

8.3 Derivation of $(ax + b)^n$

$$\text{If } y = (ax + b)^n, \text{ then } \frac{dy}{dx} = n(ax + b)^{n-1} \times a.$$

Example 1

If $y = (3x + 5)^4$, find $\frac{dy}{dx}$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= 4(3x + 5)^{4-1} \times 3 \\ &= 12(3x + 5)^3\end{aligned}$$

Example 2

If $y = (4x^3 + 5x)^6$, find $\frac{dy}{dx}$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= 6(4x^3 + 5x)^5 \times (12x^2 + 5) \\ &= 6(12x^2 + 5)(4x^3 + 5x)^5\end{aligned}$$

Example 3

If $f(x) = \sqrt{2-3x}$, find $f'(x)$.

Solution

$$\begin{aligned}f(x) &= (2-3x)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(2-3x)^{\frac{1}{2}-1} \times -3 \\ &= -\frac{3}{2}(2-3x)^{-\frac{1}{2}} \\ &= \frac{-3}{\frac{1}{2(2-3x)^{\frac{1}{2}}}} \\ &= \frac{-3}{2\sqrt{2-3x}} \text{ (Ans)}\end{aligned}$$

Exercise 8.3

Differentiate the following with respect to x .

1 $(2x - 1)^6$ [Ans : $12(2x-1)^5$]

2 $(3x + 2)^8$ [Ans : $24(3x + 2)^7$]

3 $(2x^2 + 3)^5$ [Ans : $20x(2x^2 + 3)^4$]

4 $(x^3 + 2)^7$ [Ans : $21x^2(x^3 + 2)^6$]

5 $(1 - 4x)^{-3}$ [Ans : $\frac{12}{(1-4x)^4}$]

6 $(1 + x^2)^{-2}$ [Ans : $\frac{-4x}{(1+x^2)^3}$]

7 $\sqrt{2+3x}$ [Ans : $\frac{3}{2\sqrt{2+3x}}$]

8 $\sqrt{1-x^2}$ [Ans : $\frac{-x}{\sqrt{1-x^2}}$]

9 $\frac{1}{4x-5}$ [Ans : $\frac{-4}{(4x-5)^2}$]

10 $\frac{3}{5x+1}$ [Ans : $\frac{-15}{(5x+1)^2}$]

11 $\frac{2}{(3x^2+1)^6}$ [Ans : $\frac{-12x}{(3x^2+1)^7}$]

12 $\frac{1}{\sqrt{2x+1}}$ [Ans : $\frac{-1}{(2x+1)^{\frac{3}{2}}}$]

13 $\frac{1}{\sqrt{4x-3}}$ [Ans : $\frac{-2}{(4x-3)^{\frac{3}{2}}}$]

14 $\frac{1}{(2x+5)^2}$ [Ans : $\frac{-4}{(2x+5)^3}$]

8.4 Gradients

To find the gradient of a curve at a particular point, we have to differentiate the given function and replace the given point to obtain the gradient.

Example 1

Calculate the gradient of the tangent to the curve $y = 3x^2 - 5x + 2$ at the point $(-2, 3)$.

Solution

$$y = 3x^2 - 5x + 2$$

$$\frac{dy}{dx} = 6x - 5$$

$$\text{At } (-2, 3), x = -2, \quad \frac{dy}{dx} = 6(-2) - 5 = -17$$

Therefore the gradient at $(-2, 3)$ is -17 .

Example 2

Find the coordinates of the points on the curve $y = x^3 - \frac{7}{2}x^2 + x$ at which the gradient is -1 .

Solution

$$y = x^3 - \frac{7}{2}x^2 + x$$

$$\frac{dy}{dx} = 3x^2 - 7x + 1$$

Since the gradient $= -1$,

$$3x^2 - 7x + 1 = -1$$

$$3x^2 - 7x + 2 = 0$$

$$(3x - 1)(x - 2) = 0$$

Either $x = \frac{1}{3}$ or $x = 2$

$$\text{When } x = \frac{1}{3}, y = \left(\frac{1}{3}\right)^3 - \frac{7}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{3} = -\frac{1}{54}$$

$$\text{When } x = 2, y = (2)^3 - \frac{7}{2}(2)^2 + 2 = -4$$

Hence the required points are $\left(\frac{1}{3}, -\frac{1}{54}\right)$ and $(2, -4)$.

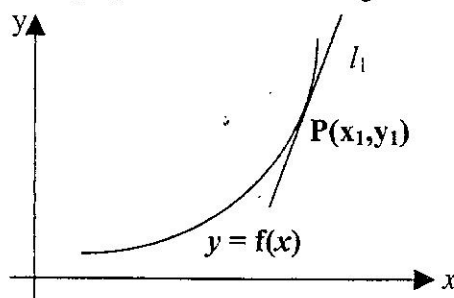
Exercise 8.4

Calculate the gradient of the tangent to the following curves at the given point.

- 1 $y = 3x^2 - 7x + 6$, $x = 2$ [Ans : 5]
- 2 $y = x^3 - 5x^2 + 2x - 5$, $x = -3$ [Ans : 59]
- 3 $y = x - \sqrt{x+1}$, $x = 3$ [Ans : $\frac{3}{4}$]
- 4 $y = \frac{x^2+1}{x}$, $x = \frac{1}{2}$ [Ans : -3]
- 5 $y = (2x+5)(x-2)$, $x = 1$ [Ans : 5]
- 6 $y = 2x^2 + 3x$, $y = 2$ [Ans : 5, -5]
- 7 $y = x^2 - 2x$, $y = -1$ [Ans : 0]
- 8 $y = \frac{x-2}{x}$, $y = 2$ [Ans: $\frac{1}{2}$]
- 9 $y = \frac{x^2+4}{x^2}$, $y = 5$ [Ans: -8, 8]
- 10 Find the coordinates of the point on the curve $y = x^3 - 3x^2 + 6x + 2$ at which the gradient is 3. [Ans: (1, 6)]
- 11 Find the coordinates of the point on the curve $y = 2x^3 - 4x^2 + x + 1$ at which the gradient is -1. [Ans: (1, 0), $(\frac{1}{3}, \frac{26}{27})$]
- 12 Given that the curve $y = ax^2 + \frac{b}{x}$ has gradient 4 at the point (1,5), calculate the value of a and b . [Ans : $a = 3$, $b = 2$]

8.5 Tangents

A line that touches a curve at a single point is called a tangent.



In the diagram the l_1 is a tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$.

We have seen in the previous section that its gradient, m , is given by

$$m = \text{value of } \frac{dy}{dx} \text{ at } (x_1, y_1).$$

Hence the equation of the tangent is given by $y - y_1 = m(x - x_1)$.

Example 1

Find the equation of the tangent to the curve $y = x^2 + 3x + 1$ at the point $(1, 2)$.

Solution

$$\text{Given } y = x^2 + 3x + 1$$

$$\frac{dy}{dx} = 2x + 3$$

$$\text{At } (1, 2), x = 1, \frac{dy}{dx} = 2(1) + 3 = 5$$

This is by definition the gradient of the tangent at $(1, 2)$.

So, we have to find the equation of a line whose gradient is 5 and which passes through $(1, 2)$.

$$\text{We have } y - 2 = 5(x - 1)$$

$$y - 2 = 5x - 5$$

$$y = 5x - 3$$

Therefore the equation of the tangent is $y = 5x - 3$.

Example 2

Find the equation of the tangent to the curve $y = x^3 - x$ at the point where $x = 1$.

Solution

$$\text{When } x = 1, y = (1)^3 - 1 = 0$$

$$\text{Now } \frac{dy}{dx} = 3x^2 - 1$$

$$\text{At } (1, 0), x = 1, \frac{dy}{dx} = 3(1)^2 - 1 = 2$$

So, we have to find the equation of a line whose gradient is 2 passing through $(1, 0)$.

$$\text{We have } y - 0 = 2(x - 1)$$

$$\therefore 0 = 2x - 2$$

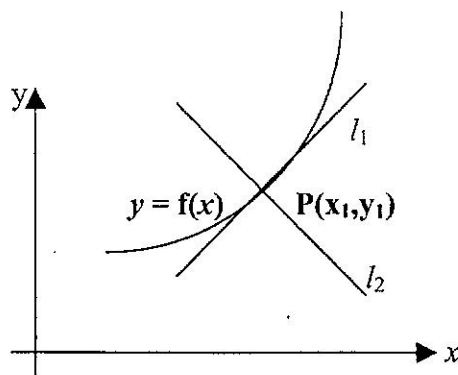
$$y = 2x - 2 \text{ (Ans)}$$

Exercise 8.5

Find the equation of tangent(s) at the given point.

- 1 $y = x^2 - 5x + 6$; at $(1, 2)$ [Ans : $y + 3x = 5$]
- 2 $y = 3x^2 - 7x + 2$; at $(-1, 3)$ [Ans : $y + 13x + 10 = 0$]
- 3 $y = x + \frac{2}{x}$; at $x = -1$ [Ans : $y + x + 4 = 0$]
- 4 Using differentiation find the equation of tangent at the point $(2, 1)$ on the curve $y = \sqrt{x^2 - 3}$. (C) [Ans : $y = 2x - 3$]
- 5 Find the equations of the tangents to the curve $y = x^3 - 11x$ which are parallel to the line $y = x$. (C) [Ans : $y = x - 16, y = x + 16$]
- 6 The tangent to the curve $y = 4x - \frac{8}{x}$ at the point $(2, 12)$ meets the x -axis at A and the y -axis at B. Find the coordinates of the midpoint of AB. (C) [Ans : $(-2, 4)$]
- 7 Find the equation of tangent to the curve $y = x^3 - 8x^2 + 15x$ at the point $(4, -4)$. Calculate the coordinates of the point where the tangent meets the curve again. [Ans: $y + x = 0$; $(0, 0)$]
- 8 Find the equation of tangent to the curve $y = 3x^2 - 2x + 5$ which is perpendicular to the line $4y + x = 2$. [Ans: $y = 4x + 2$]
- 9 The equation of a curve is $\frac{1}{x} + \frac{1}{y} = \frac{1}{a}$, where a is a constant.
Prove that $\frac{dy}{dx} = -\frac{y}{x^2}$, and find the equation of the tangent to the curve at the point $(2a, 2a)$. (C) [Ans : $2ay + x = 2a + 4a^2$]
- 10 The tangent to the curve $y = px^3$ at the point where $x = 2$ passes through the point $(1, -10)$. Find the value of p . (C) [Ans : $p = \frac{5}{2}$]

8.6 Normals



In the diagram the line l_1 is a tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$. The line l_2 is perpendicular to the tangent l_1 and is called the normal to the curve at $P(x_1, y_1)$.

Hence its gradient is $-\frac{1}{m}$ and its equation is given by $y - y_1 = -\frac{1}{m}(x - x_1)$.

Example 1

Find the equation of the normal to the curve $y = x^2 + 2x - 3$ at the point $(1, 3)$ giving your answer in the form $ay + bx = c$, where a , b and c are constants.

