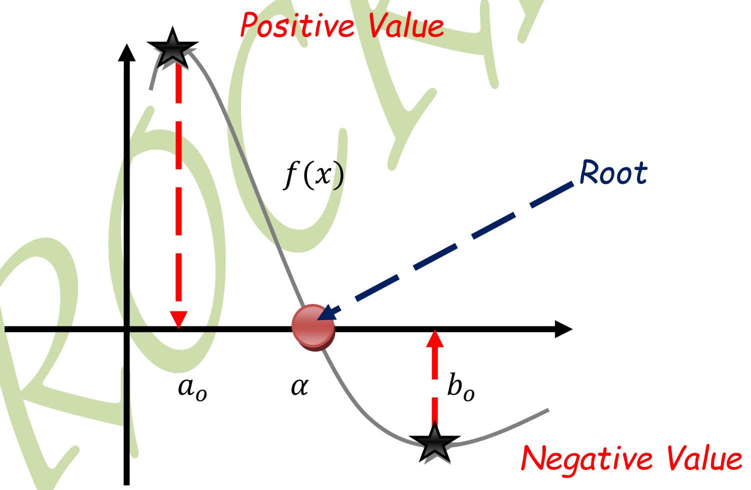
Then substituting $x = a_o$ and $x = b_o$ into $f(x)$

$f(a_o) = positive\ value$

$f(b_o) = negative\ value$

- If change of sign occurs between $x = a_o$ and $x = b_o$ then the root lies in the interval $a_o < x < b_o$ because the function crosses the $x - axis$ (*Remember the function is continuous over the same interval*)

NB: FOR TRIGONOMETRIC FUNCTIONS USE THE RADIAN MODE



Worked Examples

Question 1

Show that the root of the equation $\cos x = 2x$. lies between $x = 0.2$ and $x = 0.6$.

Suggested Solution

Start by rearranging the equation into the form $f(x) = 0$

Now let $f(x) = cos\, x - 2x$

Substituting $x = 0.2$ and $x = 0.6$ into $f(x)$. (USE RADIANS)

$f(0.2\ ) = cos\ (0.2)\ - 2(0.2\ ) = 0.580066577$ (Positive)

$f(0.6) = cos\ (0.6) - 2(0.6) = -0.374664385$ (Negative)

Since there is a change of sign therefore there is a root between $x = 0.2$ and $x = 0.$

Question 2

Show that the equation $x^3 + 2 = 4x$ has a root between $x = 1$ and $x = 2$.

Suggested Solution

Start by rearranging the equation into the form $f(x) = 0$

$x^3 - 4x + 2 = 0$

Now let $f(x) = x^3 - 4x + 2$.

Substituting $x = 1$ and $x = 2$ into $f(x)$. (NO NEED TO USE RADIANS)

$f(1) = (1)^3 - 4(1) + 2 = -1$ (Negative)

$f(2) = (2)^3 - 4(2) + 2 = 2$ (Positive)

Since $f(1) < 0$ and $f(2) > 0$, there must be a root to the equation $f(x) = 0$ between $x = 1$ and $x = 2$.

GRAPHICAL CONSIDERATION

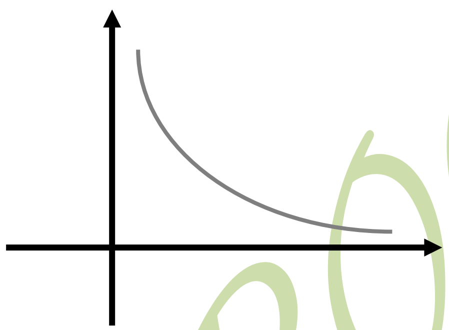
- Suppose a function $f(x)$ is continuous. In other words it has no breaks in it. The first step in solving most equations of this type is to sketch a graph.
- The graph will tell us how many roots there are and their approximate locations.
- The solution (or solutions) to $f(x) = 0$ are given by the x −values of the points where $y = f(x)$ crosses the x–axis, or if there are two functions, where the two graphs intersect.

Graphical Illustration

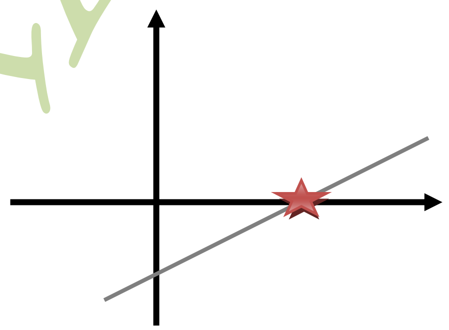
Case 1: Single graph

The number of times the graph cuts the x −axis indicates the number of roots of the equation.

No real root

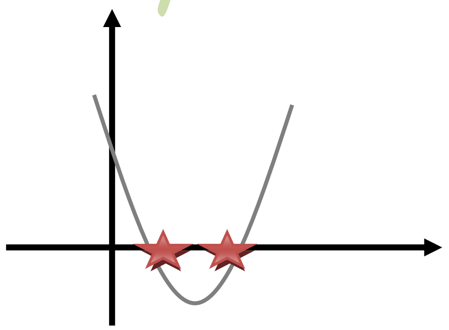


One real root

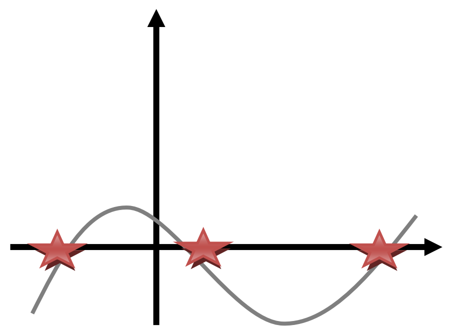


The number of times the graph cuts the x −axis indicates the number of roots of the equation.

Two real root



Three real roots

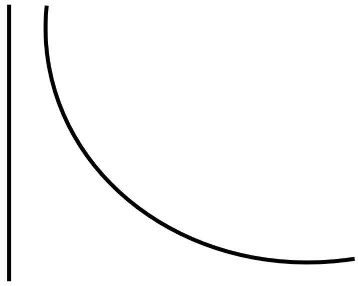


Case 2: Two graphs

The number of times the graphs intersect indicates the number of roots of the equation.

