1^{st} Order Differential Equations

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SYLLABUS (6042) REQUIREMENTS

- formulate a statement involving a rate of change as a differential equation
- solve differential equations by integration in the case where variables are separable
- sketch typical examples of a family of curves representing a general solution of a differential equation
- find a particular solution of a differential equation given initial conditions
- solve problems involving 1^{st} order differential equation

1st ORDER DIFFERENTIAL EQUATIONS

Definition: A differential equation (or DE) is any equation which contains a function and a differential coefficient e.g. $\frac{dy}{dx} = e^y cos2x$.

A differential equation is called a 1st order ODE if it has a linear differential coefficient i.e. $\frac{dy}{dx}$.

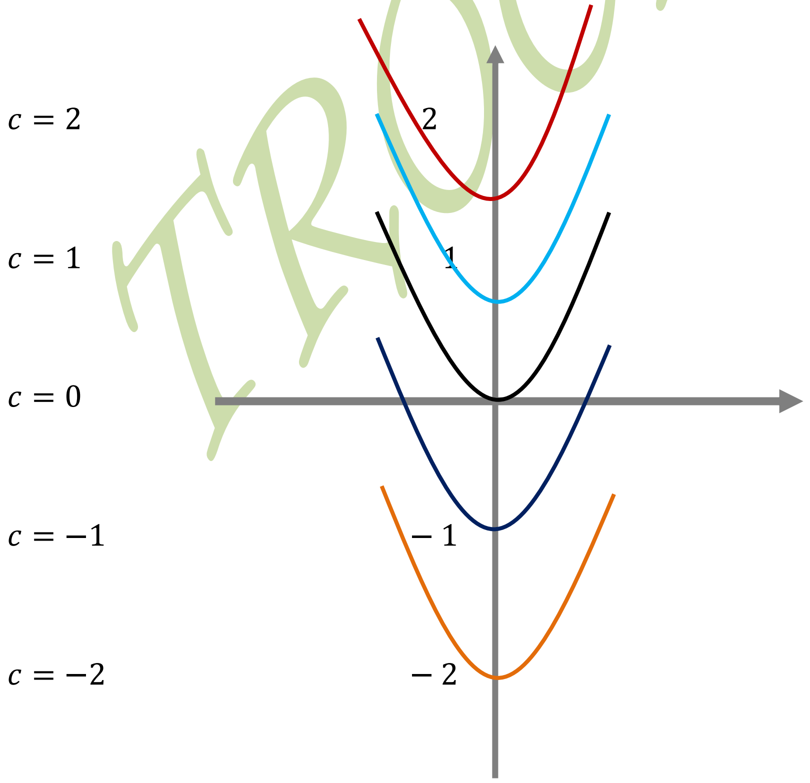
FAMILY OF CURVES

Suppose we have a differential equation $\frac{dy}{dx} = 2x$, the by integration $y = x^2 + c$.

$y = x^2 + c$ is called the general solution of the differential equation $\frac{dy}{dx} = 2x$.

If we are to draw the graphs of $y = x^2 + c$ for various values of c, we would obtain a 'family of curves'

NB: All the curves would give the gradient $\frac{dy}{dx} = 2x$.



If more information is given, the value of the constant can be found by substituting them into the general solution. Hence the particular solution is obtained.

e.g. For the above equation find c if $x = 1$ and $y = 0$.

$$\Rightarrow 0 = 1 + c \Rightarrow c = -1$$

$\therefore y = x - 1$ is the particular solution.

SEPARABLE 1st ORDER ODEs

A 1st order ODE is called separable if it can be written in general as:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

Steps to solve 1st order ODEs by separation of variable

- Make the DE look like $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (Unless it's already been done for you – in which case you can just identify the various parts or you may have to do some algebra to get it into the correct form.)
- Separate the variables: Set all the $y's$ on the LHS by multiplying both sides by $g(y)$ and get all the $x's$ on the RHS by multiplying by dx i.e. $g(y)dy = f(x)dx$.
- Integrate both sides:

$$\int g(y)dy = \int f(x)dx.$$

- This gives us an implicit solution. Solve for y (if possible). This gives us an explicit solution. From this stage the solution is expressed in terms of arbitrary constants and we have the general solution.
- If there are initial conditions, use them to solve for unknown parameters in the solution. This gives us a particular solution.

Worked Problems

Example 1

Find the general solution of the equation $\frac{dy}{dx} = \frac{y}{x}$.

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$Iny = Inx + c$$

$$e^{Iny} = e^{Inx+c}$$

$$e^{Iny} = e^{Inx}.e^{c}$$

$$y = Ax, \quad where\ A = e^{c}$$

Example 2

Solve the equation $\frac{dy}{dx} = \frac{y+1}{x-1}$, given that $y = 1$ when $x = 0$.

Solution

$$\frac{dy}{dx} = \frac{y+1}{x-1}$$

$$\int \frac{1}{y+1} dy = \int \frac{1}{x-1} dx$$

$$In|y+1| = In|x-1| + c$$

$$e^{In|y+1|} = e^{In|x-1|+c}$$

$e^{In|y+1|} = e^{In|x-1|} . e^{c}$

$y + 1 = A(x - 1) \quad [general\ solution]$

When $\ x = 0, y = 1:$

$1 + 1 = A(0 - 1) \Rightarrow A = -2$

$y + 1 = -2(x + 1)$

$\therefore y = -2x - 3\ [particular\ solution]$

<u>Example 3</u>

Find the general solution of $\frac{dy}{dx} = 3x^2e^{-y}$ and a particular solution that satisfies the condition $y = 1$ when $x = 0$.

<u>Solution</u>

$\frac{dy}{dx} = 3x^2e^{-y}$

$\int \frac{1}{e^{-y}} dy = \int 3x^2 dx$

$\int e^{y} dy = \int 3x^2 dx$

$e^{y} = x^3 + c$

$In(e^{y}) = In(x^3 + c)$

$y = In(x^3 + c) \quad [general\ solution]$

When $\ x = 0, y = 1:$

$e^1 = 0 + c \Longrightarrow c = e$

$\therefore y = In(x^3 + e) \quad [particular\ solution]$

Example 4

Cambridge Question

The variables x and θ are related by the differential equation

$$sin2\theta \frac{dx}{d\theta} = (x+1)cos2\theta$$

where $0 < \theta < \frac{\pi}{2}$. When $\theta = \frac{\pi}{12}, x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible.

Solution

$sin2\theta \frac{dx}{d\theta} = (x+1)cos2\theta$

$\int \frac{1}{x+1} dx = \int \frac{cos2\theta}{sin2\theta} d\theta$

Remember, using integration by recognition that:

$$\int \frac{f'(x)}{f(x)} dx = In|f(x)| + c$$

$\int \frac{1}{x+1} dx = \frac{1}{2} \int \frac{2cos2\theta}{sin2\theta} d\theta$

$In|x+1| = \frac{1}{2} In|sin2\theta| + c$

When $\theta = \frac{\pi}{12}, x = 0$:

$\Rightarrow In|0+1| = \frac{1}{2} In\left|sin2\left(\frac{\pi}{12}\right)\right| + c$

$\Rightarrow In|1| = \frac{1}{2} In\left|sin\left(\frac{\pi}{6}\right)\right| + c$

$\Rightarrow 0 = \frac{1}{2} In\left|\frac{1}{2}\right| + c$

$\Rightarrow 0 = \frac{1}{2} In|2^{-1}| + c$

$\Rightarrow 0 = -\frac{1}{2} In|2| + c$

$\Rightarrow 0 = -In\left|(2)^{\frac{1}{2}}\right| + c$

$\Rightarrow 0 = -In|\sqrt{2}| + c$

$\Rightarrow c = In|\sqrt{2}|$

$$\Rightarrow In|x+1| = \frac{1}{2}In|sin2\theta| + In|\sqrt{2}|$$

$$\Rightarrow In|x+1| = In\left|(sin2\theta)^{\frac{1}{2}}\right| + In|\sqrt{2}|$$

$$\Rightarrow In|x+1| = In|\sqrt{sin2\theta}| + In|\sqrt{2}|$$

$$\Rightarrow In|x+1| = In|\sqrt{2sin2\theta}|$$

$$\Rightarrow e^{In|x+1|} = e^{In|\sqrt{2sin2\theta}|}$$

$$\Rightarrow x+1 = \sqrt{2sin2\theta}$$

$$\therefore x = \sqrt{2sin2\theta} - 1$$

Example 5

Cambridge Question

(i) Express $\frac{100}{x^2(10-x)}$ in partial fractions.

(ii) Given that $x = 1$ and $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{100}x^2(10-x),$$

obtaining an expression for t in terms of x.

Answer

(i) Let $\frac{100}{x^2(10-x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{10-x}$

$$\equiv \frac{A(x)(10-x) + B(10-x) + C(x^2)}{x^2(10-x)}$$

$$\Rightarrow 100 = A(x)(10-x) + B(10-x) + C(x^2)$$

When $x = 0$

$10B = 100$

$B = 10$

When $x = 10$

$100C = 100$

$C = 1$

Comparing coefficients of x^2

$0 = -A + C$

$A = C$

$A = 1$

$$\Rightarrow \frac{100}{x^2(10-x)} \equiv \frac{1}{x} + \frac{10}{x^2} + \frac{1}{10-x}$$

(ii) $\frac{dx}{dt} = \frac{1}{100}x^2(10-x)$

$$\int \frac{100}{x^2(10-x)}dx = \int dt$$

$$\int \left(\frac{1}{x} + \frac{10}{x^2} + \frac{1}{10-x}\right)dx = \int dt$$

$$\int \frac{1}{x}dx + \int \frac{10}{x^2}dx + \int \frac{1}{10-x}dx = \int dt$$

$$\int \frac{1}{x}dx + \int 10x^{-2}dx + (-1)\int \frac{-1}{10-x}dx = \int dt$$

$$In|x| - \frac{10}{x} - In|10-x| = t + c$$

Given that $x = 1$ and $t = 0$

$$\Rightarrow In|1| - \frac{10}{1} - In|10-1| = 0 + c$$

$$\Rightarrow c = -10 - In|9|$$

$$\Rightarrow In|x| - \frac{10}{x} - In|10-x| = t - 10 - In|9|$$

$$\Rightarrow t = In|x| - \frac{10}{x} - In|10-x| + 10 + In|9|$$

$$\therefore t = In\left|\frac{9x}{10-x}\right| - \frac{10}{x} + 10 \quad \textit{or equivalent}$$

<u>Example 6</u>

If $\frac{dy}{dx} = -0.4y$ and $y = 2$ when $x = 0$, solve the differential equation to obtain an expression for y in terms of x

Suggested Solution

$$\frac{dy}{dx} = -0.4y$$

$$\int \frac{1}{y} dy = \int -0.4dx$$

$$Iny = -0.4x + c$$

$y = 2$ when $x = 0$

$$In2 = -0.4(0) + c$$

$$\therefore c = In2$$

$$\Rightarrow Iny = -0.4x + In2$$

$$\Rightarrow e^{Iny} = e^{-0.4x+In2}$$

$$e^{Iny} = e^{-0.4x}e^{In2}$$

$$\therefore y = 2e^{-0.4x}$$

Example 6

ZIMSEC November 2004 Paper 2

A car travels a distance of $50km$ with an initial speed of $80kmh^{-1}$.