Solution

Given $y = x^2 + 2x - 3$

$$\frac{dy}{dx} = 2x + 2$$

$$\text{At } (1, 3), x = 1, \frac{dy}{dx} = 2(1) + 2 = 4$$

The gradient of the tangent at $(1, 3) = 4$

So the gradient of the normal at $(1, 3) = -\frac{1}{4}$

So, we have to find the equation of a line whose gradient is $-\frac{1}{4}$ and which passes through $(1, 3)$.

$$\text{We have } y - 3 = -\frac{1}{4}(x - 1)$$

$$4(y - 3) = -x + 1$$

$$4y - 12 = -x + 1$$

$$4y + x = 13$$

Therefore the equation of the line normal is $4y + x = 13$.

Exercise 8.6

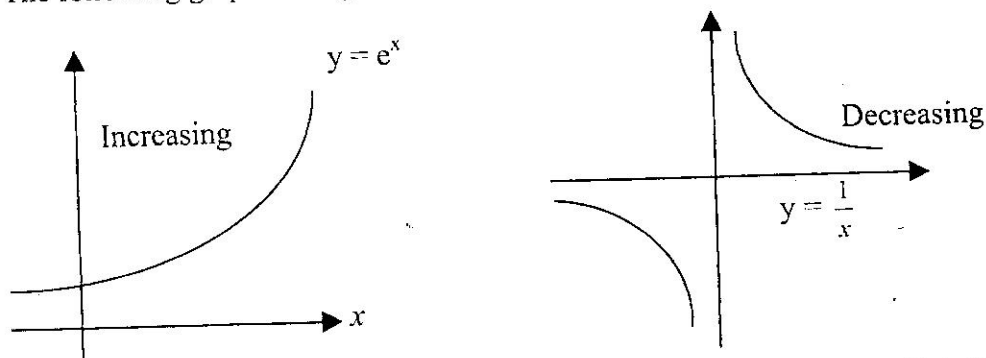
- 1 Find the equation of the normal to the curve $y = x^3 - x$ at the point (2,1).
[Ans : $11y + x = 3$]
- 2 Find the equation of the normal to the curve $y = x^3 + x + 1$ at the point where $x = 1$.
[Ans : $4y + x = 13$]
- 3 Find the equation of normal to the curve $y = -3x^3 + 5$ at the point (-1,8). (C)
[Ans : $9y = x + 73$]
- 4 Find the equation of the normal to the curve $y = 3x^2 - 4x + 5$ at the point where $x = 1$.
[Ans : $2y + x = 9$]
- 5 Find the equation of normal to the curve $y = -2x^5 + 3x$ at the point (-1,5).
[Ans : $7y = x + 36$]
- 6 Find the equation of the normal to the curve $y = 2x^2 - 7x + 3$ at the point where $x = 2$.
[Ans : $y + x + 1 = 0$]
- 7 Find the equation of normal to the curve $y = \frac{4}{\sqrt[3]{x}}$ at the point where $x = 8$.
[Ans : $y = 12x - 94$]
- 8 Find the equation of normal to the curve $y = \frac{2}{\sqrt{x}}$ at the point where $x = 4$.
[Ans : $y = 8x - 31$]
- 9 The normal to the curve $y = 2x + \frac{1}{x}$ at $x = 2$ cuts the x-axis at A. Find the coordinates of A.
[Ans : $(\frac{79}{8}, 0)$]
- 10 The normals to the curve $y = 2x + \frac{8}{x}$ at $x = 1$ and $x = 4$ intersect at the point P. Show that the coordinates of P is $(\frac{17}{5}, \frac{52}{5})$.

8.7 Increasing and Decreasing functions

A function $y = f(x)$ is said to be an increasing function at a particular point $P(x, y)$ if the value of $\frac{dy}{dx}$ or $f'(x)$ at P is positive.

Conversely, a function $y = f(x)$ is said to be a decreasing function at a particular point $Q(x, y)$ if the value of $\frac{dy}{dx}$ or $f'(x)$ at Q is negative.

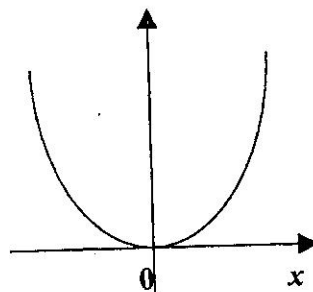
The following graphs will give a better idea of increasing and decreasing functions.



The graphs of $y = e^x$ has a positive gradient throughout. So, we say that $f(x) = e^x$ is an increasing function.

Similarly $f(x) = \frac{1}{x}$ has negative gradient throughout. Therefore, $f(x) = \frac{1}{x}$ is an example of a decreasing function.

However there are some functions which are increasing and decreasing at the same time depending on the domain that we choose. The diagram shows the graph of the function $f(x) = x^2$. It is an increasing function for $x > 0$ and a decreasing function for $x < 0$.



Example

A curve has equation $y = x + \frac{4}{x} - 2$. Find the value of $\frac{dy}{dx}$ at $x = 1$. Hence state whether the function is increasing or decreasing at $x = 1$.

Solution

$$\begin{aligned} y &= x + \frac{4}{x} - 2 \\ &= x + 4x^{-1} - 2 \\ \frac{dy}{dx} &= 1 - 4x^{-2} \\ &= 1 - \frac{4}{x^2} \end{aligned}$$

$$\text{When } x = 1, \frac{dy}{dx} = 1 - \frac{4}{(1)^2} = -3.$$

Since the value of $\frac{dy}{dx}$ at $x = 1$ is negative, the function is a decreasing function at $x = 1$.

Exercise 8.7

By making a sketch or otherwise, state which of the following functions are increasing, decreasing or both.

- 1 $f(x) \rightarrow e^{2x}$ [Ans : increasing]
- 2 $f(x) \rightarrow \ln 5x$ [Ans : decreasing]
- 3 $f(x) \rightarrow x^2 + 2$ [Ans : both]
- 4 $f(x) \rightarrow x^3$ [Ans : both]
- 5 $f(x) \rightarrow \ln x + 1$ [Ans : decreasing]
- 6 $f(x) \rightarrow 4 - x^2$ [Ans : both]
- 7 $f(x) \rightarrow \frac{1}{x^2}$ [Ans : both]
- 8 $f(x) \rightarrow e^{-3x}$ [Ans : decreasing]
- 9 $f(x) \rightarrow \frac{2}{x}$ [Ans : decreasing]
- 10 A curve has equation $y = 5x + \frac{2}{x} - 3$. Find the value of $\frac{dy}{dx}$ at $x = 2$. Hence state whether the function is increasing or decreasing at $x = 2$. [Ans : 4.5, increasing]
- 11 Given that $f(x) \rightarrow (3x^2 + 5x + 4)^5$, find $f'(-1)$ and state whether the function f is increasing or decreasing at $x = -1$. [Ans: -80 , decreasing]
- 12 A curve has equation $y = 2x^2 - \frac{5}{x^2}$. Find the value of $\frac{dy}{dx}$ at $x = 1$. Hence state whether the function is increasing or decreasing at $x = 1$. [Ans : 14, increasing]

8.8 Connected rates of change

Differentiation of the function of a function can be applied to deduce the rate of change of one variable as compared to another. This is illustrated in the examples below:

Example 1

The radius of a circular blot of ink is increasing at a constant rate of 0.2 cm/s. Find the rate at which the area is increasing when the radius is 10 cm.

Solution

Let r be the radius of the circular blot and A cm² be the area of the circular blot. Given that the rate of change of the radius is 0.2 cm/s.

$$\frac{dr}{dt} = 0.2$$

Also $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

Required: rate of change of area $\Rightarrow \frac{dA}{dt}$

But $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$
 $= 2\pi r \times 0.2 = 0.4\pi r$

When $r = 10$, $\frac{dA}{dt} = 0.4 \times \pi \times 10 = 2\pi$

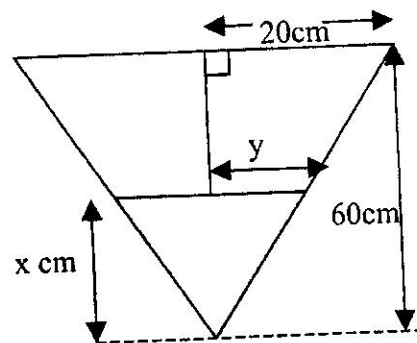
Therefore the rate of change of the area = 2π cm²/s.

Example 2

The diagram shows a vertical cross-section of a container in the form of an inverted cone of height 60 cm and base radius 20 cm. The circular base is held horizontal and uppermost. Water is poured into the container at a constant rate of 40 cm³/s.

- (i) Show that, when the depth of water in the container is x cm, the volume of water in the container is $\frac{\pi x^3}{27}$ cm³.

- (ii) Find the rate of increasing of x at the instant when $x = 2$. (C)



Solution

Let y be the radius of the water

Using similarity, $\frac{y}{20} = \frac{x}{60} \Rightarrow y = \frac{x}{3}$

Volume of water = $\frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{x}{3}\right)^2 x = \frac{\pi x^3}{27}$ (shown)

(ii) Now $V = \frac{\pi x^3}{27}$.

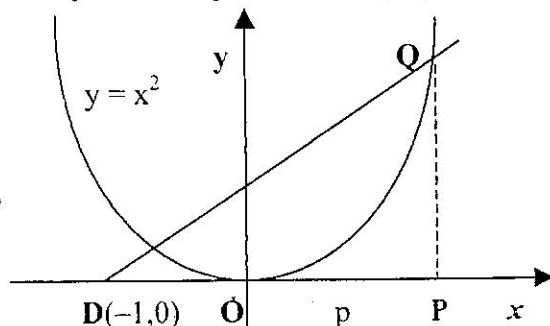
$$\frac{dV}{dx} = \frac{\pi x^2}{9}$$

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt} \Rightarrow \frac{dx}{dt} = \frac{9}{\pi x^2} \times 40$$

When $x = 2$, $\frac{dx}{dt} = \frac{9}{\pi(2)^2} \times 40 = \frac{90}{\pi}$ cm/s.