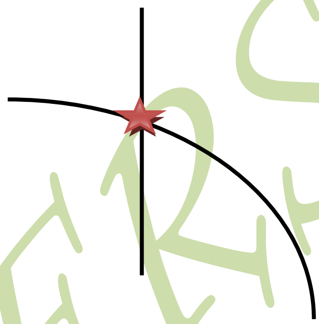
No real roots

No point of intersection



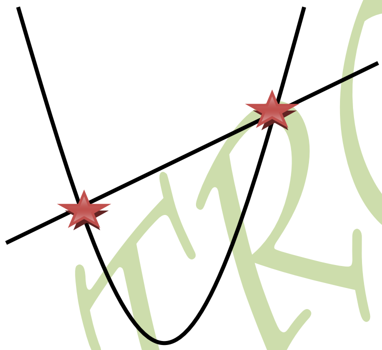
One real root

One point of intersection



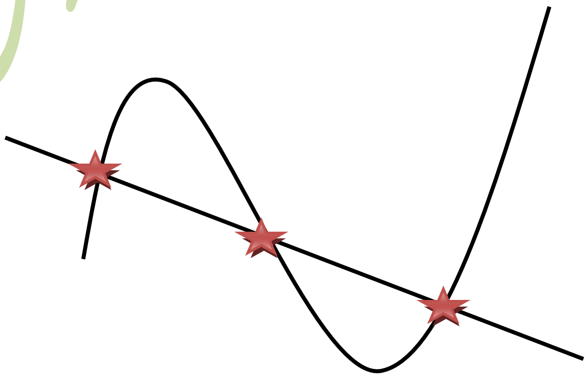
Two real roots

Two points of intersection



Three real roots

Three points of intersection

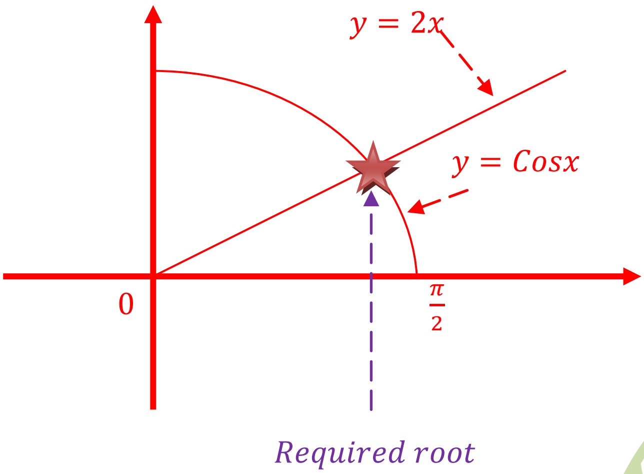


Worked Examples

Question 1

Sketch on a single diagram, the graphs of $y = \cos x$ and $y = 2x$, for $0 \leq x \leq \frac{\pi}{2}$. Hence show that there is only 1 real root for the equation $\cos x = 2x$.

Suggested Solution



Since the graphs intersect once, thus there is only one real root.

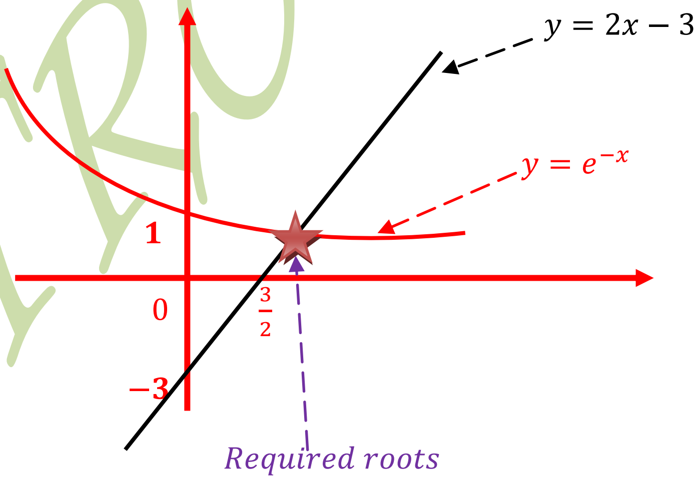
Question 2

ZIMSEC November 2017 Paper

Show that equation $e^{-x} - 2x + 3 = 0$ has only on real root, by sketching the graphs of $y = e^{-x}$ and $y = 2x - 3$, on the same axes.

Suggested Solution

$y = e^{-x}$ is the reflection of the graph of $y = e^{x}$ on the y −axis.

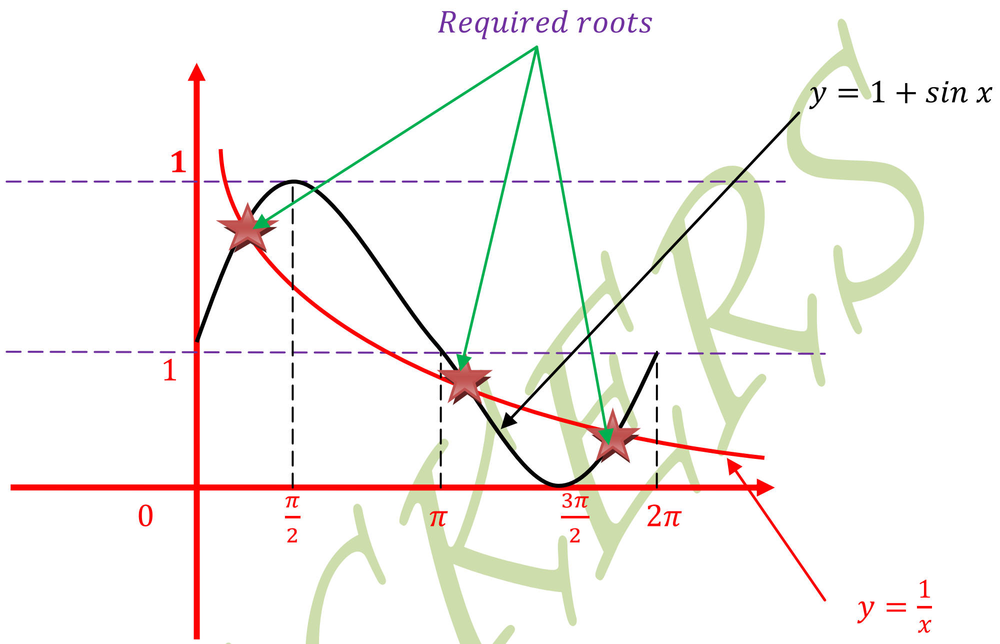


Since the graphs intersect once, thus there is only one real root.

Question 3

Sketch on a same axes, the graphs of $y = \frac{1}{x}$ and $y = 1 + sinx$ for $0 \leq x \leq 2\pi$, where x is in radians. Hence show that equation $\frac{1}{x} = 1 + sin\,x$ has three roots in the interval $[0,2\pi]$

Suggested Solution



Since the graphs intersect thrice, thus there is three real roots.