Given that the car's speed is proportional to the distance that remains to be covered, find the time to the nearest minute that the car takes to cover three-fifths of the distance. **[9]**

Suggested Solution

Let the covered distance at any instant be x thus the remaining distance is $50 - x$

$$\Rightarrow \frac{dx}{dt} \propto (50 - x)$$

$\Rightarrow \frac{dx}{dt} = k(50 - x)$

Aside:

$x = 0; \quad \frac{dx}{dt} = 80km$

$\Rightarrow 80 = 50k$

$\Rightarrow k = \frac{8}{5}$

$\Rightarrow \frac{dx}{dt} = \frac{8}{5}(50 - x)$

$\Rightarrow \int \frac{1}{50 - x} dx = \int \frac{8}{5} dt$

$\Rightarrow -In|50 - x| = \frac{8}{5}t + c$

Now when $t = 0, x = 0km$

$\Rightarrow -In|50 - 0| = \frac{8}{5}(0) + c$

$\Rightarrow c = -In|50|$

$\Rightarrow c = -In|50|$

$\Rightarrow -In|50 - x| = \frac{8}{5}t - In50$

Now when $t = ?\, minute, x = \frac{3}{5}(30)km = 30km$

$\Rightarrow -In|50 - x| = \frac{8}{5}t - In50$

$\Rightarrow -In|50 - 30| = \frac{8}{5}t - In50$

$$\Rightarrow In50 - In(20) = \frac{8}{5}t$$

$$\Rightarrow t = \frac{5}{8}In\left(\frac{50}{20}\right)$$

$$\Rightarrow t = 0.572681707hours$$

$$\Rightarrow t = 0.572681707hours \times 60minutes$$

$$\Rightarrow t = 34.36090245minutes$$

$$\therefore t = 34minutes$$

Example 7

Cambridge Question

A curve is defined by the differential equation

$$\frac{dy}{dx} = \frac{Inx}{coty}.$$

Given that the curve passes through the point where $y = \frac{\pi}{6}$ and $x = e$, find the equation of the curve and hence find the value of y when $x = 1$, giving your answer correct to 3 significant figures.

Solution

$$\frac{dy}{dx} = \frac{Inx}{coty}$$

$$\int cot\, dy = \int Inx\, dx$$

$$\int \frac{coy}{siny} dy = \int 1.Inx\, dx$$

Aside: RHS

Let $u = Inx \Rightarrow \frac{du}{dx} = \frac{I}{x}$

$$\frac{dv}{dx} = 1 \Rightarrow v = x$$

Also remember that

$$\int \frac{f'(x)}{f(x)} dy = In|f(x)| + c$$

Now:

$$In|siny| = xInx - \int x\left(\frac{1}{x}\right) dx$$

Use integration by parts on the RHS

$$In|siny| = xInx - \int 1\, dx$$

$$In|siny| = xInx - x + c$$

$y = \frac{\pi}{6}$ and $x = e$:

$$\Rightarrow In\left|sin\frac{\pi}{6}\right| = eIne - e + c$$

$$\Rightarrow c = In\left(\frac{1}{2}\right)$$

$$\Rightarrow In|siny| = xInx - x + In\left(\frac{1}{2}\right)$$

$$\Rightarrow In|siny| = Inx^x + In\left(\frac{1}{2}\right) - x$$

$$\Rightarrow In|siny| = In\left(\frac{x^x}{2}\right) - x$$

$$\Rightarrow e^{In|siny|} = e^{In\left(\frac{x^x}{2}\right) - x}$$

$$\Rightarrow siny = \frac{x^x}{2}e^{-x}$$

$\Rightarrow y = sin^{-1}\left(\frac{x^x}{2}e^{-x}\right)$

$y =?$ when $x = 1$:

$\Rightarrow y = sin^{-1}\left(\frac{1^1}{2}e^{-1}\right)$

$\Rightarrow y = sin^{-1}\left(\frac{1}{2e}\right)$

$\Rightarrow y = 0.18499306878$

$\therefore y = 0.185\ (to\ 3\ significant\ figures)$

PRACTICE QUESTIONS

Question 1

Find the general solution of $x\frac{dy}{dx} = y^2 - 1$.

$[ANS: y - 1 = kx^2(y + 1)]$

Question 2

Find the general solution of $xsin^2y\frac{dy}{dx} = (x + 1)^2$.

$\left[ANS: \frac{y}{2} - \frac{1}{4}sin2y = \frac{x^2}{2} + 2x + Inx + c\right]$

Question 3

Find the general solution of $\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+1}$.

$[ANS: y^2 = k(x^2 + 1)]$

Question 4

Find the general solution of $\frac{dy}{dx} = \frac{Cosx}{Sin2y}$.

$\left[ANS: y = \frac{1}{2}Cos^{-1}(A - 2Sinx)\right]$

Question 5

Find the general solution of $\frac{dy}{dx} = \frac{e^{-x}}{y}$.

$\left[ANS: y = \pm\sqrt{A - 2e^{-x}}\right]$

Question 6

Solve the equation $\frac{dy}{dx} = -2xtany$, given that $y = \frac{\pi}{2}$ when $x = 0$.

$\left[ANS: y = e^{-x^2}\right]$

Question 7

Solve the equation $\frac{dy}{dx} = y^2 - e^{3x}y^2$, given that $y = 1$ when $x = 0$.

$\left[ANS: y = \frac{1}{\frac{2}{3} - x + \frac{1}{3}e^{3x}}\right]$

Question 8

Solve the equation $\frac{dy}{dx} = 5xy - 2x$, given that $y = 1$ when $x = 0$.

$\left[ANS: y = \frac{3}{5}e^{\frac{5x^2}{2}} + \frac{2}{5}\right]$

Question 9

Solve the equation $(1 + x^2)\frac{dy}{dx} + xy = 0$, given that $y = 2$ when $x = 0$.

$\left[ANS: y = \frac{2}{\sqrt{1+x^2}}\right]$

Question 10

Solve the equation $\frac{dy}{dx} = e^{3x+2y}$, given that $y = 4$ when $x = 0$.

$\left[ANS: y = -\frac{1}{2} In\left(-\frac{2}{3}e^{3x} + e^{-8} + \frac{2}{3}\right)\right]$

Question 11

Solve the equation $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$, given that $y = 2$ when $x = 4$.

HINT: Factorise both numerator and denominator

$\left[ANS: \frac{(y-1)^2}{(x-3)^5} = e^{x-y-2}\right]$.

Question 12

Solve the equation $x\frac{dy}{dx} = y^2 + 1$, given that $y = 1$ when $x = 1$.

$\left[ANS: tan^{-1}y = Inx + \frac{\pi}{4}\right]$

PAST EXAMINATION QUESTIONS

Question 1

ZIMSEC NOVEMBER 2004 PAPER 2

Find the general solution of the differential equation

$$cosx\frac{dy}{dx} + 2sinx = 3tanx$$

[4]

$[ANS\text{: y} = 3\sec x + 21\text{n}|cosx| + \text{C}]$

Question 2

ZIMSEC NOVEMBER 2013 #7

Solve the differential equation $\frac{dy}{dx} = 1 - y$, giving the general solution in the form

$y = Ae^{-x} + C$, where A is an arbitrary constant and C is a constant.

Find the particular solution when the y-intercept is 3. **[6]**

$[ANS\text{: y} = 1 - \text{A}e^{-x}\ (general\ solution)]$

$[ANS\text{: y} = 1 + 2e^{-x}\ (particular\ solution)]$

Question 3

Cambridge

a) Express $\frac{2}{4-y^2}$ in partial fractions. **[3]**

$\left[ANS\text{:}\frac{1}{2(2-y)} + \frac{1}{2(2+y)}\right]$

b) Hence obtain the solution of $2cotx\frac{dy}{dx} = 4 - y^2$ for which $y = 0$ when $x = \frac{\pi}{3}$, giving your answer in the form $sec^2x = g(y)$. **[4]**

$\left[ANS: sec^2x = \frac{8+4y}{2-y}\right]$

Question 4

Cambridge November 2012 Paper 3

The variables x and y are related by the differential equation

$$(x^2+4)\frac{dy}{dx} = 6xy$$

It is given that $y = 32$ when $x = 0$. Find the expression for y in terms of x. **[6]**

$\left[ANS: y = \frac{1}{2}(x^2+4)\right]$

Question 5

Cambridge November 2014 Paper 3

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{1}{5}xy^{\frac{1}{2}}sin\left(\frac{1}{3}x\right).$$

(i) Find the general solution, giving y in terms of x. **[6]**

(ii) Given that $y = 100$ when $x = 0$, find the value of y when $x = 25$. **[3]**

(i) $\left[ANS: y = \left\{-\frac{3}{10}x\,cos\left(\frac{1}{3}x\right) + \frac{9}{10}sin\left(\frac{1}{3}x\right) + c\right\}^2\right]$

(ii) $[ANS: y = 203]$

Question 6

Cambridge June 2016 Paper 3

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = e^{-2y}tan^2x,$$

for $0 \leq \theta < \frac{\pi}{2}$ and it is given that $y = 0$ when $x = 0$. Solve the differential equation and calculate the value of y when $= \frac{\pi}{2}$.

$\left[ANS: \frac{1}{2}e^{-2y} = tanx - x + \frac{1}{2}\right]$

$[ANS: \text{y} = 0.179]$

Question 7

Cambridge November 2017 Paper 3

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = 4cos^2 ytanx,$$

for $0 \leq \theta < \frac{\pi}{2}$ and $x = 0$ when $y = \frac{\pi}{2}$. Solve this differential equation and find the value of x when $y = \frac{\pi}{2}$.