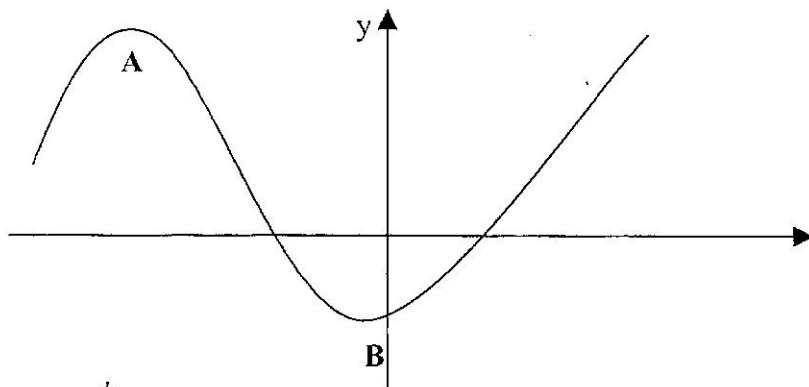
Exercise 8.8

- Given that $y = \frac{(3x-2)^{10}}{6}$, find the value of $\frac{dy}{dx}$ when $x = \frac{1}{3}$. The rate of increase of x , when $x = \frac{1}{3}$, is 2 units per second. Calculate the corresponding rate of change of y . (C) [Ans : -5, 10 units/sec]
- The radius of a circular disc is increasing at a constant rate of 0.003 cm/s. Find the rate at which the area is increasing when the radius is 20 cm. [Ans : 0.12π]
- A rectangular box of ice has dimensions x cm by x cm by $4x$ cm.
 - Express the total surface area A cm², in terms of x and write down an expression for $\frac{dA}{dx}$. [Ans : $A = 18x^2$, $\frac{dA}{dx} = 36x$]
 - Given that the ice is melting so that the total surface area, A cm², is decreasing at a constant rate of 3 cm²/s, calculate the rate of decrease of x when $x = 5$. [Ans : $1/60$ cm/s]
- A metal cube is being expanded by heat. At the instant when the length of an edge is 2 cm, the volume of the cube is increasing at the rate of 0.012 cm³ per second. At what rate is the length of the edge increasing at that instant? [Ans : 0.001]
- The figure shows the curve $y = x^2$ and the point $D(-1,0)$. $P(p,0)$ is a variable point on the x -axis and PQ is parallel to the y -axis. Express the area, A , of the triangle DPQ in terms of p . [Ans : $\frac{1}{2}(p^3 + p^2)$]
 If p is increasing at the rate of 5 units per second, calculate the rate at which A is increasing at the instant when $p = 4$ units. [Ans : 140]



8.9 Stationary points

Consider the graph of $y = f(x)$



Points A and B where $\frac{dy}{dx} = 0$ are called **stationary points**. To be more specific, point A is called a **maximum** point because $y = f(x)$ has a maximum value at A as compared to the neighbouring points around A. We can refer to the point A as a **local maximum**. However, it does not mean that the point A represents the maximum value of the whole curve.

Point B is called a **minimum** point because $y = f(x)$ has a minimum value at B as compared to the neighbouring points around B. We can refer to the point B as a **local minimum**. Again it does not mean that the point B represents the minimum value of the whole curve.

A **turning point** is a stationary point which is either a maximum or a minimum point.

To find the nature of a point, we have to find $\frac{d^2y}{dx^2}$.

For a maximum point $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

For a minimum point $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

However if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, then there is no turning point but a stationary point which is termed as a **point of inflexion**. The latter is not examinable, so we advise the reader not to identify points of inflexion on given curves.

Example 1

Find the differentiation, the coordinates of the turning points on the curve $y = x^3 - 2x^2 - 4x + 5$, stating the nature of each point. (C)

Solution

$$y = x^3 - 2x^2 - 4x + 5$$

$$\frac{dy}{dx} = 3x^2 - 4x - 4$$

For turning points, $\frac{dy}{dx} = 0$

$$3x^2 - 4x - 4 = 0$$

$$(3x + 2)(x - 2) = 0$$

Either $x = -\frac{2}{3}$ or $x = 2$

$$\text{When } x = -\frac{2}{3}, \quad y = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) + 5 = 6\frac{13}{27}$$

$$\text{When } x = 2, \quad y = 2^3 - 2(2)^2 - 4(2) + 5 = -3$$

So the turning points are $\left(-\frac{2}{3}, 6\frac{13}{27}\right)$ and $(2, -3)$.

$$\text{Now } \frac{d^2y}{dx^2} = 6x - 4$$

$$\text{At } \left(-\frac{2}{3}, 6\frac{13}{27}\right), x = -\frac{2}{3}, \quad \frac{d^2y}{dx^2} = 6\left(-\frac{2}{3}\right) - 4 = -8$$

Since $\frac{d^2y}{dx^2} < 0$, $\left(-\frac{2}{3}, 6\frac{13}{27}\right)$ is a maximum point.

$$\text{At } (2, -3), x = 2, \quad \frac{d^2y}{dx^2} = 6(2) - 4 = 8$$

Since $\frac{d^2y}{dx^2} > 0$, $(2, -3)$ is a minimum point.

Example 2

Find the differentiation, the coordinates of the turning points on the curve $y = 2x^3 + 3x^2 - 12x + 7$, stating the nature of each point. Also sketch the curve.

Solution

$$y = 2x^3 + 3x^2 - 12x + 7$$

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$

For turning points, $\frac{dy}{dx} = 0$

$$6x^2 + 6x - 12 = 0$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

Either $x = -2$ or $x = 1$

$$\text{When } x = -2, \quad y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 7 = 27$$

$$\text{When } x = 1, \quad y = 2(1)^3 + 3(1)^2 - 12(1) + 7 = 0$$

So the turning points are $(-2, 27)$ and $(1, 0)$.

$$\text{Now } \frac{d^2y}{dx^2} = 12x + 6$$

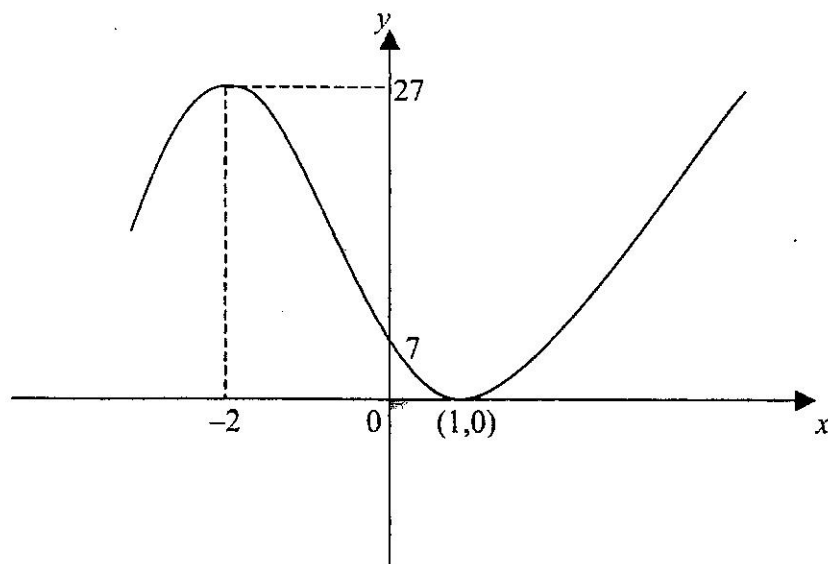
$$\text{At } (-2, 27), x = -2, \quad \frac{d^2y}{dx^2} = 12(-2) + 6 = -18$$

Since $\frac{d^2y}{dx^2} < 0$, $(-2, 27)$ is a maximum point.

$$\text{At } (1, 0), x = 1, \quad \frac{d^2y}{dx^2} = 12(1) + 6 = 18$$

Since $\frac{d^2y}{dx^2} > 0$, $(1,0)$ is a minimum point.

Also when $x = 0$, $y = 7$



Exercise 8.9

Find the stationary points of each of the following and determine the nature of the stationary points.

- | | | |
|---|------------------------------|---|
| | $y = x^2 - 2x - 3$ | [Ans : (1, -2) min ; (-1, 2) max] |
| 2 | $y = 2x^3 - 3x^2 - 12x + 18$ | [Ans : (-1, 25) max ; (2, -2) min] |
| 3 | $y = 8x + \frac{1}{2x^2}$ | [Ans : ($\frac{1}{2}$, 6) min] |
| 4 | $y = x(4 - x)^3$ | [Ans : (1, 27) max ; (4, 0) inflexion point] |
| 5 | $y = x^3 - 8x + 2$ | [Ans : (-1.63, 13.4) max ; (1.63, -6.7) min] |
| 6 | $y = 4x + \frac{9}{x}$ | [Ans : ($\frac{3}{2}$, 12) min ; ($-\frac{3}{2}$, -12) max] |
| 7 | $y = x^2 + \frac{16}{x}$ | [Ans : (2, 12) min] |

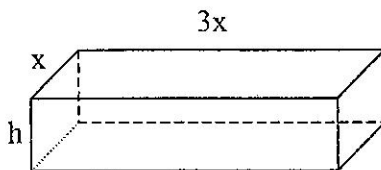
Find the coordinates of the stationary points on the following curves and hence sketch the graphs.

- 8 $y = 3x^3 - 2x^2 - 5x - 4$
- 9 $y = 2x^3 - 9x^2 + 12x$
- 10 $y = x^3 - 6x^2 + 9x + 1$
- 11 $y = x^3 + x^2 - 5x - 5$
- 12 $y = x(x - 2)^2$
- 13 $y = x(x + 3)^2$

8.10 Problems on maximum and minimum values

Example 1

The diagram shows a closed cardboard box in the shape of a cuboid with base $3x$ cm by x cm and height h cm. The total surface area of the cardboard (i.e. the outside of the box) is 1152 cm^2 .



- (i) By first expressing h in terms of x , show that the volume of the box, $V \text{ cm}^3$, is given by $V = 9x(48 - \frac{1}{4}x^2)$.
- (ii) Given that x can vary, find the maximum value of V , explaining why it is a maximum and not a minimum. (C)

Solution

(i) Total surface area $= 2(3xh + 3x^2 + xh) = 8xh + 6x^2$

Given that total surface area $= 1152 \text{ cm}^2$

$$8xh + 6x^2 = 1152$$

$$8xh = 1152 - 6x^2$$

$$h = \frac{1152 - 6x^2}{8x} = \frac{576 - 3x^2}{4x}$$

Now volume of box, $V = 3x \times x \times \frac{576 - 3x^2}{4x} = \frac{3}{4}x(576 - x^2)$

$$= \frac{9}{4}(192 - x^2)$$

$$= 9x(48 - \frac{1}{4}x^2) \text{ (shown)}$$

(ii) $V = 432x - \frac{9}{4}x^3 \Rightarrow \frac{dV}{dx} = 432 - \frac{27}{4}x^2$

At maximum value, $\frac{dV}{dx} = 0 \Rightarrow 432 - \frac{27}{4}x^2 = 0$

$$\Rightarrow x^2 = 64$$

$$\Rightarrow x = 8$$

$\therefore$ Maximum value of $V = 9(8) \left[48 - \frac{1}{4}(8)^2 \right] = 2304$

Now $\frac{d^2V}{dx^2} = -\frac{27}{2}x$

When $x = 8$, $\frac{d^2V}{dx^2} = -\frac{27}{2}(8) = -108 (< 0)$

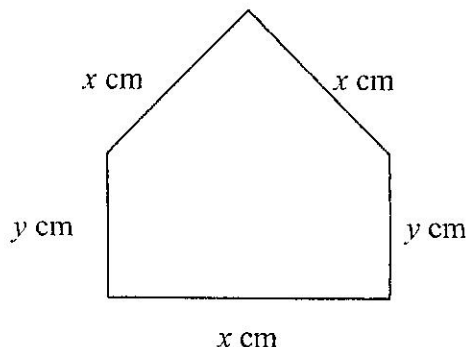
Since $\frac{d^2V}{dx^2} < 0$, we have a maximum value of V and not a minimum.