Question 4

ZIMSEC June 2017 Paper

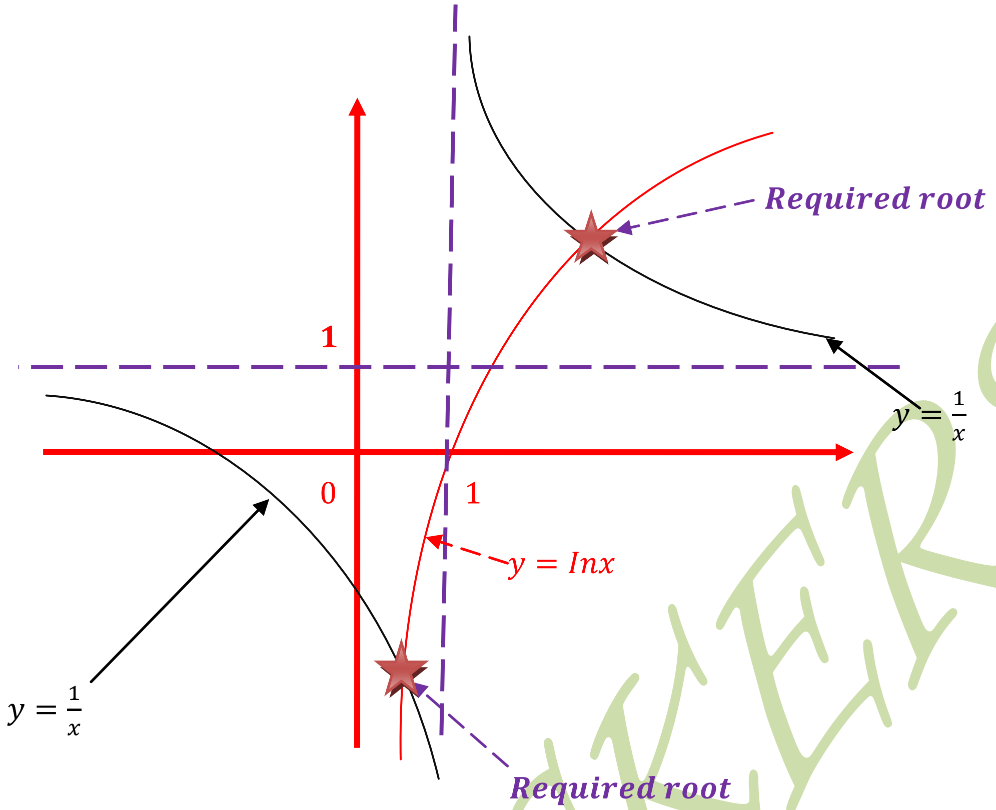
By sketching the graphs of $y = Inx$ and $= \frac{x}{x-1}$, on the same axes, show that equation $e^{-x} - 2x + 3 = 0$ has exactly two real roots.

Suggested Solution

By performing a long division the function $y = \frac{x}{x-1}$ becomes $= 1 + \frac{1}{x-1}$. Therefore the graph of $y = \frac{1}{x}$ undergoes the following transformation to yield a graph of $= 1 + \frac{1}{x-1}$.

(a) Translation, x −axis, moving 1 unit rightwards.

(b) Translation, y −axis, moving 1 unit upwards.



Since the graphs intersect twice, thus there are two real roots.

ITERATION

Notes

- This method works by using a recurrence relation to generate a sequence of approximations that get closer to the root each time. The first step is to write the equation we are trying to solve in the form $x = f(x)$. We then write this as an iterative formula of the form:

$$x_{n+1} = f(x_n).$$

- If the resulting sequence converges then this limiting value will be a root of x = f(x). Note that several different formulae are usually possible. Some may produce convergent sequences and some may not. For example, suppose we want to solve the equation

$$5x - In(x-1) = 0$$

- One way to write this equation in the form x = f(x) is:

$$x = \frac{In(x-1)}{5}$$

- This gives us the iterative formula:

$$x_{n+1} = \frac{In(x_n - 1)}{5}$$

- Now we can substitute an initial value for x_0 into this formula in the hope of generating a sequence that converges towards the root of the original equation.
- Repeating this process, we obtain a sequence of numbers, iterates, x_1, x_2, x_3,… hopefully approaching the root α.

Using the Calculator

Input the initial value (x_0) in your Calculator and press an equal sign$(=)$.

[NB: THAT VALUE IS NOW STORED IN THE CALCULATOR].

Now use the answer (ANS) stored in your calculator as follows:

$$x_{n+1} = f(ANS).$$

[NB: DO NOT INPUT ANY VALUE IN THE CALCULATOR UNTIL YOU ARE DONE]

To obtain the next root simply press an equal sign $(=)$ only and write down the displayed value as it is and then round off to the required degree of accuracy. Continue to do that up to the required root.

Worked Examples

Question 1

Write down the value of x_1 for which the iteration formula $x_{n+1} = \sqrt{\frac{4x_n+1}{x_n-1}}$, does *not* produce a valid value for x_2. Justify your answer.

Suggested Solution

$x_1 = 1$ because the iteration becomes undefined.

However, consider all values which makes $\frac{4x_n+1}{x_n-1} < 0$ eg $x_1 = 0$.

Question 2

a) Show that the equation $e^x + x = 8$ has a root between and 1 and 2.

b) Show that this equation can be arranged to give the iterative formula

$$x_{n+1} = In(8 - x_n)$$

c) Use the iterative formula found in part b) with $x_0 = 2$ to find the value of x_1, x_2, x_3, x_4 and x_5 to 5 decimal places.

Suggested Solution

a) Start by rearranging the equation into the form $f(x) = 0 \Longrightarrow e^x + x - 8 = 0$.

Now let $f(x) = e^x + x - 8$

Substituting $x = 0.2$ and $x = 0$ into $f(x)$.

$f(1\ ) = e^1 + 1 - 8 = -4.281718172$ (Negative)

$f(2) = e^2 + 2 - 8 = 1.3890560995$ (Positive)

Since there is a change of sign therefore there is a root between $x = 1$ and $x = 2$.