$[ANS: tany = 4In|secx| + 1]$

$[ANS: \text{y} = 0.587]$

Question 8

Cambridge June 2018 Paper 3

(i) Express $\frac{1}{4-y^2}$ in partial fractions. **[2]**

(ii) The variables x and y are related by the differential equation$x$

$$x\frac{dy}{dx} = 4 - y^2,$$

and $y = 1$ when $x = 1$. Solve the differential equation, obtaining an expression for y in terms of x. **[6]**

(i) $\left[ANS: \frac{1}{4(2+y)} + \frac{1}{4(2-y)}\right]$

(ii) $\left[ANS: \text{y} = \frac{2(3x^4 - 1)}{3x^4 + 1}\right]$

Question 9

Cambridge June 2019 Paper 3

The variables x and y are related by the differential equation$x$

$$(x+1)\frac{dy}{dx} = y^2 + 5.$$

It is given that $y = 2$ when $x = 0$. Solve the differential equation, obtaining an expression for y^2 in terms of x. **[7]**

$[ANS: y^2 = 9(x+1)^2 - 5]$

Question 10

Cambridge June 2019 Paper 3

The variables x and t satisfy the differential equation$x$

$$5\frac{dx}{dt} = (20-x)(40-x).$$

It is given that $x = 2$ when $t = 0$.

(i) Using partial fractions, solve the differential equation, obtaining an expression for x in terms of t. **[9]**

(ii) State what happens to the value of x when t becomes large. **[1]**

(i) $\left[ANS: x = \frac{60e^{4t} - 40}{3e^{4t} - 1}\right]$

(ii) $[ANS: x \text{ approaches } 20]$

RATES OF CHANGE

Definition: Rate of change is the rate that describes how one quantity changes in relation to another. If t is the independent variable and y is the dependant variable then:

$$rate\ of\ change = \frac{change\ in\ y}{change\ in\ t}.$$

- Rates of change can be positive or negative. This corresponds to an increase or decrease in dependant value between the two data points.
- For rates of change, the differential coefficient has $'dt'$ in the denominator i.e.

$$\frac{d(dependant\ variable)}{dt},$$

since the change is with respect to time.

- When the rate of change is increasing, the differential coefficient is positive i.e.

$$\frac{d(dependant\ variable)}{dt}.$$

- When the rate of change is decreasing, the differential coefficient is negative i.e.

$$-\frac{d(dependant\ variable)}{dt}.$$

RELATED RATES PROBLEMS

Worked Examples

Example 1

Find the rate of decrease of an area of circle when its radius is $8cm$ and is decreasing at $0.4cm/s$.

Solution

$\frac{dA}{dt} = ?$ $\quad\quad\quad r = 8cm \quad\quad\quad \frac{dr}{dt} = -0.4cm/s$

$A = \pi r^2$

$$\Rightarrow \frac{dA}{dr} = 2\pi r$$

$$= 2(8)\pi$$

$$= 16\pi$$

$$\therefore \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$= 16\pi cm \times -\frac{0.4cm}{s}$$

$$= -20.1cm^2/s$$

$\therefore$ The rate of decrease of the area of the circle is $20.1cm^2/s$.

Example 2

The radius rcm of a sphere is $10cm$ and it is increasing at the rate $0.25cm/s$. Find the rate of increase of

a) the volume,

b) the surface area.

$$\left[Volume = \frac{4}{3}\pi r^3, \quad Surface\ Area = 4\pi r^2\right]$$

Solution

a) $\frac{dv}{dt} = ?$ $\quad r = 10cm \quad \frac{dr}{dt} = 0.25cm/s$

$V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dr} = 4\pi r^2$

$= 4(10)^2\pi$

$= 400\pi$

$\therefore \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 400\pi cm^2 \times \frac{0.25cm}{s}$

$= \frac{100\pi cm^3}{s}$

$= 314cm^3/s.$

$\therefore$ The rate of increase of the volume of the circle is $314cm^3/s.$

b) $\frac{dA}{dt} = ?$ $\quad r = 10cm \quad \frac{dr}{dt} = 0.25cm/s$

$A = 4\pi r^2 \Rightarrow \frac{dA}{dr} = 8\pi r = 8(10)\pi = 80\pi$

$\therefore \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

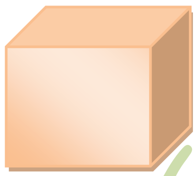
$$= 80\pi cm \times \frac{0.25cm}{s}$$

$$= \frac{20\pi cm^2}{s}$$

$$= 62.8cm^2/s$$

$\therefore$ The rate of decrease of the area of the circle is $62.8cm^2/s$.

Example 3



At time t $seconds$ the length of the side of a cube is xcm, the surface area is Scm^2 and the Volume of the cube is Vcm^3.

The surface area of the cube is increasing at a constant rate of $8cm^2/s$. Show that

a) $\frac{dx}{dt} = \frac{k}{x}$, where k is a constant to be found.

b) $\frac{dV}{dt} = 2V^{\frac{1}{3}}$

Given that $V = 8$ when $t = 0$,

c) solve the differential equation in part (b), and find the value of t when $V = 16\sqrt{2}$.

Solution

$$\frac{dA}{dt} = 8cm^2/s$$

a) $A = 6 \times x^2$

$\Rightarrow A = 6x^2$

$\Rightarrow \frac{dA}{dx} = 12x$

$\therefore \frac{dx}{dt} = \frac{dA}{dr} \div \frac{dA}{dx}$

$= 8cm^2/s \ \div 12xcm$

$\Rightarrow \frac{dx}{dt} = \frac{\left(\frac{2}{3}\right)}{x}$, where $k = \frac{2}{3}$.

b) $V = lbh = x^3$

$\therefore \frac{dV}{dx} = 3x^2$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= \frac{3x^2cm^2}{s} \times \frac{\left(\frac{2}{3}\right)}{x}cm$

$= 2xcm^3/s$

Now $V = x^3 \Rightarrow x = V^{\frac{1}{3}}$

$\therefore \frac{dV}{dt} = 2V^{\frac{1}{3}}cm^3/s$

c) Separating variables and solving:

$\frac{dV}{dt} = 2V^{\frac{1}{3}}$

$$\int \frac{1}{V^{\frac{1}{3}}} dV = \int 2dt$$

$$\int V^{-\frac{1}{3}} dV = 2t + c$$

$$\frac{3}{2} V^{\frac{2}{3}} = 2t + c$$

Given that $V = 8$ when $t = 0$,

$$\frac{3}{2}(8)^{\frac{2}{3}} = 0 + c \Rightarrow c = 6$$

$$\therefore \frac{3}{2} V^{\frac{2}{3}} = 2t + 6$$

$$\Rightarrow t = \frac{3}{4} V^{\frac{2}{3}} - 3$$

Finding t when $16\sqrt{2}$,

$$t = \frac{3}{4}(16\sqrt{2})^{\frac{2}{3}} - 3$$

$$t = \frac{3}{4}[(512)^{\frac{1}{2}}]^{\frac{2}{3}} - 3$$

$$t = \frac{3}{4}(512)^{\frac{1}{3}} - 3$$

$$t = \frac{3}{4} \times 8 - 3$$

$$\therefore t = 3$$

Example 4

A spherical balloon is blown. The volume increases at $200 cm^3/minute$.

(i) Show that $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$.

(ii) Find the rate of increase of the radius when the volume is $1000cm^3$.

Suggested Solution

(i) $V = \frac{4}{3}\pi r^3$

$3V = 4\pi r^3$

$\frac{3V}{4\pi} = r^3$

$r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

(ii) $\frac{dv}{dt} = \frac{200cm^3}{minute}$

$r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

$\frac{dr}{dv} = \frac{1}{3}\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}-1} \times \frac{\frac{3}{4\pi}cm}{cm^3}$

$= \left(\frac{3V}{4\pi}\right)^{-\frac{2}{3}} \times \frac{1}{4\pi}$

$= \frac{1}{4\pi}\left(\frac{3V}{4\pi}\right)^{-\frac{2}{3}}\left(\frac{1}{cm^2}\right)$

$\frac{dr}{dt} = \frac{dv}{dt} \times \frac{dr}{dv}$

$= \frac{1}{4\pi}\left(\frac{3 \times 1000}{4\pi}\right)^{-\frac{2}{3}}\left(\frac{1}{cm^2}\right) \times 200\frac{cm^3}{minute}$

$= 0.413566994cm/minute$

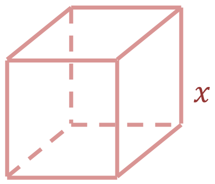
$= 0.41cm/minute(\text{to } 2s.f.)$

Example 5

The length of a cube is increasing at a rate of $5cm$ per $second$. Find the rate of change of the cube's volume when the length is equal to $20cm$.

Suggested Solution

Let the length be x.



$\frac{dx}{dt} = \frac{5cm}{s}$

Now:

$V = x^3$

$\Rightarrow \frac{dV}{dx} = \frac{3x^2cm^3}{cm} = 3(20)^2cm^2 = 1200cm^2$

Now:

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= 1200cm^2 \times 5cm/s$

$= 6000cm^3/s$

Example 6

A watermelon is assumed to be spherical in shape while it is growing. Its mass, Mkg, and radius, $r\ cm$, are related by the formula $M = kr^3$, where k is a constant. It is also assumed that the radius is increasing at a constant rate of 0.1 centimetres per day. On a particular day the radius is 10 cm and the mass is 3.2 kg.

Find the value of k and the rate at which the mass is increasing on this day.

<u>Suggested Solution</u>

$M = kr^3$

$3.2 = k(10)^3$

$3.2 = 1\,000k$

$\frac{3.2}{1\,000} = k$

$\therefore k = \frac{2}{625}$

Now

$\frac{dr}{dt} = 0.1$

Also

$M = \frac{2}{625}r^3$

$\Rightarrow \frac{dM}{dr} = \frac{6}{625}r^2$

Now

$\frac{dM}{dt} = \frac{dM}{dr} \times \frac{dr}{dt}$

$\frac{dM}{dt} = \frac{6}{625}r^2 \times 0.1$

$\frac{dM}{dt}_{r=10} = \frac{6}{625}(10)^2 \times 0.1$

$= \frac{6}{625}(10)^2 \times 0.1$

$= \frac{12}{125}$ or 0.096

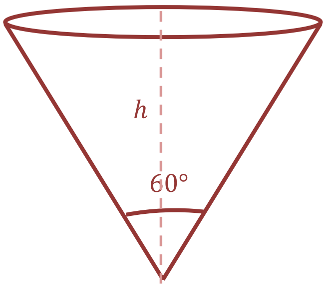
<u>Example 7</u>

<u>Cambridge Question</u>

Sand falls on to the horizontal ground at the rate of $9m^3$per minute and forms a heap in the shape of a right circular cone with vertical angle $60°$. Show that 10 $seconds$ after the sand begins to fall, the rate at which the radius of the base of the pile s increasing is $\sqrt{3}\left(\frac{4}{\pi}\right)^{\frac{1}{3}} m$ per minute.