Exercise 8.10

- 1 A rectangular block has a base which measures $2x$ cm by $3x$ cm. Given that its volume is 1800 cm^3 , prove that the total surface area, $A \text{ cm}^2$, of the block is given by $A = 12x^2 + \frac{3000}{x}$.

Calculate the value of x for which A has a stationary value. Give a reason why it is a minimum value and not a maximum. (C) [Ans : $x = 5$]

- 2 A piece of wire, 40 cm long, is bent to form the shape shown in the diagram.

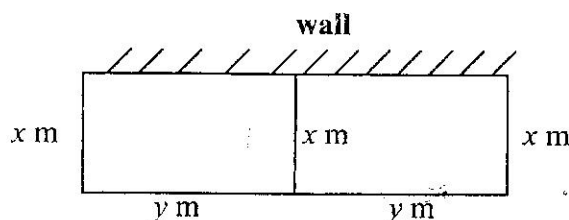


Express y in terms of x . Show that the area enclosed, $A \text{ cm}^2$, is given approximately by $A = 20x - 1.07x^2$.

Hence determine the value of x for which A is maximum.

[Ans: $y = 20 - \frac{3}{2}x$; 9.4]

- 3 A farmer has 120m of fencing to make two identical enclosures using an existing wall as one side of each enclosure. The dimensions of each enclosures are x cm and y cm as shown.



Obtain an expression, in terms of x only, for the total area of the two enclosures, and show that the maximum value of this area is 1200 m^2 .

[Ans: $y = \frac{1}{2}(120 - 3x)$]

- 4 An open tank with a square base is to be made from a thin sheet of metal. Find the length of the square base and the height of the tank so that the least amount of metal is used to make a tank with a capacity of 8 cm^3 . [Ans : $2\sqrt[3]{2}, \sqrt[3]{2}$]

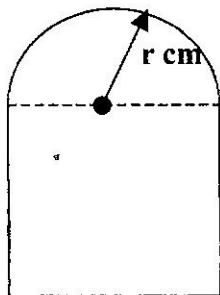
- 5 A cuboid has a total surface area of 150 cm^2 and is such that its base is a square of side $x \text{ cm}$. Show that the height, $h \text{ cm}$, of the cuboid is given by

$$h = \frac{75 - x^2}{2x}$$

Express the volume of the cuboid in terms of x . Hence determine, as x varies, its maximum volume and show that this volume is a maximum. (C)

$$\left[\text{Ans: } v = \frac{75}{2}x - \frac{1}{2}x^3, v_{\max} = 250 \right]$$

- 6 A piece of wire, 60 cm long, is bent to form the shape shown in the figure.



This shape consists of a semi-circular arc, radius $r \text{ cm}$, and three sides of a rectangle of height $x \text{ cm}$. Express x in terms of r and hence show that the area enclosed, $A \text{ cm}^2$, is given by

$$A = 60r - 2r^2 - \frac{\pi r^2}{2} \quad \left[\text{Ans: } x = 30 - r - \frac{1}{2}\pi r; r = \frac{60}{4 + \pi} \right]$$

Hence determine, to 3 significant figures, the value of r for which A is stationary. Determine whether this value of r makes A a maximum or a minimum. (C)

- 7 A piece of wire, 100 cm in length, is divided into two parts. One part is bent to form an equilateral triangle of side $x \text{ cm}$ and the other part is bent to form a square of side $y \text{ cm}$. Show that

$$y = \frac{100 - 3x}{4}$$

and that the area $A \text{ cm}^2$, enclosed by the two shapes, is such that

$$A = \frac{\sqrt{3}}{4}x^2 + \frac{(100 - 3x)^2}{16}$$

Calculate the value of x for which A has a stationary value. Determine whether this value of x makes A a maximum or a minimum. [Ans: $x = 19$]

CHAPTER 9

INTEGRATION

Integration as we all know is the reverse process of differentiation. The process of obtaining y from $\frac{dy}{dx}$ is known as integration. The expression for y involving a constant is called an **indefinite integral**.

9.1 Integral of ax^n of $\int ax^n dx$

If $\frac{dy}{dx} = ax^n$, then $y = \frac{ax^{n+1}}{n+1} + c$, $n \neq -1$ and a and c are constants.

The steps for obtaining the indefinite integral of ax^n are:

Step 1 : Add 1 to the power.

Step 2 : Divide by the new power.

Example

$$(a) \int 2x^4 dx \qquad (b) \int \frac{2}{x^3} + x^2 dx$$

Solution

$$(a) \int 2x^4 dx = \frac{2x^5}{5} + c$$

$$\begin{aligned}(b) \int \frac{2}{x^3} + x^2 dx &= \int 2x^{-3} + x^2 dx \\ &= \frac{2x^{-2}}{-2} + \frac{x^3}{3} + c \\ &= -\frac{1}{x^2} + \frac{x^3}{3} + c\end{aligned}$$

Exercise 9.1

Integrate the following with respect to x :

1 $3x^2 + 2x$ [Ans : $x^3 + x^2 + c$]

2 $8x^3 + 9x^2 - x$ [Ans : $2x^4 + 3x^3 - \frac{1}{2}x^2 + c$]

3 $\frac{4}{x^2}$ [Ans : $-\frac{4}{x} + c$]

4 $\sqrt{x}$ [Ans : $\frac{2}{3}x^{\frac{3}{2}} + c$]

5 $\frac{6x^5 - 4x^3}{x^3}$ [Ans : $x^3 - 2x + c$]

6 $(2x + 3)(3x - 2)$ [Ans : $2x^3 + \frac{5}{2}x^2 - 6x + c$]

7 $x^3 - \frac{2}{\sqrt{x}}$ [Ans : $\frac{x^4}{4} + 4x^{\frac{1}{2}} + c$]

8 $3x^2 + \sqrt{x}$ [Ans : $x^3 + \frac{2}{3}x^{\frac{3}{2}} + c$]

9 $\sqrt[3]{27x}$ [Ans : $\frac{3}{4}x^{\frac{4}{3}} + c$]

10 $\sqrt{x}(\sqrt{x} + 3)$ [Ans : $\frac{x^2}{2} + 2x^{\frac{3}{2}} + c$]

11 $x^2\left(3 + \frac{5}{x^4}\right)$ [Ans : $x^3 - \frac{5}{x} + c$]

12 $\sqrt{x}\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right)$ [Ans : $\frac{x^2}{2} + 2x + c$]

13 $x^2\left(x^2 + \frac{3}{x^6}\right)$ [Ans : $\frac{x^4}{4} - \frac{3}{2x^2} + c$]

9.2 Integral of the type $(ax + b)^n$

$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{(n+1) \times a} + c$, where $n \neq -1$ and a , b and c are constants.

Example

(a) $\int (2x + 3)^7 dx$

(b) $\int (1 - 3x)^5 dx$

(c) $\int \frac{1}{\sqrt{4+3x}} dx$

(d) $\int \sqrt[3]{2x+5} dx$

Solution

$$\begin{aligned} \text{(a)} \quad \int (2x + 3)^7 dx &= \frac{(2x+3)^8}{8 \times 2} + c \\ &= \frac{(2x+3)^8}{16} + c \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int (1 - 3x)^5 dx &= \frac{(1-3x)^6}{6 \times -3} + c \\ &= \frac{(1-3x)^6}{-18} + c \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int \frac{1}{\sqrt{4+3x}} dx &= \int (4+3x)^{-\frac{1}{2}} dx \\ &= \frac{(4+3x)^{\frac{1}{2}}}{\frac{1}{2} \times 3} + c \\ &= \frac{2}{3} (4+3x)^{\frac{1}{2}} + c \\ &= \frac{2}{3} \sqrt{4+3x} + c \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int \sqrt[3]{2x+5} dx &= \int (2x+5)^{\frac{1}{3}} dx \\ &= \frac{(2x+5)^{\frac{4}{3}}}{\frac{4}{3} \times 2} + c \\ &= \frac{3}{8} (2x+5)^{\frac{4}{3}} + c \end{aligned}$$

Exercise 9.2

Integrate the following with respect to x .

1 $(2x + 1)^5$ [Ans : $\frac{(2x+1)^6}{12} + c$]

2 $(3x + 2)^7$ [Ans : $\frac{(3x+2)^8}{24} + c$]

3 $(5x + 3)^4$ [Ans : $\frac{(5x+3)^5}{25} + c$]

4 $(2-3x)^5$ [Ans : $\frac{(2-3x)^6}{-18} + c$]

5 $(1-4x)^{-3}$ [Ans : $\frac{1}{8(1-4x)^2} + c$]

6 $(1+x)^{-2}$ [Ans : $\frac{-1}{1+x} + c$]

7 $\sqrt{2+3x}$ [Ans : $\frac{2}{9}(2+3x)^{\frac{3}{2}} + c$]

8 $\sqrt{1-2x}$ [Ans : $-\frac{1}{3}(1-2x)^{\frac{3}{2}} + c$]

9 $\frac{1}{(4x-5)^3}$ [Ans : $\frac{-1}{8(4x-5)^2} + c$]

10 $\frac{1}{(2x+5)^4}$ [Ans : $\frac{-1}{6(2x+5)^3} + c$]

11 $\frac{2}{(3x-4)^6}$ [Ans : $\frac{2}{15(3x-4)^5}$]

12 $\frac{1}{\sqrt{2x+1}}$ [Ans : $\sqrt{2x+1} + c$]

9.3 Problems involving the evaluation of a constant of integration

In this case we are going to deal with equations of the type $\frac{dy}{dx} = f(x)$. To find y in terms of x , we have to integrate both sides. By substituting the value of x and y , we can find the value of the constant of integration, c .

Example 1

The curve for which $\frac{dy}{dx} = 2x(x-3)$ passes through the point $(3,6)$. Find the equation of the curve.

Solution

$$\frac{dy}{dx} = 2x(x-3)$$

integrating both sides, we have

$$y = \int 2x^2 - 6x dx$$

$$y = \frac{2x^3}{3} - 3x^2 + c$$

Since the curve passes through $(3,6) \Rightarrow x=3, y=6$

$$6 = \frac{2(3)^3}{3} - 3(3)^2 + c$$

$$6 = -9 + c$$

$$c = 15$$

Hence the equation of the curve is $y = \frac{2x^3}{3} - 3x^2 + 15$

Example 2

A curve is such that $\frac{dy}{dx} = ax - 3$, where a is a constant. Given that the gradient of the normal at the point $(2, -1)$ on the curve is $-\frac{1}{5}$, find

(a) the value of a ,

(b) the equation of the curve. (C)

Solution

(a) gradient of normal $= -\frac{1}{5} \Rightarrow$ gradient of tangent $= 5$

$$\frac{dy}{dx} = ax - 3 = 5$$

$$a(2) - 3 = 5$$

$$2a = 8$$

$$\Rightarrow a = 4$$

(b) $\frac{dy}{dx} = 4x - 3$,

integrating both sides, we have

$$y = \int 4x - 3 dx$$

$$y = 2x^2 - 3x + c$$

Since the curve passes through $(2, -1) \Rightarrow x = 2, y = -1$

$$-1 = 2(2)^2 - 3(2) + c$$

$$c = -3$$

Hence the equation of the curve is $y = 2x^2 - 3x - 3$

Exercise 9.3

- 1 A curve is such that $\frac{dy}{dx} = (3x - 2)^2$. Given that the curve passes through the point $(1, 2)$, find its equation. (C) [Ans : $y = \frac{(3x - 2)^3}{9} + \frac{17}{9}$]

- 2 A curve is such that $\frac{dy}{dx} = \frac{12}{(2x + 1)^2}$ and $P(1, 5)$ is a point on the curve.