$e^x + x = 8$

b) $e^x = 8 - x$

$In(e^x) = In(8 - x)$

$x = In(8 - x)$

$x_{n+1} = In(8 - x_n)$ [as required]

c) $x_{n+1} = In(8 - x_n)$

$x_0 = 2$

$x_1 = x_{0+1} = In(8 - x_0) = In(8 - 2)$

$= 1.791759469 = 1.79175\ (5d.p.)$

$x_2 = x_{1+1} = In(8 - x_1) = In(8 - 1.791759469)$

$= 1.825877527 = 1.82588\ (5d.p.)$

$x_3 = 1.820366763 = 1.82037\ (5d.p.)$

$x_4 = 1.821258923 = 1.82126\ (5d.p.)$

$x_5 = 1.821114541 = 1.82111\ (5d.p.)$

Question 3

The iterative formula

$$x_{n+1} = \sqrt{\frac{(x_n)^3 + 12}{3x_n}}$$

is used to approximate the root α.

(a) Taking $x_1 = 1$, find to 6 decimal places, the values of x_2, x_3 and x_4.

(b) Given that the iteration converges, find and simplify an equation in x whose real root is being approximated by this iteration.

(c) Hence

(i) write down the exact value of this root,

(ii) Find the absolute errors e_2, e_3 and e_4, in the first three approximations x_2, x_3 and x_4, giving your answers to 6 decimal places.

Suggested Solution

(a)

$$x_{n+1} = \sqrt{\frac{(x_n)^3 + 12}{3x_n}}$$

$x_1 = 1$

$$x_2 = x_{1+1} = \sqrt{\frac{(x_1)^3 + 12}{3x_1}} = \sqrt{\frac{(1)^3 + 12}{3(1)}}$$

$= 2.08166599$

$= 2.081666\ (6\ d.p.)$

$x_3 = x_{2+1} = 1.834661356 = 1.834661\ (6\ d.p.)$

$x_4 = x_{3+1} = 1.817204712 = 1.817205\ (6\ d.p.)$

(b)

$$x_{n+1} = \sqrt{\frac{(x_n)^3 + 12}{3x_n}}$$

replacing both x_{n+1} and x_n by x:

$x = \sqrt{\frac{(x)^3+12}{3x}}$

$x^2 = \frac{(x)^3+12}{3x}$

$3x^3 = x^3 + 12$

$2x^3 = 12$

$2x^3 - 12 = 0$

$x^3 - 6 = 0$

(c)

(i) $x = \sqrt[3]{6}$

(ii) $Absolute\ Error = |Actual - Estimate|$

$$e_2 = |\sqrt[3]{6} - 2.081665199| = 0.264545397 = 0.264545$$

$$e_3 = |\sqrt[3]{6} - 1.834661356| = 0.017540763 = 0.017541$$

$$e_4 = |\sqrt[3]{6} - 1.817204712| = 0.000084119 = 0.000084$$

Question 4

ZIMSEC NOVEMBER 2003 Paper 1

The relation $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{3}{x_n}\right)$ gives a set of values converging to the root of an equation.

Find the equation in its simplest form. **[2]**

Using $x_1 = 1$, calculate x_2, x_3 and x_4 correct to 4 decimal places and give the absolute errors in each case. **[6]**

Suggested solution

Steps: Replace both x_{n+1} and x_n in $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{3}{x_n}\right)$ by x.

$\Rightarrow x = \frac{1}{3}\left(2x + \frac{3}{x}\right)$

$3x = \left(2x + \frac{3}{x}\right)$

$x = \frac{3}{x}$

$x^2 = 3$

$\therefore x^2 - 3 = 0$ is the required equation.

Now $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{3}{x_n}\right)$

$x_1 = 1$

$$x_2 = x_{1+1} = \frac{1}{3}\left(2x_1 + \frac{3}{x_1}\right) = \frac{1}{3}\left[2(1) + \frac{3}{1}\right] = \frac{5}{3} = 1.666666666667 = 1.6667\ (4d.p.)$$

$$x_3 = x_{2+1} = \frac{1}{3}\left(2x_2 + \frac{3}{x_2}\right) = \frac{1}{3}\left[2\left(\frac{5}{3}\right) + \frac{3}{\left(\frac{5}{3}\right)}\right] = 1.711111111111 = 1.7111\ (4d.p.)$$

$x_4 = 1.725156325 = 1.7252\ (4d.p.)$

Calculating the errors

Let the errors for x_2, x_3 and x_4 be e_2, e_3 and e_4 respectively.

Since $x^2 - 3 = 0$ is the required equation $\therefore$ the exact value is $x = \sqrt{3}$.

$Absolute\ error = |Actual\ Value - Estimate|$

$e_2 = \left|\sqrt{3} - 1.6667\right| = 0.065350807 = 0.0654\ (4d.p.)$

$e_3 = |\sqrt{3} - 1.7111| = 0.020950807 = 0.0210\ (4.d.p.)$

$e_4 = |\sqrt{3} - 1.7252| = 0.0068500807 = 0.0069\ (4d.p.)$

Past Examination Questions

Question 1

CAMBRIDGE June 2015

i) Given that

$$\int_0^a \left(3e^{\frac{1}{2}x} + 1\right) dx = 10$$

show that the positive constant a satisfies the equation

$$a = 2In\left(\frac{16 - a}{6}\right).$$

ii) Use an iterative formula $a_{n+1} = 2In\left(\frac{16-a_n}{6}\right)$ with $a_1 = 2$ to find the value of a correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

Answer: 1.732

Question 2

ZIMSEC June 2001 Paper 1

Use the iterative formula

$$x_{n+1} = \frac{2({x_n}^3 + 1)}{3{x^2}_n},$$

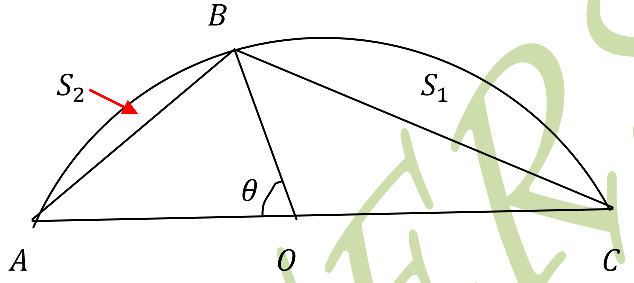
where $x_0 = 1$, to calculate x_1, x_2 and x_3. Give your answers to 6 decimal places.

Given that the iteration converges, find and simplify an equation in x whose real root is being approximated by this iteration.

Hence write down the exact value of this root. Find the absolute errors e_1, e_2 and e_3, in the first three approximations x_1, x_2 and x_3, giving your answers to 6 decimal places.