Suggested Solution

$$\frac{dv}{dt} = 9m^3/min$$



$$tan30° = \frac{r}{h}$$

$$\Rightarrow h = \frac{r}{tan30°}$$

$$\Rightarrow h = \frac{r}{\left(\frac{1}{\sqrt{3}}\right)}$$

$$\Rightarrow h = \sqrt{3}r$$

Now:

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow V = \frac{1}{3}\pi r^2\left(\sqrt{3}r\right)$$

$$\Rightarrow V = \frac{\sqrt{3}}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dr} = \sqrt{3}\pi r^2 m^2$$

Now:

$$\frac{dr}{dr}=\frac{dV}{dt}\div\frac{dV}{dr}$$

$$\Rightarrow\frac{dr}{dr}=\frac{9m^3/min}{\sqrt{3}\pi r^2m^2}$$

$$\Rightarrow\frac{dr}{dr}=\frac{9}{\sqrt{3}\pi r^2}m/min$$

$$\Rightarrow\frac{dr}{dr}=\frac{9\sqrt{3}}{3\pi r^2}m/min$$

$$\Rightarrow\frac{dr}{dr}=\frac{3\sqrt{3}}{\pi r^2}m/min$$

Calculating r:

$$\frac{dv}{dt}=9m^3/60seconds$$

$$\frac{dv}{dt}=\frac{9}{60}m^3/second$$

After 10 $seconds, V=\frac{9}{60}\times 10$

$$=\frac{9}{6}=\frac{3}{2}m^3$$

Since $V=\frac{\sqrt{3}}{3}\pi r^3$

$$\Rightarrow\frac{3}{2}=\frac{\sqrt{3}}{3}\pi r^3$$

$$\Rightarrow r^3=\frac{9}{2\sqrt{3}\pi}$$

$$\Rightarrow r^3=\frac{9\sqrt{3}}{2(3)\pi}$$

$$\Rightarrow r^3 = \frac{3\sqrt{3}}{2\pi}$$

$$\Rightarrow r = \left(\frac{3\sqrt{3}}{2\pi}\right)^{\frac{1}{3}}$$

But

$$\frac{dr}{dr} = \frac{3\sqrt{3}}{\pi r^2} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \frac{3\sqrt{3}}{\pi\left[\left(\frac{3\sqrt{3}}{2\pi}\right)^{\frac{1}{3}}\right]^2} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \frac{3\sqrt{3}}{\pi\left[\frac{(3)^{\frac{2}{3}}(\sqrt{3})^{\frac{2}{3}}}{(2)^{\frac{2}{3}}(\pi)^{\frac{2}{3}}}\right]} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \frac{3\sqrt{3}}{(\pi)^{\frac{1}{3}}\left[\frac{(3)^{\frac{2}{3}}(\sqrt{3})^{\frac{2}{3}}}{(2)^{\frac{2}{3}}}\right]} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \frac{3\sqrt{3}}{(\pi)^{\frac{1}{3}}}\left[\frac{(2)^{\frac{2}{3}}}{(3)^{\frac{2}{3}}(\sqrt{3})^{\frac{2}{3}}}\right] m/minute$$

$$\Rightarrow \frac{dr}{dr} = \frac{(3)^{\frac{1}{3}}(\sqrt{3})^{\frac{1}{3}}}{(\pi)^{\frac{1}{3}}}\left[(2)^{\frac{2}{3}}\right] m/minute$$

$$\Rightarrow \frac{dr}{dr} = \frac{(3)^{\frac{1}{3}}(\sqrt{3})^{\frac{1}{3}}}{(\pi)^{\frac{1}{3}}}\left[(4)^{\frac{1}{3}}\right] m/minute$$

$$\Rightarrow \frac{dr}{dr} = \left(\frac{12\sqrt{3}}{\pi}\right)^{\frac{1}{3}} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \left(\frac{4 \times 3 \times \sqrt{3}}{\pi}\right)^{\frac{1}{3}} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \left(\frac{4 \times \sqrt{3}^2 \times \sqrt{3}}{\pi}\right)^{\frac{1}{3}} m/minute$$

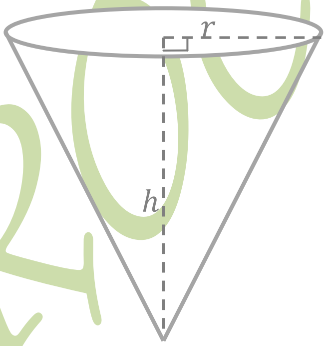
$$\Rightarrow \frac{dr}{dr} = \left(\frac{4 \times \sqrt{3}^3}{\pi}\right)^{\frac{1}{3}} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \left(\sqrt{3}^3\right)^{\frac{1}{3}} \left(\frac{4}{\pi}\right)^{\frac{1}{3}} m/minute$$

$$\Rightarrow \frac{dr}{dr} = \sqrt{3} \left(\frac{4}{\pi}\right)^{\frac{1}{3}} m/minute\ (Shown)$$

Example 8

ZIMSEC November 2019 Paper 2



The diagram shows a container in the form of a right circular cone of height h and radius r. The height of the cone is 10 cm and the diameter is $2cm$. The container catches water from a tap leaking at a constant rate of 0.1 cm^3/s, the height and the radius are always proportional.

Find the rate at which the top surface area of water is increasing when the water is halfway up the cone. **[9]**

$$\left[\text{Volume of a cone} = \frac{1}{3}\pi r^2 h\right]$$

$$\frac{dV}{dt} = 0.1cm^3/sec$$

The height of the cone is $10\ cm$

The diameter is $2cm \Rightarrow r = 1$

The height and the radius are always proportional $\Rightarrow h \propto r$

$$\Rightarrow 10 = kr$$

$$\Rightarrow 10 = k(1)$$

$$\Rightarrow 10 = k$$

$$\Rightarrow h = 10r$$

When the water is halfway up the cone $\Rightarrow h = 5cm$

$$\Rightarrow r = \frac{1}{10}h = \frac{1}{10}(5) = \frac{1}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi r^2(10r)$$

$$V = \frac{10}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dr} = 10\pi r^2$$

$$\Rightarrow \frac{dV}{dr}_{r=\frac{1}{2}} = 10\pi\left(\frac{1}{2}\right)^2$$

$$\Rightarrow \frac{dV}{dr}_{r=\frac{1}{2}} = 10\pi^2\left(\frac{1}{4}\right)$$

$$\Rightarrow \frac{dV}{dr} = \frac{5\pi}{2}\ cm^2$$

The top surface area of water = circle

$A = \pi r^2$

$\Rightarrow \frac{dA}{dr} = 2\pi r$

$\Rightarrow \frac{dA}{dr}_{r=\frac{1}{2}} = 2\pi\left(\frac{1}{2}\right)$

$\Rightarrow \frac{dA}{dr} = \pi cm$

The rate at which the top surface area of water is increasing $= \frac{dA}{dt}$

$\Rightarrow \frac{dA}{dt} = \frac{dV}{dt} \times \frac{dr}{dV} \times \frac{dA}{dr}$

$\Rightarrow \frac{dA}{dt} = 0.1\ cm^3/sec\ \times \frac{2}{5\pi\ cm^2} \times \pi\ cm$

$\Rightarrow \frac{dA}{dt} = \frac{1}{10} \times \frac{2}{5\pi} \times \pi\ cm^2/sec$

$\therefore \frac{dA}{dt} = \frac{1}{25} cm^2/sec$

<u>Example 9</u>

A cylindrical glass with radius of $4cm$ is being filled with water from a tap at a rate of $40\ cm^3/s$. How fast is the water level in the glass rising?

[Volume of a cylinder $= \pi r^2 h$]

<u>Suggested Solution</u>

$\frac{dV}{dt} = 40cm^3/s$

$V = \pi r^2 h$

$\Rightarrow \frac{dV}{dh} = \pi r^2$

$\Rightarrow \frac{dV}{dh_{r=4}} = \pi(4)^2 cm^2$

$\Rightarrow \frac{dV}{dh} = 16\pi cm^2$

Now:

$\frac{dh}{dt} = \frac{dV}{dt} \therefore \frac{dV}{dh}$

$\Rightarrow \frac{dh}{dt} = \frac{40\ cm^3/s}{16\pi\ \ cm^2}$

$\Rightarrow \frac{dh}{dt} = \frac{40}{16\pi} cm^3/s$

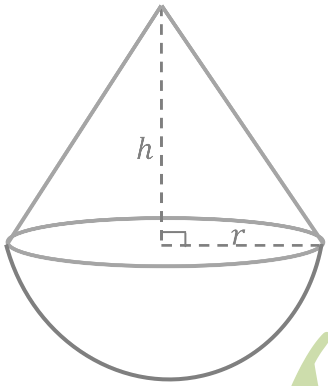
$\therefore \frac{dh}{dt} = \frac{5}{2\pi} cm/s$

PRACTICE QUESTIONS

Question 1

ZIMSEC November 2008 Paper 1

A solid consists of a hemisphere of radius r joined to a cone of constant height $60cm$. The base of the cone is the plane face of a hemisphere (See diagram).