(i) the normal to the curve at P crosses the x -axis at Q . Find the coordinates of Q .

(ii) Find the equation of the curve. (C)

$$[\text{Ans : (i) } \left(\frac{23}{3}, 0\right); \text{ (ii) } y = -\frac{6}{2x + 1} + 7]$$

- 3 The curve for which $\frac{dy}{dx} = \sqrt{1 + 2x}$, passes through the point $(4, 30)$. Find the equation of the curve. [Ans : $y = \frac{1}{3}(1 + 2x)^{\frac{3}{2}} + 21$]

- 4 A curve for which $\frac{dy}{dx} = 2x - 5$, passes through the point $P(4, -2)$.

(i) the equation of the normal at P ,

(ii) the equation of the curve. [Ans : (i) $x + 3y + 2 = 0$; (ii) $y = x^2 - 5x + 2$]

- 5 Given that $\frac{dy}{dx} = \frac{a}{x^2} + 1$, and that when $x = 1$, $\frac{dy}{dx} = 3$ and $y = 3$. Find the value of y when $x = 2$. (C) [Ans : 5]

- 6 In an experiment with a cube and a stop-watch it is found that when the stop watch registers t minutes the side of the cube is increasing at the rate of $0.3 t^2$ cm/min. Given that the side of the cube is 6 cm when $t = 0$, calculate the length of the side when $t = 2$. (C) [Ans : 6.8]

9.4 Evaluating definite integrals

Consider the expression $\int_a^b f(x) dx$. The expression is called the definite integral of $f(x)$ with respect to x from a to b . Hence, a is the lower limit of the integral and b the upper limit.

To evaluate the expression $\int_a^b f(x) dx$, we have to find the value of $\int f(x) dx$ when $x = b$ and minus the value of $\int f(x) dx$ when $x = a$.

Example

Evaluate (a) $\int_0^1 \frac{1}{\sqrt{x}} dx$ (b) $\int_1^{\infty} \frac{1}{x^2} dx$ (c) $\int_0^4 \sqrt{2x+1} dx$

Solution

$$\begin{aligned} \text{(a)} \quad \int_0^1 \frac{1}{\sqrt{x}} dx &= \int_0^1 x^{-\frac{1}{2}} dx \\ &= \left[2x^{\frac{1}{2}} \right]_0^1 = \left[2(1)^{\frac{1}{2}} - 2(0)^{\frac{1}{2}} \right] = 2 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_1^{\infty} \frac{1}{x^2} dx &= \int_1^{\infty} x^{-2} dx \\ &= \left[\frac{x^{-1}}{-1} \right]_1^{\infty} \\ &= \left[\frac{-1}{x} \right]_1^{\infty} \\ &= \left[\left(\frac{-1}{\infty} \right) - \left(\frac{-1}{1} \right) \right] = 0 + 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_0^4 \sqrt{2x+1} dx &= \int_0^4 (2x+1)^{\frac{1}{2}} dx \\ &= \left[\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2} \times 2} \right]_0^4 = \left[\frac{(2x+1)^{\frac{3}{2}}}{3} \right]_0^4 \\ &= \left[\frac{(2(4)+1)^{\frac{3}{2}}}{3} \right] - \left[\frac{(2(0)+1)^{\frac{3}{2}}}{3} \right] \\ &= \frac{26}{3} \end{aligned}$$

Exercise 9.4

Evaluate the following definite integrals:

1 $\int_0^2 x^2 dx$ [Ans : $\frac{7}{3}$]

2 $\int_2^4 \frac{2}{x^4} dx$ [Ans : $\frac{19}{324}$]

3 $\int_0^1 x^2(2x+1)dx$ [Ans : $\frac{5}{6}$]

4 $\int_0^2 (2x+3)(3x-1)dx$ [Ans : 24]

5 $\int_1^4 \frac{x^3 + 3x^2}{x} dx$ [Ans : $\frac{87}{2}$]

6 $\int_1^3 \frac{x^3 + 1}{x^2} dx$ [Ans : 2]

7 $\int_1^2 3 - 2x^2 + \frac{4}{x^3} dx$ [Ans : $-\frac{1}{6}$]

8 $\int_1^4 \frac{2 + \sqrt{x}}{x^2} dx$ [Ans : $\frac{5}{2}$]

9 $\int_0^1 (3x-2)^5 dx$ [Ans : $\frac{1}{18}$]

10 $\int_0^3 \sqrt{x+1} dx$ [Ans : $\frac{14}{3}$]

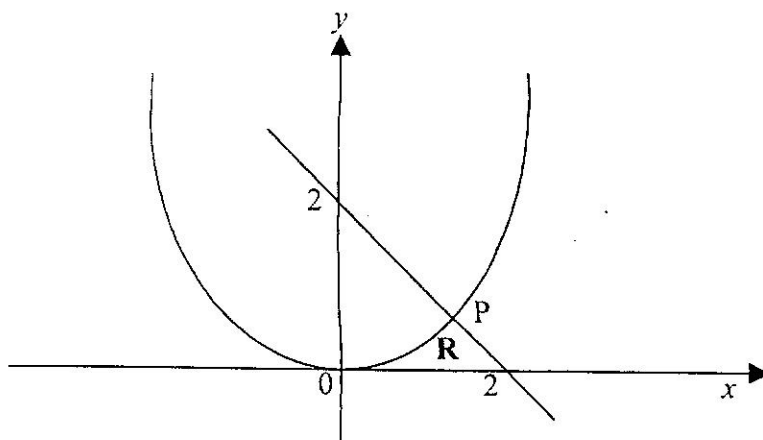
11 $\int_1^4 \sqrt{x} - \frac{1}{\sqrt{x}} dx$ [Ans : $\frac{8}{3}$]

9.5 Plane Areas by integration

Example 1

A region R is found in the first quadrant and bounded by the curve $y = x^2$, the x-axis and the line $y = 2 - x$. Show this region clearly on a sketch and hence find the area of R.

Solution



To find the coordinates of the point of intersection P, we solve the equations

$$y = x^2 \text{ and } y = 2 - x \text{ simultaneously}$$

$$x^2 = 2 - x$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

Either $x = -2$ or $x = 1$

Since R is found in the first quadrant, $x = 1$

$$y = 1^2 = 1$$

Therefore P has coordinates (1,1)

To find the area of R, we integrate the curve $y = x^2$ from 0 to 1 and then we find the area of the triangle from $x = 1$ to $x = 2$.

$$\text{Now area of the region R} = \int_0^1 x^2 \, dx + \text{Area of } \Delta$$

$$= \left[\frac{x^3}{3} \right]_0^1 + \frac{1}{2} \times 1 \times 1$$

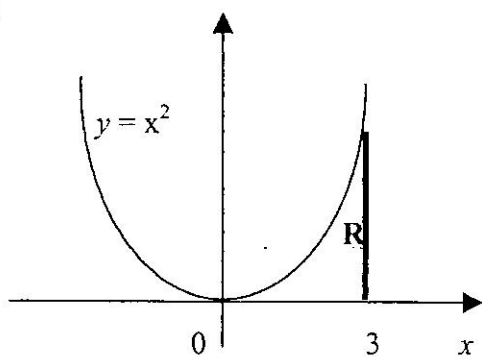
$$= \frac{1}{3} + \frac{1}{2}$$

$$= \frac{5}{6}$$

Exercise 9.5

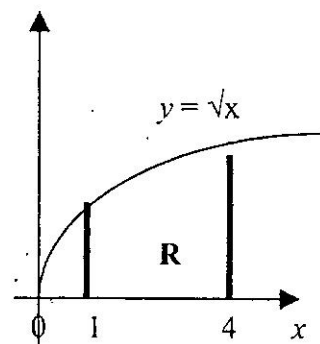
Find the area represented by the letter R

1



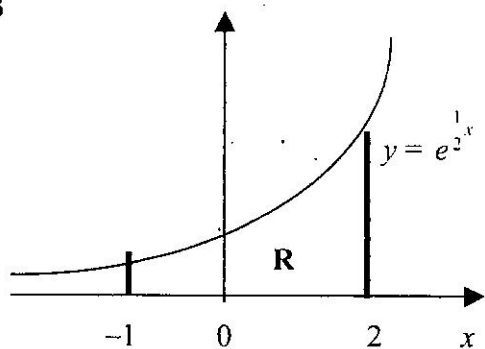
[Ans : 9]

2



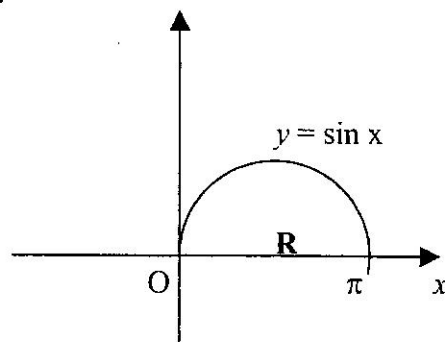
[Ans : $14/3$]

3



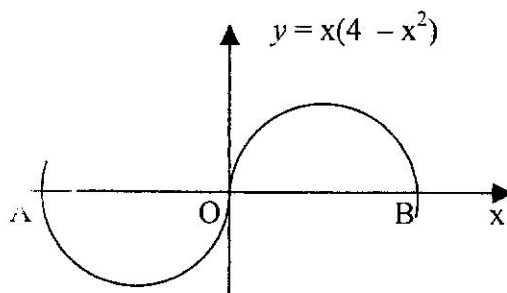
[Ans : 4.22]

4



[Ans: 2]

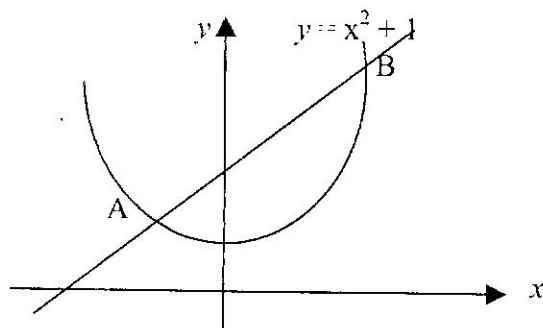
- 5 The diagram shows the graph of $y = x(4 - x^2)$. State the coordinates of A and B. Hence find the area enclosed by the curve and the x-axis.