Question 3

UCLES November 1998 Paper 1



The diagram shows a semicircle ABC on AC as diameter. The midpoint of AC is O, and angle $AOB = \theta$ radians, where $0 < \theta < \frac{\pi}{2}$. The area of the segment S_1 bounded by the chord BC is twice the area of the segment S_2 bounded by the chord AB. Show that $3\theta = \pi + sin\theta$.

Using the iterative formula

$$\theta_n = \frac{1}{3}(\pi + sin\theta_n),$$

together with a suitable starting value, to find θ correct to 3 significant figures. You should show the value of each approximation that you calculate.

NEWTON RAPHSON METHOD

- It is also called the Newton Method, and actually it is a powerful technique for solving equations numerically. The Newton-Raphson method is another iterative method that can potentially converge on a root very rapidly.
- Newton's method may converge slowly at first. However, as the iteration come closer to the root, the speed of convergence increases.

Newton Raphson Method: Derivation

- Several methods are used to derive the Newton Raphson Formula. In this case we are going to use the Taylor series method.

- The Taylor polynomial for $f(x)$ is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n$$

- As the function approaches a root, higher-order terms of the Taylor polynomial will approach zero. Therefore, we can consider the higher-order terms trivial. We truncate these terms to get a linear approximate to (x) , given as:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- Since we are solving for a root, we set $f(x) = 0$ and solve for x:

$$0 = f(x_0) + f'(x_0)(x - x_0)$$

$$\frac{0}{f'(x_0)} = \frac{f(x_0)}{'(x_0)} + \frac{f'(x_0)(x - x_0)}{f'(x_0)}$$

$$0 = \frac{f(x_0)}{f'(x_0)} + (x - x_0)$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- From this we can derive the Newton Raphson method, given as:

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right]$$

- This creates an iterative process to more accurately approximate a root.

Using the Calculator

Input the initial value $(\boldsymbol{x_0})$ in your Calculator and press an equal sign$(=)$.

[N.B.: THAT VALUE IS NOW STORED IN THE CALCULATOR].

Now use the answer $(\boldsymbol{ANS})$ stored in your calculator as follows:

$$\boldsymbol{x_{n+1} = ANS - \left[\frac{f(ANS)}{f'(ANS)}\right]}$$

[N.B.: DO NOT INPUT ANY VALUE IN THE CALCULATOR UNTIL YOU ARE DONE].

To obtain the next root simply press an equal sign $(=)$ only and write down the displayed value as it is and then round off to the required degree of accuracy. Continue to do that up to the required root.

Worked Examples

Question 1

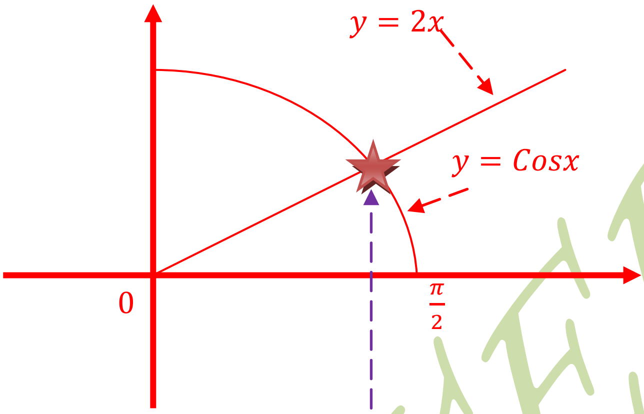
a) Sketch on a single diagram, the graphs of $y = \cos x$ and $y = 2x$, for $0 \le x \le \frac{\pi}{2}$. Hence show that there is only 1 real root for the equation $\cos x = 2x$.

b) Show that the root lies between $x = 0.2$ and $x = 0.6$.

c) Starting with $x_0 = 0.5$, use the Newton Rhaphson method to find the root correct to 4 decimal places.

<u>Suggested Solution</u>

a)



Required root

Since the graphs intersect once, thus there is only one real root.

.b) Let $f(x) = cos\ x - 2x$

Substituting $x = 0.2$ and $x = 0$ into $f(x)$. (USE RADIANS)

$f(0.2\ ) = cos\ (0.2) - 2(0.2\ ) = 0.580066577$ (Positive)

$f(0.6) = cos\ (0.6) - 2(0.6) = -0.374664385$ (Negative)

Since there is a change of sign therefore there is a root between $x = 0.2$ and $x = 0$.

c) $f(x) = cos\ x - 2x$

$f'(x) = -sin\ x - 2 = -(sinx + 2)$

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right]$$

$x_0 = 0.5$

$$x_1 = x_{0+1} = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 0.5 - \frac{cos\ 0.5 - 2(0.5)}{-(\sin(0.5) + 2)}$$

$$= 0.5 + \frac{cos\ 0.5 - 2(0.5)}{(\sin(0.5) + 2)}$$

$$= 0.5 + \frac{cos\ 0.5 - 2(0.5)}{(\sin(0.5) + 2)}$$

$\therefore x_1 = 0.450626693$

$$x_2 = x_{1+1} = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 0.5 + \frac{\cos(0.450626693) - 2(0.450626693)}{(\sin(0.450626693) + 2)}$$

$$= 0.450183647$$

$x_3 = x_{2+1} = 0.450183611$

$\therefore$ The root is 0.45012 (to $2d.p.$)

Question 2

The function $f(x) = e^{2x} - x - 6$

a) Show that the Newton Raphson formula for finding the solution of $f(x)$ can be reduced to

$$x_{n+1} = \frac{(2x_n - 1)e^{2x_n} + 6}{2e^{2x_n} - 1}.$$

b) Hence use the structure in (a) to estimate the root correct to 5 decimal places. Start with $x_0 = 0.8$