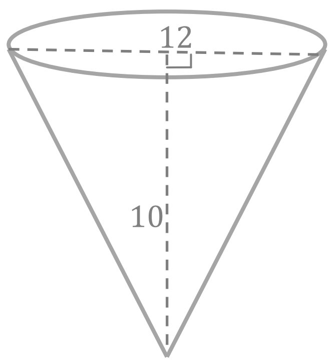


If r increases at a rate of $2cm$ per $minute$, calculate the rate of increase of the volume of the solid when $r = 6ocm$. Leave your answer in terms of π. **[4]**

$$\left[\text{Volume of a cone} = \frac{1}{3}\pi r^2 h \quad ; \quad \text{Volume of a sphere} = \frac{4}{3}\pi r^3\right]$$

Answer: $\frac{dA}{dt} = 19\ 00\pi\ cm^3/minute$

Question 2

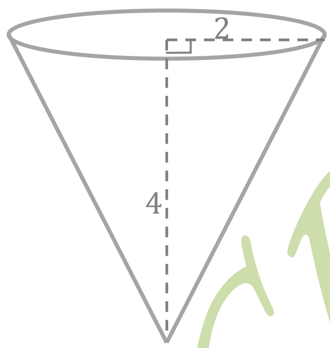


A $10cm$ tall funnel is being drained with water at a constant rate of $10cm^3$ per $second$. The mouth of the funnel is $12cm$ in diameter. How far is the solution level dropping when there is $200cm^3$ left in the funnel?

$$\left[\text{Volume of a cone} = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = -0.13cm/sec$

Question 3

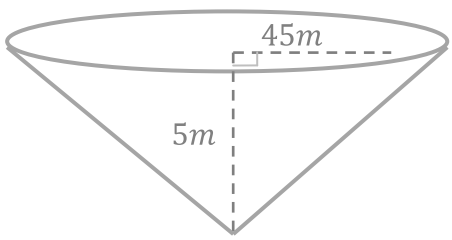


A water tank (inverted cone) with radius $2m$ and height $4m$. If water is pumped in at a rate of $2\ m^3/minute$, find the rate at which the water level is rising when the water is $3m$ deep.

$$\left[\text{Volume of a cone} = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = \frac{8}{9\pi}m/sec$ or $0.28m/sec$

Question 4



Water is draining from a conical reservoir at a rate of 50 $m^3/minute$. How fast is the water level falling when the water is $2m$ deep? The tank is $5m$ tall with a radius is $5m$ deep.

$$\left[\text{Volume of a cone } = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = \frac{25}{102\pi}\, m/minute$

Question 5

Sand is poured into a conical pile at a rate of $4cm^3/s$. The radius is half the height of the cone. When the cone is $16cm$ high, how fast is the height of the sand changing at this instant.

$$\left[\text{Volume of a cone } = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = \frac{1}{16\pi}\, cm/sec$ or $0.0199 cm/sec$

Question 6

Water is being poured into a cone-shaped cup at a rate of $50cm^3/s$. The radius is half the height of the cone. If the cup has a height of $8cm$ and maximal radius of $3cm$, how fast is the water rising when it is $4cm$ full.

$$\left[\text{Volume of a cone } = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = \frac{200}{9\pi}\, cm/sec$

Question 7

River sand is being is dumped from a conveyor belt a rate of 25 $m^3/minute$. It forms a pile of the shape of a right circular cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is $18m$ high?

$$\left[\text{Volume of a cone } = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = \frac{75}{243\pi} m/minute \approx 0.0982m/minute$

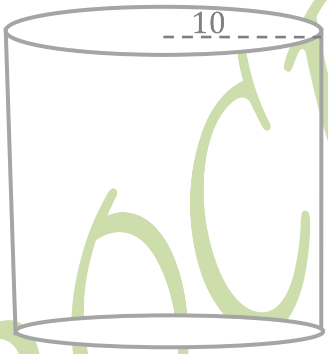
Question 8

An inverted conical container has a diameter of $6inches$ and a depth of $2.5inches$. If water is flowing out of the vertex of the container at a rate of $2\pi\ inches^3/s$, how fast is the depth of the water dropping when the height of the water is $10inches$?

$$\left[\text{Volume of a cone} = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = -\frac{1}{2} inches/sec$

Question 8

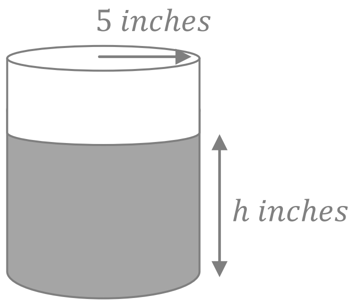


A cylindrical water tank with a radius of $10metres$ is being drained. The fluid is being drained from the tank at a rate of $1.2m^3/minute$. At what rate is the fluid level inside the tank changing?

$$[\text{Volume of a cylinder} = \pi r^2 h]$$

Answer: $\frac{dh}{dt} = -\frac{3}{250} m/minute$

Question 9

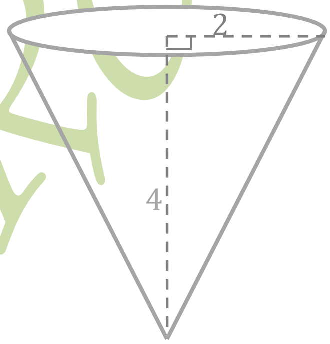


A coffeepot has the shape of a cylinder with radius 5 *inches* as shown in the diagram above. Let h be the depth of the coffee in pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. Show that

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$$

[Volume of a cylinder $= \pi r^2 h$]

Question 10



A water tank has the shape of an inverted circular cone with the base radius $2m$ and height $4m$. If water is being pumped into the tank at the rate of $2m^3/minute$, find the rate at which the water level is rising when the water is $3m$.

$$\left[\text{Volume of a cone} = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = \frac{8}{9\pi}\, m/minute$

Question 11

Air is being pumped into a spherical balloon so that the volume increases at a rate of $100\ m^3/s$. How fast is the radius of the balloon increasing when the diameter is $50cm$?

$$\left[\text{Volume of a sphere} = \frac{4}{3}\pi r^3\right]$$

Answer: $\frac{dr}{dt} = \frac{1}{25\pi}\, cm/sec$

Question 12

Gravel is being is dumped from a conveyor belt a rate of $20\ m^3/minute$ and its coarseness is such that it forms a pile of the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is $10m$ high?

$$\left[\text{Volume of a cone} = \frac{1}{3}\pi r^2 h\right]$$

Answer: $\frac{dh}{dt} = \frac{4}{5\pi}\, m/minute \ \ or \ \ 0.255m/minute$

FORMULATION OF SIMPLE STATEMENTS INVOLVING RATES OF CHANGE AS A DIFFERENTIAL EQUATION

NOTES

- The proportional sign ($\propto$) is replaced by an equal sign ($=$) and a constant k or any other letter.
- Initially means $t = 0$.
- Transforming a constant yields another constant e.g. $e^k = A$ or $-2k = B$ or $(2k + c = D$.
- The differential coefficient is negative$\left(-\frac{d(dependant\ variable)}{d(independent\ variable)}\right)$ when there is a decrease.
- Examples: Death, decay, fishing, melting, harvesting, destroying, grazing leakages, exhausting something, burning, decreasing something etc.
- The differential coefficient is negative$\left(\frac{d(dependant\ variable)}{d(independent\ variable)}\right)$ when there is an increase.
- Examples: Birth, replenishing, planting, heating, cooling, addition of substances, refilling of liquid/gas/substances, pouring of liquid/sand, increasing something, planting, breeding etc.

STEPS

- Express the information involving a rate of change in algebraic form (as a differential equation)
- Make the DE look like $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (Unless it's already been done for you – in which case you can just identify the various parts or you may have to do some algebra to get it into the correct form.)

- Separate the variables: Set all the $y's$ on the LHS by multiplying both sides by $g(y)$ and get all the $x's$ on the RHS by multiplying by dx i.e. $g(y)dy = f(x)dx$.
- Integrate both sides: $\int g(y)dy = \int f(x)dx$. This gives us an implicit solution. Solve for y (if possible). This gives us an explicit solution. From this stage the solution is expressed in terms of arbitrary constants and we have the general solution.
- If there are initial conditions, use them to solve for unknown parameters in the solution. This gives us a particular solution.

GENERAL ODES – Rates of Change

Worked Example

ZIMSEC November 2006 #14

(a) Solve the differential equation $\frac{dy}{dx} = x^2y^2$, given that $y = -\frac{1}{2}$ when $x = 3$, expressing y in terms of x. **[4]**

(b) A liquid is being kept in an oven maintained at a temperature of 200°C.. It is assumed that the rate of increase in temperature of the liquid, is proportional to $(200 - \theta)$, where θ°C $(\theta < 200)$ is the temperature of the liquid at time t minutes.

Form a differential equation relating to θ and t. **[2]**

Show that the general solution of the differential equation is

$$\theta = 200 - Ae^{-kt}$$

where A is a constant. **[3]**

If the temperature of the liquid rises from 0°C to 100°C in 6 minutes, find the temperature of the liquid after 10 minutes, giving your answer to the nearest °C. **[6]**

Solution

a) $\frac{dy}{dx} = x^2y^2$

$\int \frac{1}{y^2}dy = \int x^2dx$

$\int y^{-2}dy = \int x^2dx$

$\frac{-1}{y} = \frac{x^3}{3} + c$

$y = -\frac{3}{x^3+b}$ $[general\ solution]$

When $y = -\frac{1}{2}, x = 3$

$-\frac{1}{2} = -\frac{3}{3^3+b} \Rightarrow 27 + b = 6 \Longrightarrow b = -21.$

$\Rightarrow y = -\frac{3}{x^3-21}$

$\therefore y = \frac{3}{21-x^3}$ $[particular\ solution]$

b) $\frac{d\theta}{dt} \propto 200 - \theta$

$\frac{d\theta}{dt} = k(200 - \theta)$

$\int \frac{1}{200-\theta}d\theta = \int kdt$

$-1\int \frac{-1}{200-\theta}d\theta = kt + c$

$-In(200 - \theta) = kt + c$

$In(200-\theta)=-kt$

$e^{(In(200-\theta))}=e^{-kt-c}$

$200-\theta=e^{-kt}\times e^{-c}$

$200-\theta=Ae^{-kt}$

$\therefore \theta=200-Ae^{-kt}\quad [as\ required]$

When $=0℃$, $t=0$ minutes

$0=200-Ae^{0}$

$\Rightarrow A=200.$

$\Rightarrow \theta=200-200e^{-kt}$

When $\theta=100℃,\quad t=6$ minutes

$100=200-200e^{-6t}$

$200e^{-6t}=100$

$e^{-6t}=\frac{100}{200}$

$e^{-6t}=\frac{1}{2}$

$In(e^{-6t})=In\left(\frac{1}{2}\right)$

$-6t=In(2^{-1})$

$-6t=-In(2)$

$\therefore t=\frac{1}{6}In2$

$\Rightarrow \theta=200-200e^{-\left(\frac{1}{6}In2\right)t}$

When $t=10$ minutes, $\theta=?℃$

$\theta = 200 - 200e^{-\left(\frac{1}{6}In2\right)10}$

$\theta = 137.0039475°C$

$\therefore\ \theta = 137°C \quad (to\ the\ nearest\ °C)$

RELATED ODES PAST EXAMINATION QUESTIONS

ZIMSEC June 2012 #16

(a) Solve the differential equation $(4-x)\frac{dy}{dx} = y$, given that $y = 4$ when $x = 1$, expressing y in terms of x. **[4]**

(b) A liquid is kept in a cold room. It is assumed that the rate of decrease in temperature of the liquid, is proportional to θ, where θ°C is the temperature of the liquid at time t minutes. When $t = 0$, $\theta = 60$°C and after 5 minutes $\theta = 40$°C.