[Ans : 8]

- 6 The region R is bounded by the curve $y = x^2$ and $y = \sqrt{x}$. Show the region R clearly on a sketch and hence find the area of R. [Ans : $1/3$]

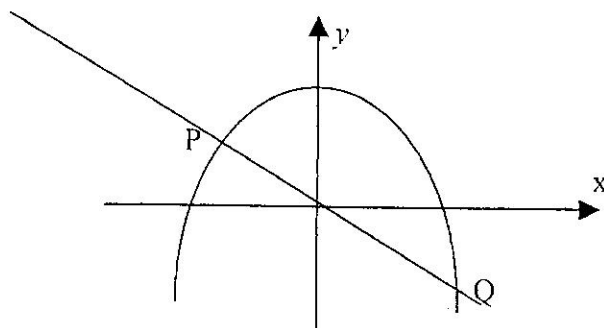
- 7 The diagram shows the graph of $y = x^2 + 1$ and the line $y = x + 3$. Find
 (a) the coordinates of A and B.
 (b) the area enclosed by the curve and the line.
 [Ans: (a) A(-1,2) B(2,5) ; (b) 4.5]



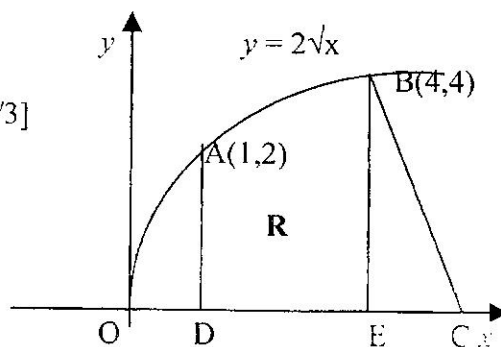
- 8 Sketch the curve $y = x^3 - 4x$. Hence find the area bounded by the curve and the x-axis.
 [Ans : 8]

- 9 The region R is bounded by the curve $y = x^2$ and $y = x + 2$. Show the region R clearly on a sketch and hence find the area of R.
 [Ans : 4.5]

- 10 The diagram shows the graph of $y = 2 - x^2$ and the line $y = -x$. Find
 (a) the coordinates of P and Q.
 (b) the area enclosed by the curve and the line.
 [Ans: (a) A(-1,1) B(2,-2) ; (b) 4.5]

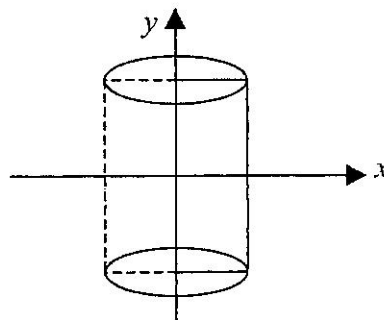
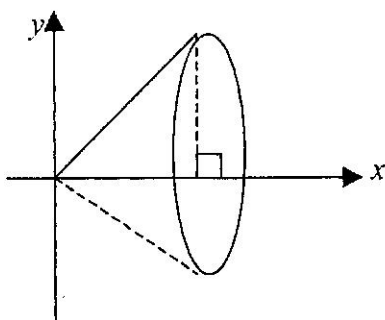


- 11 The diagram shows the points A(1,2) and B(4,4) on the curve $y = 2\sqrt{x}$. The line BC is the normal to the curve at B, and C lies on the x-axis. Lines AD and BE are perpendicular to the x-axis. Find
 (a) the equation of the normal BC,
 (b) the area of R.
 [Ans: (a) $y = -2x + 12$; (b) $28/3$]



9.6 Volume of Revolution

When an area bounded by a curve and the x -axis or the y -axis is rotated through 360° about that axis, a solid of revolution is formed. For example, when a right angle triangle is rotated through 360° about the x -axis, a cone will be generated as shown below. Similarly, a rectangle lying vertically, when rotated through 360° about the y -axis, will generate a cylinder as shown below.



To find the volume of revolution about the x -axis, we use the following formula

$$V = \pi \int_a^b y^2 dx$$

where a and b are the lower and upper limits respectively.

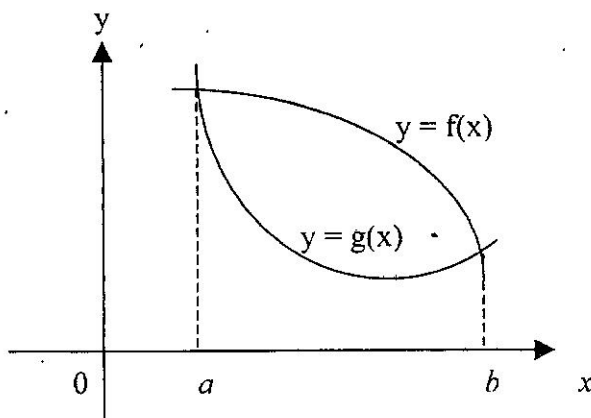
Again, to find the volume of revolution about the y -axis, we use the following formula

$$V = \pi \int_a^b x^2 dy$$

where a and b are the lower and upper limits respectively.

When the area bounded by any two curves $y = f(x)$ and $y = g(x)$ [see figure 1] is rotated through 360° about the x -axis, and if their points of intersection occur at $x = a$ and $x = b$,

the volume of solid formed is given by
$$V = \pi \int_a^b \{f(x)\}^2 - \{g(x)\}^2 dx.$$



Example 1

The region R is bounded by part of the curve $y = (x-2)^{\frac{3}{2}}$ for which $2 \leq x \leq 4$, the x -axis, and the line $x = 4$. Find, in terms of π , the volume of solid obtained when R is rotated through four right angles about the x -axis. (C)

Solution

First we make a sketch

Now using $V = \pi \int_a^b y^2 dx$, we have

$$\text{But } y^2 = (x-2)^3$$

$$\text{Thus } V = \pi \int_2^4 (x-2)^3 dx$$

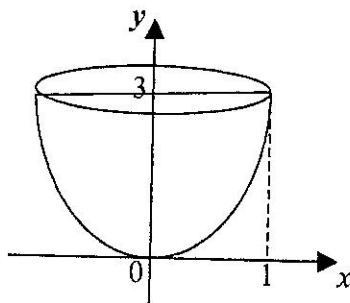
$$= \pi \left[\frac{(x-2)^4}{4} \right]_2^4$$

$$= \pi \left[\frac{(4-2)^4}{4} \right] - \pi \left[\frac{(2-2)^4}{4} \right] = 4\pi \text{ (Ans)}$$

Example 2

A cup has the shape made by rotating the graph of $y = 3x^2$, for $0 \leq x \leq 1$, through four right angles about the y -axis. Find the volume of the cup, giving your answer in terms of π . (C)

Solution



$$\text{When } x = 1, y = 3(1)^2 = 3$$

$$\text{Using } V = \pi \int_a^b x^2 dy$$

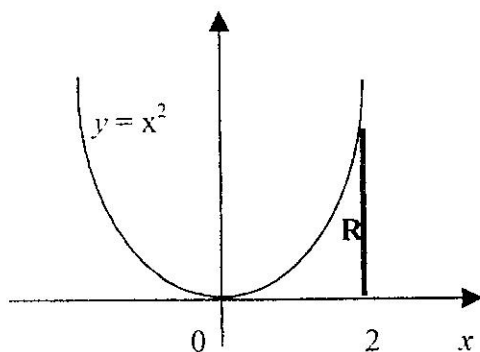
$$\text{But } x^2 = \frac{y}{3}$$

$$\begin{aligned} V &= \pi \int_0^3 \frac{y}{3} dy = \pi \left[\frac{y^2}{6} \right]_0^3 \\ &= \pi \left[\frac{3^2}{6} \right] - \pi \left[\frac{0^2}{6} \right] \\ &= \frac{3}{2} \pi \text{ (Ans)} \end{aligned}$$

Exercise 9.6

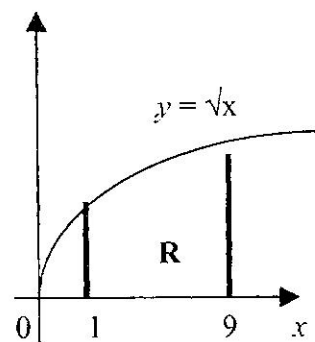
Find the volume of revolution obtained when the region R is rotated about the x-axis.

1



[Ans : $32/5\pi$]

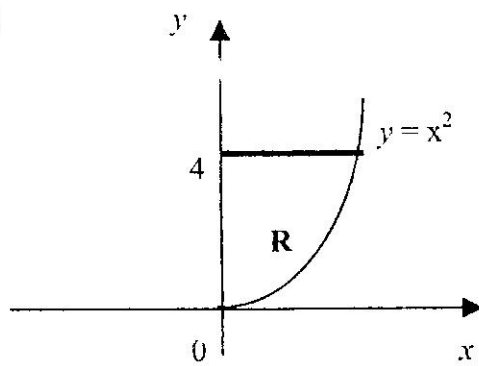
2



[Ans : 40π]

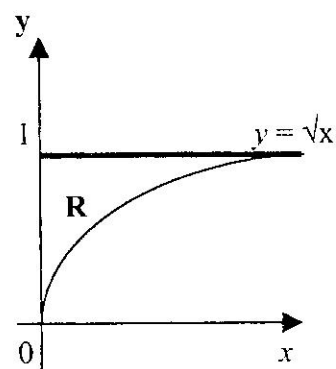
Find the volume of revolution obtained when the region R is rotated about the y-axis.

3



[Ans : 8π]

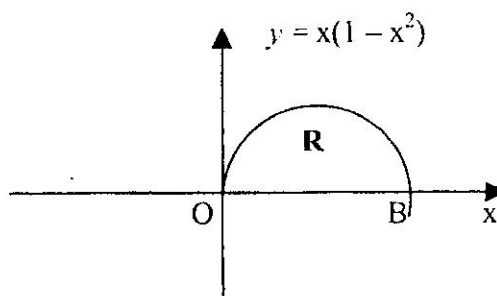
4



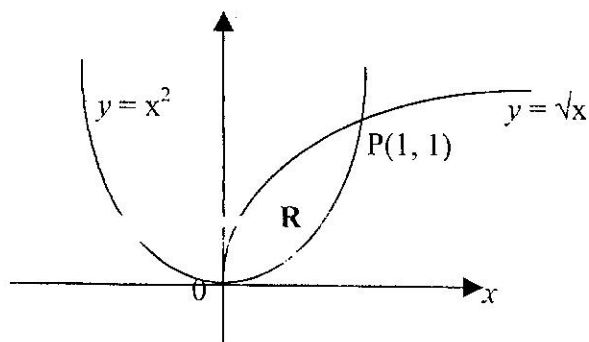
[Ans : $1/5\pi$]

- 5 The diagram shows part of the graph of $y = x(1 - x^2)$. State the coordinates of B. Hence find the volume of revolution obtained when the region R is rotated through four right angles about the x-axis.