Suggested Solution

a) $f(x) = e^{2x} - x - 6$

$f'(x) = 2e^{2x} - 1$

$$x_{n+1} = x_n - \left[\frac{e^{2x_n} - x_n - 6}{2e^{2x_n} - 1}\right]$$

$$x_{n+1} = \frac{x_n(2e^{2x_n} - 1) - (e^{2x_n} - x_n - 6)}{2e^{2x_n} - 1}$$

$$x_{n+1} = \frac{2x_n e^{2x_n} - x_n - e^{2x_n} + x_n + 6}{2e^{2x_n} - 1}$$

$$x_{n+1} = \frac{2x_n e^{2x_n} - e^{2x_n} + 6}{2e^{2x_n} - 1}$$

$$x_{n+1} = \frac{(2x_n - 1)e^{2x_n} + 6}{2e^{2x_n} - 1} \; [as\ required]$$

b) $x_0 = 0.8$

$$x_1 = x_{0+1} = \frac{(2x_0 - 1)e^{2x_0} + 6}{2e^{2x_0} - 1}$$

$$= \frac{[2(0.8) - 1]e^{2(0.8)} + 6}{2e^{2(0.8)} - 1}$$

$\therefore x_1 = 1.007383127$

$$x_2 = x_{1+1} = \frac{(2x_1 - 1)e^{2x_1} + 6}{2e^{2x_1} - 1}$$

$$= \frac{[2(1.007383127) - 1]e^{2(1.007383127)} + 6}{2e^{2(1.007383127)} - 1}$$

$= 0.972264324$

$x_3 = x_{2+1} = 0.970872112$

$x_4 = x_{3+1} = 0.97087002$

$\therefore$ The root is 0.97087 (to $5d.p.$)

Past Examination Questions

Question 1

ZIMSEC June 2000 Paper 1

On a single diagram sketch the graphs of $y = In(10x)$ and $y = \frac{6}{x}$, explaining how you can deduce that the equation $In(10x) = \frac{6}{x}$ has exactly one real root.

Given that the root is close to 2, use the iteration $x_{n+1} = \frac{6}{In(10x_n)}$ to evaluate the root correct to 3 decimal places.

The same equation may be written in the form $xIn(10x) - 6 = 0$. Taking $f(x)$ to be $xIn(10x) - 6$, find $f'(x)$, and show that the Newton-Raphson iteration for the root of $f(x) = 0$ may be simplified to the form

$$x_{n+1} = \frac{x_n+6}{1+In(10x_n)}.$$

Question 2

ZIMSEC June 2005 Paper 1

By using a graphical argument, or otherwise, show that the equation $In(x+2) = 2x - 8$ has exactly two real roots

Show that one of the roots lies between 4 and 5.

Apply the Newton-Raphson method once starting $x_1 = 5$ to obtain a second approximation, giving your answer to 4 decimal places.

Question 3

ZIMSEC November 2013 Paper 1

(i) Sketch on the same axes, the graphs of $y = e^x$ and $y = \frac{2}{1+x}$.

(ii) State the number of real roots of the equation $e^x(1 + x) = 2$.

(iii) Taking $x_1 = 5$ as a first approximations to the root, use the Newton Raphson method twice to find the root of the equation correct to 3 decimal places.

Answer: (ii) 1 real root (iii) 0.375

Question 4

ZIMSEC November 2016 Paper 1

(i) By sketching the graphs of $y = tanx$ and $y = 4x$ on a single diagram for $0 < x < \frac{\pi}{2}$, show that the equation $tanx = 4x$ has exactly one positive root.

(ii) Show, by calculation, that the root of the equation lies between 1 and 1.5.

(iii) Show that the Newton-Raphson structure can be written in the form

$$\frac{x_n tan^2 x_n - tanx_n + x_n}{tan^2 x_n - 3}$$

(iv) Use the structure in **(iii)** once starting with $x_1 = 1.4$, to estimate the root correct to 4 decimal places.

Answer: (iv) 01.3935

Question 5

ZIMSEC November 2018 Paper 1

The function $f(x) = x^3 - 4e^{\frac{1}{2}x}$.

(i) Show that the equation $f(x) = 0$ has a root between 1.9 and 2.5.

(iii) Show that the Newton-Raphson formula for finding the root of the equation in **(i)** above can be reduced to

$$X_{n+1} = \frac{2{x_n}^3 + 2e^{\frac{1}{2}x_n}(2 - x_n)}{3{x_n}^2 - 2e^{\frac{1}{2}x_n}}$$

(iv) Starting with $x_1 = 2$, use the formula to find x_2, correct to three decimal places.

Answer: (iv) 2.438

Question 6

ZIMSEC November 2018 Paper 2

(a) Show by calculation that the equation $4Cos\left(\frac{\pi}{2}x\right) = 6 - x$ has a root between 3 and 4.

(b) Use the Newton Raphson method twice to find the root correct to two decimal places starting with $x_1 = 3.5$.

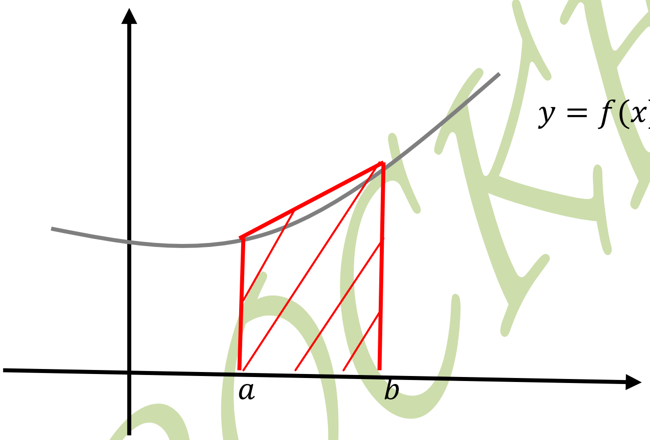
Answer: (b) 3.44

TRAPEZIUM RULE

- The trapezium rule is a method of finding the approximate integration of a function. This technique is a much more accurate way to approximate area beneath a curve.
- In general, when the graph of $y = f(x)$ is approximately linear for $a \leq x \leq b$, the value of the integral

$$\int_a^b f(x)dx$$

can be approximated by the area of a trapezium. This can be seen easily from the diagram below.



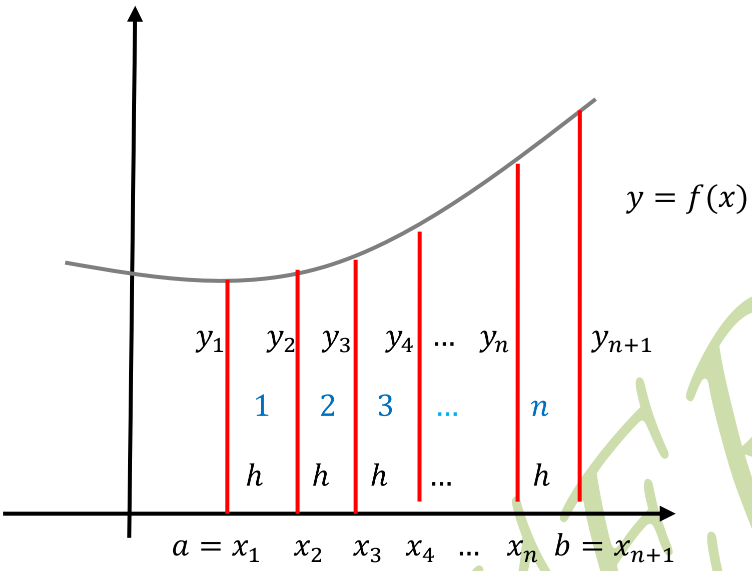
$$Area\ of\ trapezium = \frac{1}{2}(sum\ of\ parallel\ sides) \times perpendicular\ height$$

$$= \frac{1}{2}(a+b)h$$

- By increasing the number of trapezia between a and b we obtain a more accurate approximation. The increased number of trapezia gives rise to the following formula

$$\int_a^b f(x)dx = \frac{h}{2}[x_1 + x_{n+1} + 2(x_1 + x_2 + x_3 + x_4 + \cdots + x_{n-2} + x_{n-1} + x_n)]$$