(i) Form a differential equation relating to θ and t.

(ii) Solve the differential equation and find the temperature of the liquid after further 5 minutes. **[8]**

Answers: a) $y = \frac{12}{4-x}$

b)(i) $\frac{d\theta}{dt} = -k\theta$

(ii) $In\theta = \frac{1}{5}tIn\left(\frac{2}{3}\right) + In60;$ $\theta = 26\frac{2}{3}$.

ZIMSEC June 2018 #16

(a) Solve the differential equation $\frac{dy}{dx} = ytanx$, given that $y = 1$ when $x = 0$, expressing y in terms of x. **[4]**

(b) A spray for cockroaches kills at a rate which inversely proportional to the square root of the number, N, of live cockroaches at time, t minutes after spraying.

(i) Show that the information can be modelled by the differential equation

$$\frac{dN}{dt} = -\frac{k}{\sqrt{N}}$$

[2]

(ii) Solve the differential equation given that initially there were 144 live cockroaches and 36 minutes later, 81 live cockroaches remained. **[5]**

(iii) Find the number of live cockroaches left one hour after spraying. **[2]**

Answers: a) $y = \frac{1}{cosx}$ $or\ y = secx$

b)(i) $N = \sqrt[3]{(1728 - 27.25t)^2}$

(ii) $N = 16.$

ZIMSEC November 2018 Paper 2 #11

It is assumed that the length, l, in cm of a certain snake at time, t, months after birth increases at a rate proportional to $(10 - l)^{\frac{1}{2}}$.

When $t = 0$ $\quad l = 1$ $\quad$ and $\frac{dl}{dt} = 0.3$

(i) Show that l and t satisfy the differential equation.

$$\frac{dl}{dt} = 0.1(10 - l)^{\frac{1}{2}}$$

[2]

(ii) Solve the differential equation and obtain an expression for l in terms of t. **[6]**

(iii)Find the maximum length of the snake. **[2]**

Answers: (i)$l = 10 - (3 - 0.05t)^2$

(ii) $\frac{dl}{dt} = 0 \Rightarrow l = 10$

ZIMSEC November 2019 Paper 1 #14

The height , h $metres$, of a tree at time, t $years$, after being planted assumes that the rate of increase of its height is directly proportional to $(10 - h)^{\frac{1}{2}}$.

When $t = 0$ $\quad h = 1$ $\quad$ and $\frac{dh}{dt} = 0.4$

(i) Show that this satisfies the differential equation.

$$\frac{dh}{dt} = \frac{2}{15}(10 - h)^{\frac{1}{2}} \quad [3]$$

(ii) Solve the differential equation to obtain an expression for h in terms of t. **[6]**

(iii)Find the maximum height of the tree and the time taken to reach his height after planting **[2]**

(iv)Find the time taken to reach $\frac{2}{5}$ of its maximum height. **[2]**

Answers: (ii) $h = 10 - \left(3 - \frac{1}{15}t\right)^2$

(iii) $\frac{dh}{dt} = 0 \Rightarrow t = 45 \Rightarrow Maximum\ heigt = 10$

(iv) $t = 8.3\ years$

ODES INVOLVING VOLUMES & AREAS

Worked Examples

ZIMSEC NOVEMBER 2005 #13

A cylindrical water tank of a height 1 metre and a cross sectional area of 3000 cm^2 has a tap at the base

When the tank is full, the tap is opened and water flows out at the rate of $200\sqrt{h}\ cm^3$ per second. The depth of water which remains t seconds after the tap is opened is hcm.

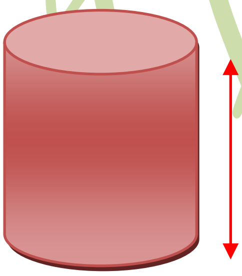
Given that no water enters the tank, show that h satisfies the differential equation

$\frac{dh}{dt} = -\frac{1}{15}\sqrt{h}.$ **[5]**

Solve the differential equation, giving t in terms of h. **[5]**

When the tap is closed, the tank is found to be three-quarters full. Find, to the nearest second, the time for which the tap has been open. **[2]**

Solution



$\pi r^2 = 3\ 000.$

$\frac{dV}{dt} = -200\sqrt{h}$

Now

$V = \pi r^2 h$

$\frac{dV}{dh} = \pi r^2$

$\Rightarrow \frac{dV}{dh} = 3\ 000$

Now $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

$\Rightarrow \frac{dh}{dt} = \frac{-200\sqrt{h}}{3\ 000}$

$\therefore \frac{dh}{dt} = -\frac{1}{15}\sqrt{h} \quad [as\ required]$

Solving the ODE:

$\frac{dh}{dt} = -\frac{1}{15}\sqrt{h}$

$\int \frac{1}{\sqrt{h}} dh = \int -\frac{1}{15} dt$

$\int h^{-\frac{1}{2}} dy = -\frac{1}{15} t + c$

$2h^{\frac{1}{2}} = -\frac{1}{15} t + c$

$2\sqrt{h} = -\frac{1}{15} t + c$

When $t = 0, h = 100$

$2\sqrt{100} = -\frac{1}{15}(0) + c$

$\Rightarrow c = 20$

$\Rightarrow 2\sqrt{h} = -\frac{1}{15}t + 20$

$\Rightarrow \frac{1}{15}t = 20 - 2\sqrt{h}$

$\therefore t = 300 - 30\sqrt{h}$

When $h = \frac{3}{4}(100) = 75, \quad t = ?$

$t = 300 - 30 \times \sqrt{75}$

$t = 300 - 30 \times \sqrt{75}$

$t = 40.19237886$

$\therefore t = 40 seconds$ (to the neasrest $second$)

Example 2

Cambridge Question

A rectangular reservoir has a horizontal base of area $1000m^2$. At time $t = 0$, it is empty and water begins to flow into it at a constant rate of $30m^3s^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where hm is the depth of the water at time ts. When $h = 1, \frac{dh}{dt} = 0.02$

(i) Show that h satisfies the differential equation $\frac{dh}{dt} = 0.01\left(3 - \sqrt{h}\right)$.

It is given that, after making the substitution $x = 3 - \sqrt{h}$, the equation in part **(i)** becomes

$$(x-3)\frac{dx}{dt} = 0.005x.$$

(ii) Using the fact that $x = 3$ when $t = 0$, solve this differential equation, obtaining an expression for t in terms of x.

(iii) Find the time at which the depth of water reaches $4m$.

Suggested Solution

(i) Inflow rate:

$$\frac{dV}{dt} = 30m^3s^{-1}$$

Outflow rate:

$$-\frac{dV}{dt} \propto \sqrt{h}$$

$$\Rightarrow \frac{dV}{dt} = -k\sqrt{h}m^3s^{-1}$$

∴ The rate of volume at any instant is given by:

$$\frac{dV}{dt} = 30 - k\sqrt{h}$$

Now:

$V = lwh$

$\frac{dV}{dh} = lw$

$\Rightarrow \frac{dV}{dh} = 1000m^2$

Also

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

$\frac{dh}{dt} = \frac{(30 - k\sqrt{h})}{1000} ms^{-1}$

When $h = 1, \frac{dh}{dt} = 0.02$

$0.02 = \frac{(30 - k\sqrt{1})}{1000}$

$20 = 30 - k$

$\Rightarrow k = 10$

$$\Rightarrow \frac{dh}{dt} = \frac{(30 - 10\sqrt{h})}{1000} ms^{-1}$$

$$\Rightarrow \frac{dh}{dt} = (0.03 - 0.01\sqrt{h})ms^{-1}$$

$$\therefore \frac{dh}{dt} = 0.01(3 - \sqrt{h})ms^{-1} \quad (as\ required)$$

(ii) $(x - 3)\frac{dx}{dt} = 0.005x$

$$\int \frac{(x-3)}{x} dx = \int 0.005dt$$

$$\int \left[\frac{x}{x} - \frac{3}{x}\right] dx = \int 0.005dt$$

$$\int \left[1 - \frac{3}{x}\right] dx = \int 0.005dt$$

$$x - 3In|x| = 0.005t + c$$

Given that $x = 3$ when $t = 0$

$$3 - 3In|3| = 0.005(0) + c$$

$$\therefore c = 3 - In|27|$$

$$\Rightarrow x - 3In|x| = 0.005t + 3 - In|27|$$

$$\Rightarrow x - 3In|x| - 3 + In|27| = 0.005t$$

$$\Rightarrow 0.005t = x + In\left|\frac{27}{x^3}\right| - 3$$

$$\Rightarrow t = \frac{x}{0.005} + \frac{1}{0.005} In\left|\frac{27}{x^3}\right| - \frac{3}{0.005}$$

$$\therefore t = 200x + 200In\left|\frac{27}{x^3}\right| - 600 \quad or\ equivalent$$

(iii) $x = 3 - \sqrt{h}$

$= 3 - \sqrt{4}$

$= 3 - 2$

$= 1$

$t = 200x + 200In\left|\frac{27}{x^3}\right| - 600$

$\Rightarrow t = 200(1) + 200In\left|\frac{27}{(1)^3}\right| - 600$

$\Rightarrow t = 259.167\ 373\ 201$

$\therefore t = 259s$

Example 3

Cambridge Question

Liquid is flowing into a small tank which has a leak. Initially the tank is empty and t minutes later, the volume liquid in the tank is Vcm^3. The liquid is flowing into the tank at a constant rate of $80cm^3$ per minute. Because of the leak, liquid is being lost from the tank at a rate which, at any instant, is equal to $kVcm^3$ per minute, where k is a positive constant.

(i) Write down a differential equation describing this situation and solve it to show that

$$V = \frac{1}{k}(80 - 80e^{-kt})$$

(ii) It is observed that $V = 500$ when $t = 15$, so that k satisfies the equation

$$k = \frac{4 - 4e^{-15k}}{25}$$

Use an iterative formula based on this equation, to find the value of k correct to 2 significant figures. Use the initial value of $k = 0.1$ and show the result of each iteration correct to 4 significant figures.

(iii)Determine how much liquid is in the tank 20 minutes after the liquid started flowing and state what happens to the volume of the liquid in the tank after a long time.