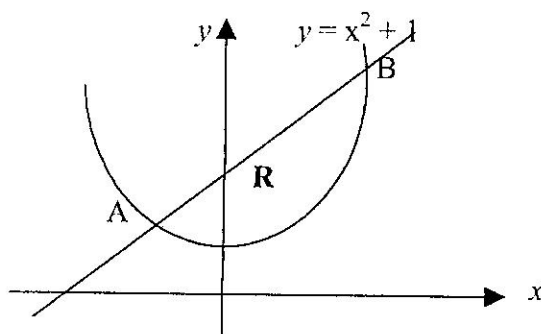
[Ans : B(1,0) ; $824/35 \pi$]



- 6 The diagram shows the graph of $y = x^2$ and $y = \sqrt{x}$ together with their point of intersection $P(1, 1)$. Show that the volume of revolution obtained when **R** is rotated through four right angles about the y -axis is $\frac{3}{10}\pi$.

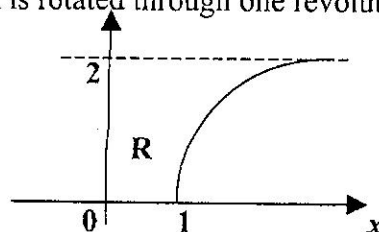


- 7 The diagram shows the graph of $y = x^2 + 1$ and the line $y = 3x + 5$. Find
 (a) the coordinates of A and B.
 (b) the volume of revolution obtained when **R** is rotated about the x -axis.
 [Ans: (a) $A(-1, 2)$ $B(4, 17)$; (b) $1412/5 \pi$]



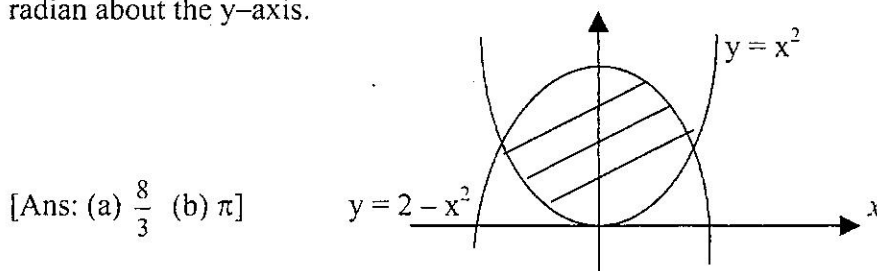
- 8 The area bounded by the curve $y = x^2 + 2$ and the line $y = x + 8$ is rotated through an angle of 2π radians about the x -axis. Find the volume of solid generated.
 [Ans : 250π]
- 9 The region **R** is bounded by the curve $y = x^2$ and $y = x$. Show the region **R** clearly on a sketch and hence find the volume of revolution obtained when **R** is rotated about the x -axis.
 [Ans : $2/15 \pi$]

- 10 The diagram shows part of the graph of $y^2 = x - 1$. The region in the first quadrant bounded by the axes, the curve, and the line $y = 2$ is denoted by **R**. Find the volume of solid generated when **R** is rotated through one revolution,
 (a) about the y -axis
 (b) about the x -axis
 [Ans : (a) $206/15\pi$ (b) 12π]



AREA AND VOLUME – PAST EXAMINATION QUESTIONS

- 1 The region enclosed by the x -axis and the part of the curve $y = x(2-x)$ between $x = 0$ and $x = 2$ is rotated about the x -axis to form a solid of revolution. Find the volume of this solid. [Ans : $\frac{16}{15}\pi$]
- 2 Find the volume formed when the region in the first quadrant bounded by $y = x$, $y = \frac{1}{x}$, $x = 2$ and $y = 0$ is rotated through 360° about the x -axis. [Ans : $\frac{5}{6}\pi$]
- 3 Prove that the area of the region bounded by the coordinate axes, the line $x = 2$, and the curve $y = \frac{1}{(3-x)(1+x)}$ is $\frac{1}{2} \ln 3$.
- 4 The finite region A is bounded by the curve $y = x^2$ and a minor arc of the circle $x^2 + y^2 = 12$.
 - (a) Find the area of A.
 - (b) Prove that the volume formed when A is rotated through an angle of π about the y -axis is $\pi(16\sqrt{3} - \frac{45}{2})$. [Ans: (a) $2\pi + \sqrt{3}$]
- 5 The region R in the first quadrant is bounded by the y -axis, the x -axis, the line $x = 3$ and the curve $y^2 = 4 - x$.
 - (a) Draw a sketch showing the region R and calculate its area.
 - (b) Calculate the volume formed when R is rotated about the y -axis through one revolution. [Ans: (a) $\frac{14}{3}$ (b) $\frac{188}{15}\pi$]
- 6 The diagram shows the graphs of $y = x^2$ and $y = 2 - x^2$. Calculate
 - (a) the area of the shaded region.
 - (b) the volume of the solid generated when the shaded region is rotated through π radian about the y -axis.



- 7 The finite region bounded by the x -axis, the curve $y = e^{-x}$, and the lines $x = \pm a$ is denoted by R. Find, in terms of a , the volume of the solid generated when R is rotated through one revolution about the x -axis. [Ans: $\frac{1}{2}\pi(e^{2a} - e^{-2a})$]

CHAPTER 10

ALGEBRA

10.1 Modulus functions

The function $f: x \rightarrow |x|$, where x is real, is the absolute value function.

We state the following

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In general if $|a| = |b|$, then $a^2 = b^2$

Example 1

Solve $|x| = |x+3|$ (C)

Solution

$$\begin{aligned} \text{We have } x^2 &= (x+3)^2 \\ x^2 &= x^2 + 6x + 9 \\ 6x + 9 &= 0 \\ x &= -\frac{3}{2} \end{aligned}$$

Example 2

Solve $|3x+1| = 2|x-2|$

Solution

$$\begin{aligned} \text{We have } (3x+1)^2 &= 2^2 (x-2)^2 \\ 9x^2 + 6x + 1 &= 4(x^2 - 4x + 4) \\ 9x^2 + 6x + 1 &= 4x^2 - 16x + 16 \\ 5x^2 + 22x - 15 &= 0 \\ (5x-3)(x+5) &= 0 \end{aligned}$$

Either $x = \frac{3}{5}$ or $x = -5$

Exercise 10.1

Solve the following

1 $|x| = |x + 1|$ [Ans: $-\frac{1}{2}$]

2 $|x| = |x - 3|$ [Ans: $\frac{3}{2}$]

3 $|2x + 3| = |x - 1|$ [Ans: $-4, -\frac{2}{3}$]

4 $|3x + 4| = |x - 2|$ [Ans: $-3, -\frac{1}{2}$]

5 $|2x - 5| = |x + 1|$ [Ans: $6, \frac{4}{3}$]

6 $|3x - 2| = 2|x - 1|$ [Ans: $0, \frac{4}{5}$]

7 $|4x - 3| = 2|x + 2|$ [Ans: $-\frac{1}{6}, \frac{7}{2}$]

8 Solve the equation $|x + a| = |x|$, giving your answer in terms of a .

[Ans: $-\frac{a}{2}$]

9 Solve the equation $|x + 2a| = |x - a|$, giving your answer in terms of a .

[Ans: $-\frac{a}{2}$]

10 Solve the equation $|2x + a| = |x - 2a|$, giving your answer in terms of a .

[Ans: $-3a, \frac{a}{3}$]

11 Solve the equation $|3x + 2a| = |x - 4a|$, giving your answer in terms of a .

[Ans: $-3a, \frac{a}{2}$]

12 $|3x - 5| = 3|x + 1|$ [Ans: $\frac{1}{3}$]

10.2.1 Modulus inequalities (I)

If $|x| < a$ then $-a < x < a$

For example if $|x| < 3$ then $-3 < x < 3$

Example 1

Solve $|2x - 5| < 1$

Solution

$$|2x - 5| < 1$$

$$\Rightarrow -1 < 2x - 5 < 1$$

Adding 5, we have

$$4 < 2x < 6$$

Dividing by 2, we obtain

$$2 < x < 3 \text{ (Ans)}$$

Example 2

Solve $|3 - 4x| < 5$

Solution

$$|3 - 4x| < 5$$

$$\Rightarrow -5 < 3 - 4x < 5$$

Subtracting 3, we have

$$-8 < -4x < 2$$

Dividing by -4 , the inequality sign changes and we obtain

$$2 > x > -\frac{1}{2} \quad \text{or} \quad -\frac{1}{2} < x < 2 \text{ (Ans)}$$

Exercise 10.2.1

Solve the following inequalities:

1 $|x - 3| < 1$ [Ans : $2 < x < 4$] 6 $|4 - 3x| < 2$ [Ans : $\frac{2}{3} < x < 2$]

2 $|x + 5| < 2$ [Ans : $-7 < x < -3$] 7 $|1 - 2x| < 3$ [Ans : $-1 < x < 2$]

3 $|4x - 5| < 3$ [Ans : $\frac{1}{2} < x < 2$] 8 $|3x + 4| < 2$ [Ans : $-2 < x < -\frac{2}{3}$]

4 $|3 - 2x| < 1$ [Ans : $1 < x < 2$] 9 $|1 - x| < 5$ [Ans : $-4 < x < 6$]

5 $|x - 2| < \frac{1}{2}$ [Ans : $\frac{3}{2} < x < \frac{5}{2}$] 10 $|2x - 1| < 0.2$ [Ans : $0.4 < x < 0.6$]

10.2.2 Modulus inequalities (II)

If $|x| > a$ then $x < -a \cup x > a$

For example if $|x| > 2$ then $x < -2 \cup x > 2$

Example 1

Solve $|2x - 5| > 1$

Solution

$$|2x - 5| > 1$$

$$\begin{aligned} \Rightarrow 2x - 5 < -1 & \quad \text{or} \quad 2x - 5 > 1 \\ \Rightarrow 2x < 4 & \quad \text{or} \quad 2x > 6 \\ \Rightarrow x < 2 & \quad \text{or} \quad x > 3 \quad (\text{Ans}) \end{aligned}$$

Example 2

Solve $|1 - 3x| > 2$

Solution

$$|1 - 3x| > 2$$

$$\begin{aligned} \Rightarrow 1 - 3x < -2 & \quad \text{or} \quad 1 - 3x > 2 \\ \Rightarrow -3x < -3 & \quad \text{or} \quad -3x > 1 \\ \Rightarrow x > 1 & \quad \text{or} \quad x < -\frac{1}{3} \quad (\text{inequality sign changes}) \end{aligned}$$

Exercise 10.2.2

Solve the following inequalities:

1 $|x - 2| > 1$ [Ans : $x < 1$ or $x > 3$] 6 $|3 - 4x| > 2$ [Ans : $x < \frac{1}{4}$ or $x > \frac{5}{4}$]

2 $|x + 3| > 2$ [Ans : $x < -5$ or $x > -1$] 7 $|5 - 2x| > 3$ [Ans : $x < 1$ or $x > 4$]

3 $|3x - 4| > 5$ [Ans : $x < -\frac{1}{3}$ or $x > 3$] 8 $|3x + 2| > 3$ [Ans : $x < -\frac{5}{3}$ or $x > \frac{1}{3}$]

4 $|1 - 2x| > 1$ [Ans : $x < 0$ or $x > 1$] 9 $|1 - x| > 5$ [Ans : $x < -4$ or $x > 6$]

5 $|2x - 1| > \frac{1}{4}$ [Ans : $x < \frac{3}{8}$ or $x > \frac{5}{8}$] 10 $|2x - 3| > 0.1$ [Ans : $x < 1.45$ or $x > 1.55$]

10.2.3 Modulus inequalities (III)

For a given modulus inequality having modulus on both sides, we have to square both sides and proceed as usual.