Derivation



- The trapezia (trapeziums) should have the same width (h).
- From the above diagram, there are $(n+1)$ y −ordinates which are $\boldsymbol{y_1}$, $\boldsymbol{y_2}$, $\boldsymbol{y_3}$, $\boldsymbol{y_4}$ ... and $\boldsymbol{y_{n+1}}$ and (n) trapezia(trapeziums) or strips or intervals named **1, 2, 3**, ... up to $\boldsymbol{n}$.
- Hence

$$Number\ of\ trapezia = Number\ of\ ordinates - 1$$

Finding the height (h):

$b - a = h \times number\ of\ trapezia,$

$b - a = h \times n$

$\dfrac{b-a}{n} = h$

$$\therefore h = \frac{Upper\ limit - Lower\ limit}{Number\ of\ Trapezia}$$

If the question specifies the number of ordinates only then:

$$h = \frac{Upper\ limit - Lower\ limit}{Number\ of\ Ordinates - 1}$$

Finding the Area (A):

We add the areas for all the trapezia **1, 2, 3, ...** up to ***n***.

$$\int_a^b f(x)dx = \sum_{r=1}^{n} area\ of\ trapezium\ \boldsymbol{r}$$

and in our case:

$$\int_a^b f(x)dx = \sum_{r=1}^{4} area\ of\ trapezium\ \boldsymbol{r}$$

$$\int_a^b f(x)dx = \frac{1}{2}(y_1 + y_2)h + \frac{1}{2}(y_2 + y_3)h + \frac{1}{2}(y_3 + y_4)h + \cdots + \frac{1}{2}(y_n + y_{n+1})h$$

$$= \frac{h}{2}(y_1 + y_2 + y_2 + y_3 + y_3 + \cdots + y_n + y_n + y_{n+1})$$

$$= \frac{h}{2}[y_1 + y_{n+1} + 2(y_2 + y_3 + \cdots + y_n)]$$

We can generalise the above formula as follows:

$$= \frac{h}{2}[first\ ordinate + last\ ordinate + 2(other\ ordinates)]$$

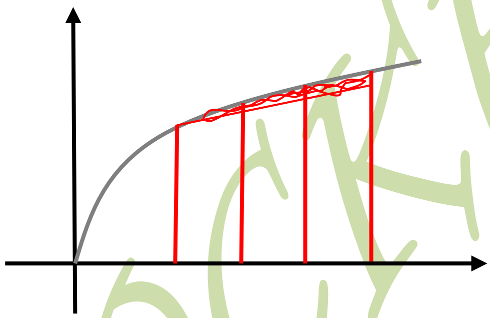
$$\therefore \int_a^b f(x)dx = \frac{h}{2}[first + last + twice(the\ rest)]$$

$$\Rightarrow \int_a^b f(x)dx = \frac{h}{2}[y_1 + y_{n+1} + 2(y_2 + y_3 + y_4 + y_5 + \cdots + y_{n-1} + y_n)]$$

WHEN DOES THE TRAPEZIUM RULE OVERESTIMATE AND UNDERESTIMATE?

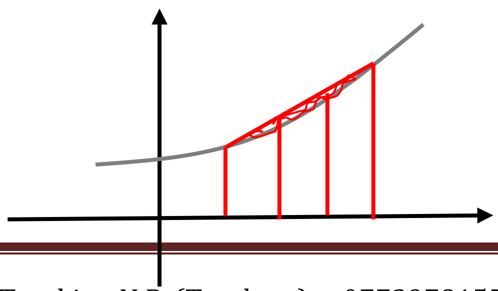
NOTE: *The Trapezium Rule overestimates a curve that is concave up and underestimates functions that are concave down*.

UNDERESTIMATE



- When the curve is concave down (concave), trapezium rule underestimate the area, because when you connect the left and right sides of the trapezium to the curve, and then connect those two points to form the top of the trapezium, you will be left with a small space above the trapezium.
- The small space is outside the trapezium, but it's still under the curve, which means that it will get missed in the trapezium rule estimate, even though it's part of the area under the curve.

OVERESTIMATE



- When the curve is concave up (convex), trapezium rule overestimate the area, because each trapezium will include a small amount of area that is above the curve.
- Since that area is above the curve, but inside the trapezium, it will get included in the trapezium rule estimate, even though it should not be because it's not part of the area under the curve.

Worked Examples

Question 1

a) Evaluate

$$\int_0^{\frac{\pi}{6}} cos^2xdx$$

b) Use the Trapezium Rule with 5 ordinates to find the approximate value of

$$\int_0^{\frac{\pi}{6}} cos^2xdx,$$

giving your answer to 3 decimal places.

c) Hence find the resulting percentage error correct to 3 decimal places.