Suggested Solution

(i) $\frac{dV_1}{dt} = 80$

$\frac{dV_2}{dt} = -kV$

Now:

$$\frac{dV}{dt} = \frac{dV_1}{dt} + \frac{dV_2}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 80 - kV$$

$$\Rightarrow \int \frac{1}{80 - kV} dV = \int dt$$

$$\Rightarrow -\frac{1}{k} \int \frac{-k}{80 - kV} dV = \int dt$$

$$\Rightarrow -\frac{1}{k} In|80 - kV| = t + c$$

Initially the tank is empty $\Rightarrow t = 0$ and $V = 0$

$$\Rightarrow -\frac{1}{k} In|80 - 0| = 0 + c$$

$$\Rightarrow c = -\frac{1}{k} In|80|$$

$$\Rightarrow -\frac{1}{k} In|80 - kV| = t - \frac{1}{k} In|80|$$

$$\Rightarrow In|80 - kV| = -kt + In|80|$$

$$\Rightarrow e^{In|80-kV|} = e^{-kt+In|80|}$$

$$\Rightarrow e^{In|80-kV|} = e^{-kt} e^{In|80|}$$

$$\Rightarrow 80 - kV = 80e^{-kt}$$

$$\Rightarrow kV = 80 - 80e^{-kt}$$

$$\therefore V = \frac{80 - 80e^{-kt}}{k} \quad (as\ required)$$

(ii) $k_{n+1} = \dfrac{4 - 4e^{-15k_n}}{25}$

$k_1 = 0.1$

$k_2 = k_{1+1} = \dfrac{4 - 4e^{-15k_1}}{25}$

$= \dfrac{4 - 4e^{-15(0.1)}}{25}$

$= 0.124299174$

$= 0.1243$ (to $4s.f$)

$k_3 = k_{2+1} = \dfrac{4 - 4e^{-15k_2}}{25}$

$= \dfrac{4 - 4e^{-15(0.124299174)}}{25}$

$= 0.135203904$

$= 0.1352$ (to $4s.f$)

$k_4 = k_{3+1} = \dfrac{4 - 4e^{-15k_3}}{25}$

$= \dfrac{4 - 4e^{-15(0.135203904)}}{25}$

$= 0.13894548$

$= 0.1389$ (to $4s.f$)

$\therefore$ The root is 0.14 (to $2s.f.$)

(iii)$V = \dfrac{80 - 80e^{-0.14t}}{0.14}$

$$= \frac{80 - 80e^{-0.14(20)}}{0.14}$$

$$= 536.6799642cm^3$$

$$= 540cm^3(\text{to } 2\ s.f.)$$

NB: After a long time $\Rightarrow t \rightarrow \infty$

Now:

$$\lim_{n\to\infty}(e^{-n}) = 0 \Rightarrow \lim_{t\to\infty}\left(\frac{80 - 80e^{-0.14t}}{0.14}\right) = \frac{80 - 80(0)}{0.14}$$

$$= \frac{80}{0.14}$$

$$= 571.4285714$$

$$= 570cm^3(\text{to } 2s.f.)$$

$\therefore$ After a long time the liquid in the tank will approach $570cm^3$.

Example 4

CAMBRIDGE NOVEMBER 2009 PAPER 3

In a model of the expansion of a sphere of radius r cm, it is assumed that, at time t seconds after the start, the rate of increase of the surface area of the sphere is proportional to its volume. When $t = 0$, $r = 5$ and $\frac{dr}{dt} = 2$.

(i) Show that r satisfies the differential equation $\frac{dr}{dt} = 0.08r^2$. **[4]**

[The surface area A and volume V of a sphere of radius r are given by the formulae

$$A = 4\pi r^2, V = \frac{4}{3}\pi r^3]$$

(ii) Solve this differential equation, obtaining an expression for r in terms of t. **[5]**

(iii)Deduce from your answer to part **(ii)** the set of values that t can take, according to this model. **[1]**

Suggested Solution

(i) $\frac{dA}{dt} \propto V$

$\frac{dA}{dt} \propto \left(\frac{4}{3}\pi r^3\right)$

$\frac{dA}{dt} = k\left(\frac{4}{3}\pi r^3\right)$

Now:

$A = 4\pi r^2$

$\Rightarrow \frac{dA}{dr} = 8\pi r$

$\therefore \frac{dr}{dt} = \frac{dA}{dt} \div \frac{dA}{dr}$

$= k\left(\frac{4}{3}\pi r^3\right) \div 8\pi r$

$= k\left(\frac{1}{6}r^2\right)$

When $t = 0, r = 5$ and $\frac{dr}{dt} = 2.$

$\frac{dr}{dt} = k\left(\frac{1}{6}r^2\right)$

$\Rightarrow 2 = k\left[\frac{1}{6}(5)^2\right]$

$\Rightarrow k = \frac{12}{25}$

Now:

$\frac{dr}{dt} = \frac{12}{25}\left(\frac{1}{6}r^2\right)$

$\frac{dr}{dt} = 0.08r^2\ (as\ required)$

(ii) $\frac{dr}{dt} = 0.08r^2$

$\int \frac{1}{r^2} dr = \int 0.08\, dt$

$\int r^{-2} dr = 0.08t + c$

$-\frac{1}{r} = 0.08t + c$

When $t = 0, r = 5$ and $\frac{dr}{dt} = 2$.

$\Rightarrow\ -\frac{1}{5} = 0.08(0) + c$

$\Rightarrow\ c = -\frac{1}{5}$

$\Rightarrow -\frac{1}{r} = 0.08t - \frac{1}{5}$

$\Rightarrow \frac{1}{r} = \frac{1}{5} - 0.08t$

$\Rightarrow \frac{1}{r} = \frac{1 - 0.4t}{5}$

$\therefore r = \frac{5}{1 - 0.4t}$

(i) $1 - 0.4t \neq 0$

$\Rightarrow 1 \neq 0.4t$

$\Rightarrow t \neq 2.5$

Also:

$t \geq 0$ and $t < 2.5$

$\therefore 0 \leq t < 2.5$

RELATED ODES PAST EXAMINATION QUESTIONS

ZIMSEC NOVEMBER 2015 #17

A rectangular tank has a square base with sides $3m$ long and height $5m$.

The tank is initially full of water. The water leaks through the base and sides of the tank at a rate proportional to the total area in contact with water. When the depth is $3m$, the level of water is falling at a rate of $0.3m/h$.

If y denoted the depth of water in the tank after t hours,

(i) show that $\frac{dy}{dt} = -\frac{3}{50}\left(\frac{4y}{3} + 1\right)$. **[5]**

(ii) solve the differential equation in (**i**), giving y in terms of t. **[6]**

(iii)find how long it takes for the tank to be half full. **[2]**

Answers: ii)$y = \frac{1}{4}\left(23e^{-\frac{2}{25}t} - 3\right)$

iii) $7.2\ hours$

ZIMSEC NOVEMBER 2010 #16

a) Some juice is tapped into a cylindrical container at a rate of $100\ cm^3$ per minute. It is sieved out through a hole at the bottom of the cylinder at a rate of $2.5h\ cm^3$ per minute, where hcm is the height of the juice in the cylinder at time t minutes. The radius of the cylinder is $5cm$.

1. Show that $\frac{dh}{dt} = \frac{(40-h)}{10\pi}$. **[4]**
2. Solve the differential equation to find h in terms of t, given that at time $t = 0, h = 0$.
   Hence or otherwise state the maximum value of h and explain why this height cannot be exceeded. **[8]**

b) The delivering tap in part (a) was closed off when the juice was at maximum height, and the juice was allowed to drain out. The differential equation satisfied by the drainage process only is $\frac{dh}{dt} = -\frac{1}{10\pi}h$.

Solve the differential equation to find t in terms of h and show that the time taken to make juice go down to a height of $0.88cm$ is nearly 2 hours. **[4]**

Answer: a)(ii)$y = 40\left(1 - e^{-\frac{1}{10\pi}t}\right)$

ZIMSEC NOVEMBER 2011 #14

a) Find the general solution of the differential equation $x\frac{dy}{dx} = y + yx$, given that the curve passes through the point $(2; 4)$. **[4]**

b) A water tank has the shape of a cuboid with base area $4\,m^2$ and a height $3m$ and is initially empty. Water is poured into the tank at a constant rate of $0.05m^3$per minute. There is a small hole at the bottom of the tank through which water leaks out. The depth of the water in the tank is h metres when water has been poured in for t minutes.

(i) In a simple model it is assumed that the water leaks out of the tank at a constant rate of $0.025m^3$ per minute.

1. Show that the variable h satsfies the differential equation$\frac{dh}{dt} = \frac{1}{160}$
2. Hence or otherwise, find the time when the tank starts to overflow.

(ii) In a more refined model the variable, h, satisfies the differential equation

$160\frac{dh}{dt} = 2 - h.$

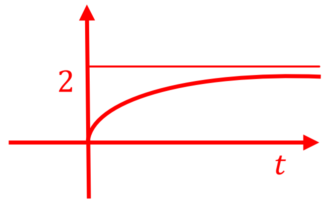
1. Solve the differential equation, expressing h in terms of t.
2. Hence sketch the graph of h against t. **[13]**

Answers: a)$Iny = Inx + x - 2 + In2$ or $Iny = In2x + x - 2$ or $y = 2xe^{x-2}$

b)(i)(2) $480\ minutes$

(ii)(1) $h = 2\left(1 - e^{-\frac{1}{160}t}\right)$

(ii)(2)



ZIMSEC JUNE 2005 #11

During a flood water is entering a lake at a constant rate of 12 cubic metres per second. The volume of water in the lake at any given time is kh^3cubic metres, where h is the depth, in metres, of the water at the dam wall and k is a constant.

Show that the rate of change of depth is given by the equation $\frac{dh}{dt} = \frac{4}{kh^2}$. **[4]**

Obtain the general solution of this equation. **[2]**

Given that when the flood starts, the depth of water at the walls is $24m$, find in terms of k, the time that it will take to increase to a depth of $30m$. **[3]**

Answers: $\frac{h^3}{3} = \frac{4}{k}t + c$

$t = 1098k \ seconds$

ZIMSEC NOVEMBER 2003 #13

A water tank with a uniform cross-section has a tap at its base. When the tap is opened, water flows out at a rate proportional to the square root of the depth of water in the tank. Given that cross-sectional area of the tank is $5m^2$ and the depth of water, t minutes after opening is h metres, show that $\frac{dh}{dt} = -\frac{k}{5}\sqrt{h}$. **[6]**

Give that the tap is opened when the depth of water is 2 metres, find an expression in term of k for the time taken for the depth to reach 1 metres. **[5]**

Answer: $T = \frac{10}{k}(\sqrt{2} - 1)$

Edexcel January 2008

Liquid is pouring into a large vertical circular cylinder at a constant rate of $1\,600cm^3/s$ and is leaking out of a hole in the base at a rate proportional to the square root of the height of the liquid already in the cylinder.

The area of the circular cross section of the cylinder is $4\,000cm^2$.

a) Show that at time t $seconds$, the height hcm of liquid in the cylinder satisfies the differential equation:

$\frac{dh}{dt} = 0.4 - k\sqrt{h}$, where k is a positive constant. **[3]**

When $h = 25$, water is leaking out of the hole at $400cm^3/s$

b) Show that $k = 0.02$. **[1]**

c) Separate the variables of the differential equation $\frac{dh}{dt} = 0.4 - 0.02\sqrt{h}$, to show that the time taken to fill the cylinder from empty to a height of $100cm$ is given by

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\,dh$$

[2]

Using the substitution $h = (20 - x)^2$, or otherwise,

d) find the exact value of

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\,dh$$

[6]

e) Hence find the time taken to fill the cylinder from empty to a height of $100cm$, giving your answer in $minutes$ and $seconds$ to the nearest $second$. **[1]**

Answers: d) $2000In2 - 1000$

e) $(2000In2 - 1000)seconds = 6minutes\ 26seconds$

ODES INVOLVING PARTIAL FRACTIONS

Worked Example

ZIMSEC NOVEMBER 2014 #11

(a) The gradient function of a curve is given by $\frac{dy}{dx} = \frac{15}{(3x+4)^2}$, and $M(2;6)$ is a point on the curve.

Find the equation of the curve. **[4]**

(b) The rate of increase of the number of people, x, who own a laptop is proportional to the product of x and $N - x$, where N is the total population.

Initially $x = \frac{1}{10}N$ and after one week $x = \frac{1}{4}N$.

(i) Form a differential equation from the above information and show that after t weeks, $\frac{x}{N-x} = 3^{t-2}$

(ii) Find the time, t , in (i) when $x = \frac{3}{4}N$

(iii)Sketch the graph x against t for the equation in (i)

[11]

Solution

(a) $\frac{dy}{dx} = \frac{15}{(3x+4)^2}$

$$\frac{dy}{dx} = 15(3x+4)^{-2}$$

$$\int dy = \int 15(3x+4)^{-2}dx$$

$$y = \frac{15(3x+4)^{-2+1}}{(-2+1)\times 3} + c$$

$$y = \frac{-5}{3x+4} + c \; [general\ solution]$$

When $x = 2, y = 6$:

$$6 = \frac{-5}{3(2)+4} + c \Longrightarrow c = \frac{13}{2}$$

$$\therefore y = \frac{13}{2} - \frac{5}{3x+4} \quad [particular\ solution]$$

(b)(i) $\frac{dx}{dt} \propto x(N-x)$

$$\frac{dx}{dt} = kx(N-x)$$

$$\int \frac{1}{x(N-x)} dx = \int kdt$$

Resolving $\frac{1}{x(N-x)}$ to partial fractions yields $\frac{1}{Nx} + \frac{1}{N(N-x)}$

$$\Rightarrow \frac{1}{N}\int \frac{1}{x} + \frac{1}{(N-x)} dx = kt + c$$

$$\frac{1}{N}[Inx - In(N-x)] = kt + c$$

$$\frac{1}{N} In\left(\frac{x}{N-x}\right) = kt + c$$

When $t = 0, x = \frac{1}{10}N$

$$\frac{1}{N} In\left(\frac{\frac{1}{10}N}{N - \frac{1}{10}N}\right) = k(0) + c$$

$$c = \frac{1}{N} In\left(\frac{1}{9}\right)$$

$$\Rightarrow \frac{1}{N} In\left(\frac{x}{N-x}\right) = kt + \frac{1}{N} In\left(\frac{1}{9}\right)$$

When $t = 1, x = \frac{1}{4}N$

$$\frac{1}{N}In\left(\frac{\frac{1}{4}N}{N-\frac{1}{4}N}\right) = k(1) + \frac{1}{N}In\left(\frac{1}{9}\right)$$

$$\frac{1}{N}In\left(\frac{1}{3}\right) = k + \frac{1}{N}In\left(\frac{1}{9}\right)$$

$$k = \frac{1}{N}In\left(\frac{1}{3}\right) - \frac{1}{N}In\left(\frac{1}{9}\right)$$

$$k = \frac{1}{N}In\left(\frac{1}{3} \times \frac{9}{1}\right)$$

$$k = \frac{1}{N}In(3)$$

$$\Rightarrow \frac{1}{N}In\left(\frac{x}{N-x}\right) = \left(\frac{1}{N}In(3)\right)t + \frac{1}{N}In\left(\frac{1}{9}\right)$$

$$In\left(\frac{x}{N-x}\right) = tIn(3) + In\left(\frac{1}{9}\right)$$

$$In\left(\frac{x}{N-x}\right) = tIn(3) + In(3^{-2})$$

$$In\left(\frac{x}{N-x}\right) = In(3^{t}) + In(3^{-2})$$

$$In\left(\frac{x}{N-x}\right) = In(3^{t} \times 3^{-2})$$

$$In\left(\frac{x}{N-x}\right) = In(3^{t-2})$$

$$\therefore \frac{x}{N-x} = 3^{t-2} \quad [as\ required]$$

(ii) $t = ?$, when $x = \frac{3}{4}N$

$$\frac{\frac{3}{4}N}{N-\frac{3}{4}N} = 3^{t-2}$$

$$\frac{\frac{3}{4}N}{\frac{1}{4}N} = 3^{t-2}$$

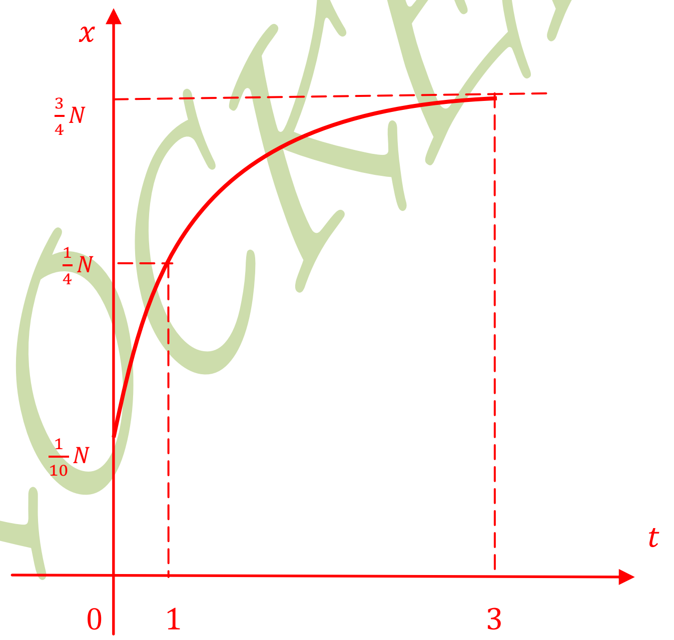
$3^1 = 3^{t-2}$

$\Longrightarrow t - 2 = 1$

$\therefore t = 3 weeks$

(iii)

Time (***weeks***)	**0**	**1**	**3**
# of population	$\frac{1}{10}N$	$\frac{1}{4}N$	$\frac{3}{4}N$



RELATED ODES PAST EXAMINATION QUESTIONS

UCLES NOVEMBER 1998 #16

a. Find the general solution of the differential equation

$$\frac{dx}{dt} = kx,$$

where k is a positive constant, expressing x in terms of k and t in your answer. **[4]**

b. Solve the differential equation

$$\frac{dx}{dt} = kx(a - x) \qquad (0 < x < a),$$

where k and a are positive constants, given that $x = \frac{1}{2}a$ when $t = 0$. Express x in terms of k and t in your answer. **[6]**

ZIMSEC JUNE 2013 #12

In any chemical reaction in which a compound P is formed from a compound Q, the masses of P and Q present at time t, are x and y respectively. The sum of the masses x and y is 10 and at any time the rate at which x is increasing is proportional to the product of the two masses at that time.

(i) Show that the situation can be modelled by the differential equation

$\frac{dx}{dt} = kx(10 - x)$ where k is a constant. **[2]**

(ii) Find the general solution of this differential equation. **[2]**

Hence find t correct to 3 significant figures when $x = 9.9$, given that $x = 2$, when $t = 0$ and $x = 5$ when $t = In2$. **[4]**

Answers: ii) $\frac{1}{10} In\left(\frac{x}{10-x}\right) = kt + A$

$t = 2.99$

ZIMSEC JUNE 2014 #9

A mathematics teacher is going on vacation leave before his students write their final examination. He has tried to keep it a secret, but the rumour is already spreading in the school, at the rate which is proportional to the product of the proportion x of those who have heard it and $(1-x)$, those who have not heard it.

The situation is modelled by the differential equation

$$\frac{dx}{dt} = kx(1-x)$$

If initially a proportion c of the population has heard the rumour, show that

$$x = \frac{c}{c+(1-c)e^{-kt}}$$

[8]

ZIMSEC JUNE 2017 #15

At any time t at a college, the number of people infected with a disease A is x and the number infected with disease B is y. The sum of the infected people with disease A and those infected with disease B is p. At any time t, the rate at which x is increasing is proportional to the product of x and y.

(a) Show that the situation can be modelled by the differential equation
$\frac{dx}{dt} = cx(p-x)$ where c is a constant. **[2]**

(b) Solve the differential equation expressing x in terms of $c,\ p$ and another constant k. **[6]**

(c) If $x = \frac{p}{10}$ at time $t = 0$, find, in terms of c and p, the time when $y = \frac{p}{10}$. **[5]**

Answers: b) $x = \frac{pe^{pk-pct}}{1+e^{pk-pct}}$

c) $t = \frac{1}{pct} In81$

ZIMSEC JUNE 2019 #12

The rate at which the quantity x is decreasing is proportional to the product x of those who and $(1-x)$.

(i) Form a differential equation which models this situation. **[1]**

(ii) Initially $x = 0.2$, and the rate of decrease is 1.6 unites per unit time. Solve the differential equation. **[8]**

Answers: i) $\frac{dx}{dt} = -kx(1-x)$

ii) $x = \frac{1}{e^{10t+In4}+1}$

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONTRIBUTIONS ARE WELCOME; FEEL FREE TO CONTACT ME SO THAT WE CAN IMPROVE THE DOCUMENT TOGETHER.

ENJOY

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