Suggested solution

a) $\int_0^{\frac{\pi}{6}} cos^2xdx$

Use the identity $cos^2x = \frac{Cos2x+1}{2}$

$$\Rightarrow \int_0^{\frac{\pi}{6}} cos^2 x dx = \int_0^{\frac{\pi}{6}} \frac{Cos2x + 1}{2} xdx$$

$$= \left[\frac{Sin2x}{4} + \frac{x}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{Sin2\left(\frac{\pi}{6}\right)}{4} + \frac{\left(\frac{\pi}{6}\right)}{2}\right) - \left(\frac{Sin2(0)}{4} + \frac{(0)}{2}\right)$$

$$= \frac{\sqrt{3}}{8} + \frac{\pi}{12}$$

b) $h = \dfrac{\frac{\pi}{6} - 0}{5 - 1} = \dfrac{\pi}{24}$

$x_0 = 0$ $\qquad$ $y_0 = (cos0)^2 = 1$

$x_1 = \dfrac{\pi}{24}$ $\qquad$ $y_1 = \left(cos\dfrac{\pi}{24}\right)^2 = 0.982962913$

$x_2 = \dfrac{\pi}{12}$ $\qquad$ $y_2 = \left(cos\dfrac{\pi}{12}\right)^2 = 0.933012701$

$x_3 = \dfrac{\pi}{8}$ $\qquad$ $y_3 = \left(cos\dfrac{\pi}{8}\right)^2 = 0.85355339$

$x_4 = \dfrac{\pi}{6}$ $\qquad$ $y_4 = \left(cos\dfrac{\pi}{6}\right)^2 = \dfrac{3}{4}$

$$A = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$A = \frac{\left(\frac{\pi}{24}\right)}{2}\left[1 + \frac{3}{4} + 2(0.982962913 + 0.933012701 + 0.85355339)\right]$$

$$A = 0.477067731$$

$\therefore A = 0.477$ (to $3d.p.$)

c) Percentage error $= \frac{|Actual\ Value - Estimate|}{Actual\ Value} \times 100\%$

$$= \frac{\left|\left(\frac{\sqrt{3}}{8} + \frac{\pi}{12}\right) - 0.477\right|}{\left(\frac{\sqrt{3}}{8} + \frac{\pi}{12}\right)} \times 100\%$$

$$= 0.0027299248\%$$

$\therefore\ Percentage\ Error = 0.272\%$

Question 2

Use the trapezium rule, with four equal intervals, to estimate the value of

$$\int_1^3 (1 + In2x)dx$$

Suggested Solution

$h = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

$x_0 = 1$ $\quad y_0 = 1 + In(2) = 1.6931471806$

$x_1 = 1\frac{1}{2}$ $\quad y_1 = 1 + In(3) = 2.0986122887$

$x_2 = 2$ $\quad y_2 = 1 + In(4) = 2.3862943611$

$x_3 = 2\frac{1}{2}$ $\quad y_3 = 1 + In(5) = 2.6094379124$

$x_4 = 3$ $\quad y_4 = 1 + In(6) = 2.7917594692$

$$A = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$A = \frac{\left(\frac{1}{2}\right)}{2}\begin{bmatrix} 1.6931471806 + 2.7917594692 \\ +2(2.0986122887 + 2.3862943611 + 2.6094379124) \end{bmatrix}$$

$A = \frac{1}{4}[18.67359577423]$

$A = 4.66839894356$

$\therefore A = 4.7$ (to 2 s.f.)

Question 3

Part of the track of a rollercoaster results in the car travelling at a velocity, in m/s, modelled using the function

$$v(t) = e^{\frac{t}{10}} - 1$$

Using the trapezium rule with 5 strips, calculate an approximate value for displacement over the first 20 seconds and explain whether this will be an underestimate or overestimate

Suggested Solution

i) Displacement:

$$\int_0^{20} \left(e^{\frac{t}{10}} - 1\right) dt$$

$$\approx \frac{1}{2} \times 4 \times [(e^0 - 1) + (e^2 - 1) + 2\{(e^{0.4} - 1) + (e^{0.8} - 1) + (e^{1.6} - 1)\}]$$

$$= 44.7401720$$

$$= 44.7m\ (3sf)$$

ii) This will be an overestimate as the tops of the trapezia will be above the curve.

PAST EXAMINATION QUESTIONS

Question 1

Using the trapezium rule with 4 strips, calculate an approximate value for the area under the curve $f(x) = In(x + 1)$ between $x = 2$ and $x = 4$. Give your answer to 4 decimal places.

$[Ans: 2.7486\ (3d.p.)]$

Question 2

Use the trapezium rule, with five ordinates, to evaluate

$$\int_0^{0.8} e^{x^2}\, dx$$

, giving your answer to four decimal places. $[Ans: 1.0192]$

Question 3

By considering four strips of equal width and considering the approximate area to be that of four trapezia, estimate the value of

$$\int_0^1 cos\sqrt{x}\, dx$$

, giving your answer to 4 $d.p.$ $[Ans: 0.7640]$

Question 4

ZIMSEC November 2003 Paper 1

The area of the region R is bounded by $y = (1 + cosx)^{-\frac{1}{2}},\ x = 0,\ x = \frac{\pi}{2}$ and the x −axis.

Use the trapezium rule with ordinates $x = 0,\ x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$ to show that R is

approximately

$$\frac{\pi}{16}\left[\sqrt{2} + 4\left(2 - \sqrt{2}\right)^{\frac{1}{2}} + 2\right].$$

Question 5

ZIMSEC November 2003 Specimen Paper 1

Use the trapezium rule with three intervals to estimate

$$\int_2^8 \frac{x}{\sqrt{1 + x^2}}\,dx$$

correct to 2 decimal places. $[Ans: 5.8]$

Evaluate

$$\int_2^8 \frac{x}{\sqrt{1 + x^2}}\,dx$$

Hence find the percentage error. $[Ans: a)\ 5.83;\ b)\ 0.5\%]$

Question 6

ZIMSEC November 2014 Paper 1

(i) Use the trapezium rule with for trapezia of equal width to estimate the value of

$$\int_1^2 \frac{1}{x} e^{\frac{1}{3}x}\,dx,$$

giving your answer correct to 4 decimal places. [Ans: 1.1304]

(ii) Explain whether the trapezium rule has underestimated or overestimated the value of this integral.

[Ans: It overestimates the value of the integral because it includes the area above the graph]

Question 7

ZIMSEC November 2018 Paper 2

a) Use the substitution $x = Cos^2\theta$ to show that

$$\int \sqrt{\frac{x}{1-x}}\,dx = -2\int 2Cos^2\theta \, \mathrm{d}\,\theta$$

b) Hence evaluate

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx,$$

correct to 3 decimal places.

Additional questions

c) Use the trapezium rule with 5 ordinates to estimate

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx,$$

correct to 3 decimal places [Ans: 0.285]

c) Hence find the percentage error.

Question 8

ZIMSEC November 2017 Paper 1

(i) Find the exact value of

$$\int_0^1 2x^2 e^x \, dx$$

$[Ans: 2e - 4]$

(ii) Use the trapezium rule with 6 ordinates to evaluate

$$\int_0^1 2x^2 e^x \, dx,$$

correct to 4 decimal places. $[Ans: 1.4908]$

(iii) Hence, find correct to 3 decimal places, the percentage error in using the trapezium rule as an approximation to the integral. $[Ans: 3.776]$

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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