Example

Solve $|2x - 5| < |x + 1|$

Solution

Squaring both sides, we have

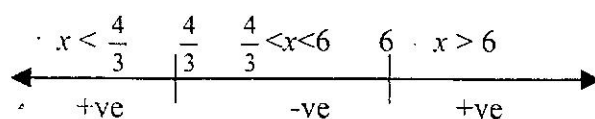
$$(2x - 5)^2 < (x + 1)^2$$

$$4x^2 - 20x + 25 < x^2 + 2x + 1$$

$$3x^2 - 22x + 24 < 0$$

$$(3x - 4)(x - 6) < 0$$

Critical values of x are $4/3$ and 6



Hence $\frac{4}{3} < x < 6$ (Ans)

Exercise 10.2.2

Solve the following inequalities:

1 $|3x - 2| < |x - 1|$ [Ans : $\frac{1}{2} < x < \frac{3}{4}$]

2 $|2x + 3| > |x - 4|$ [Ans : $x < -7$ or $x > \frac{1}{3}$]

3 $|x + 2| < |x|$ [Ans : $x < -1$]

4 $|3x + 1| > |x + 2|$ [Ans : $x < -\frac{3}{4}$ or $x > \frac{1}{2}$]

5 $|x + 3| > |x - 2|$ [Ans : $x > -\frac{1}{2}$]

6 $|x + 2a| < |x - a|$ [Ans : $x < -\frac{a}{2}$]

7 $|2x + 3a| < |x - 2a|$ [Ans : $x < -5a$ or $x > -\frac{a}{3}$]

10.3 Remainder and factor theorem

We expect the reader to be familiar with the remainder and factor theorem, chapters learnt earlier in lower classes.

Example 1

Show that $(x - 1)$ is a factor of $6x^3 + 11x^2 - 5x - 12$, and find the other two linear factors of the expression. (C)

Solution

$$\begin{aligned}\text{Let } f(x) &= 6x^3 + 11x^2 - 5x - 12 \\ f(1) &= 6(1)^3 + 11(1)^2 - 5(1) - 12 \\ &= 6 + 11 - 5 - 12 = 0\end{aligned}$$

Since $f(1) = 0$, $(x - 1)$ is a factor of $f(x)$.

We use long division to find the other quadratic factor of $f(x)$

$$\begin{array}{r} 6x^2 + 17x + 12 \\ (x-1) \overline{) 6x^3 + 11x^2 - 5x - 12} \\ \underline{6x^3 - 6x^2} \\ 17x^2 - 5x - 12 \\ \underline{17x^2 - 17x} \\ 12x - 12 \\ \underline{12x - 12} \\ 0 \end{array}$$

$$\begin{aligned}f(x) &= (x - 1)(6x^2 + 17x + 12) \\ &= (x - 1)(3x + 4)(2x + 3)\end{aligned}$$

Hence the other two linear factors of $f(x)$ are $(3x + 4)$ and $(2x + 3)$.

Example 2

One root of the equation $x^2 + kx + 11 = 0$, where k is a constant, is 1. Find the value of k . Hence find the other two roots of the equation, giving your answers in an exact form. (C)

Solution

$$\begin{aligned}\text{Let } f(x) &= x^2 + kx + 11 \\ f(1) &= (1)^2 + k(1) + 11 \\ &= k + 12\end{aligned}$$

Since 1 is a root $f(1) = 0$ and $(x - 1)$ is a factor of $f(x)$.

$$\therefore k + 12 = 0$$

$$k = -12$$

$$\text{Thus } f(x) = x^2 - 12x + 11$$

We use long division to find the other quadratic factor of $f(x)$

$$\begin{array}{r}
 x^2 + x - 11 \\
 (x-1) \overline{) \begin{array}{r} x^3 \\ x^3 - x^2 \end{array} } \\
 \hline
 x^2 - 12x + 11 \\
 x^2 - x \\
 \hline
 -11x + 11 \\
 -11x + 11 \\
 \hline
 0
 \end{array}$$

$$\therefore f(x) = (x-1)(x^2 + x - 11)$$

Consider the quadratic factor $(x^2 + x - 11)$

Here $a = 1$, $b = 1$ and $c = -11$

Using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -11}}{2}$$

$$x = \frac{-1 \pm \sqrt{45}}{2}$$

$$x = \frac{-1 \pm 3\sqrt{5}}{2}$$

Hence the other two roots of the equation are $\frac{-1+3\sqrt{5}}{2}$ and $\frac{-1-3\sqrt{5}}{2}$

Exercise 10.3

- 1 Find the value of p for which $(x - 1)$ is a factor of $2x^3 + px^2 - 5x + 1$. [Ans: 2]
- 2 Given that $(x - 2)$ is a factor of the polynomial $2x^3 + ax^2 - 5x + 6$, find the value of the constant a . (C) [Ans: $a = -3$]
- 3 Given that $(x - 2)$ and $(x + 2)$ are factors of $x^3 + ax^2 + bx + 4$, find the value of a and of b . (C) [Ans : $a = -1, b = -4$]
- 4 Show that the expression $x^3 + (a-2)x^2 + (a-7)x - 4$ has a factor $(x + 1)$ for all values of a . If the expression has a factor $(x + 2)$, find the value of a and the third factor. [Ans : $a = 3, x - 2$]
- 5 It is given that $(2x + 1)$ is a factor of $f(x)$, where $f(x) = 6x^3 + cx^2 + 3$, find the value of c and factorise $f(x)$ completely. [Ans: $c = -9$; $3(2x+1)(x-1)^2$]
- 6 Show that both $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of $x^4 + x^3 - x^2 - 3x - 6$. Hence write down one quadratic factor of $x^4 + x^3 - x^2 - 3x - 6$, and find a second quadratic factor of this polynomial. [Ans : $x^2 + 4x + 2$]
- 7(a) Show that $(x + 2)$ is a factor of the polynomial $f(x) = 2x^3 - 3x^2 - 11x + 6$.
 (b) Express $f(x)$ as a product of three linear factors.
 (c) By considering the graph of $f(x)$, or otherwise, solve the inequality $f(x) \leq 0$.
 [Ans: (b) $(x + 2)(2x - 1)(x - 3)$; (c) $x \leq -2$ or $\frac{1}{2} \leq x \leq 3$]
- 8 Given that $(2x + 1)$ is a factor of $2x^3 + ax^2 + 16x + 6$, show that $a = 9$. Find the real quadratic factor of $2x^3 + ax^2 + 16x + 6$. By completing the square, or otherwise, show that this quadratic factor is positive for all real values of x .
 [Ans: $x^2 + 4x + 6$]
- 9 The polynomial $4x^3 - 20x^2 + 13x + 12$ is denoted by $f(x)$. Show that $(x - 4)$ is a factor of $f(x)$. Hence, or otherwise, solve the equation $f(x) = 0$. (C)
 [Ans: $-\frac{1}{2}, \frac{3}{2}, 4$]
- 10 Use the factor theorem to find a linear factor of the polynomial $P(x)$, where $P(x) = 2x^3 - 7x^2 + 2x + 3$, and hence express $P(x)$ as a product of three linear factors.
 The polynomial $Q(x)$ is obtained from $P(x)$ by replacing by $x + 1$, so that $Q(x) = P(x + 1)$.
 Express $Q(x)$ as a product of three linear factors. (C)
 [Ans: $(x - 1)$; $P(x) = (x - 1)(x - 3)(2x + 1)$; $Q(x) = x(x - 2)(2x + 3)$]

10.4 Partial fractions

In this section, we are going to learn the process of expressing a single fraction as a finite sum of simpler fractions, called partial fractions.

10.4.1 Distinct linear factors

Linear factors are factors of the type $(ax + b)$.

Examples of linear factors are $(x + 1)$, $(2x - 3)$, x and so on.

To every linear factor $(ax + b)$ in the denominator of a **proper fraction**, there corresponds a partial fraction of the form $\frac{A}{ax + b}$, where A is a constant.

Example 1

Express $\frac{4}{(x+2)(x-2)}$ in partial fractions. (C)

Solution

$$\text{Let } \frac{4}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$\text{Then } 4 = A(x-2) + B(x+2)$$

$$\text{Put } x = 2, \quad 4 = 4B \Rightarrow B = 1$$

$$\text{Put } x = -2, \quad 4 = -4A \Rightarrow A = -1$$

$$\text{Hence } \frac{4}{(x+2)(x-2)} = \frac{-1}{x+2} + \frac{1}{x-2} \quad (\text{Ans})$$

Example 2

Express $\frac{2}{x(x-1)(x-2)}$ in partial fractions.

Solution

$$\text{Let } \frac{2}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$\text{Then } 2 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1)$$

$$\text{Put } x = 0 \quad 2 = A(0-1)(0-2) \\ 2A = 2$$

$$\Rightarrow A = 1$$

$$\text{Put } x = 1 \quad 2 = B(1-2) \\ -B = 2$$

$$\Rightarrow B = -2$$

$$\text{Put } x = 2 \quad 2 = C(2)(2-1) \\ 2C = 2$$

$$\Rightarrow C = 1$$

$$\text{Hence } \frac{2}{x(x-1)(x-2)} = \frac{1}{x} - \frac{2}{x-1} + \frac{1}{x-2} \quad (\text{Ans})$$

Exercise 10.4.1

Express each of the following in partial fractions

1 $\frac{5x}{(x-2)(x+3)}$ [Ans : A = 2, B = 3]

2 $\frac{3x+9}{(x^2-1)(x+2)}$ [Ans : A = 4, B = -1]

3 $\frac{3-x}{(x+1)(x+3)}$ [Ans : A = 2, B = -3]

4 $\frac{7x-5}{(2x-1)(x-2)}$ [Ans : A = 1, B = 3]

5 $\frac{-13x+12}{(3x-2)(x+1)}$ [Ans : A = 2, B = -5]

6 $\frac{7x-6}{x(x-2)}$ [Ans : A = 3, B = 4]

7 $\frac{-5x+3}{x(3x-1)}$ [Ans : A = -3, B = 4]

8 $\frac{2x+1}{x(x-1)(x+1)}$ [Ans : A = -1, B = $\frac{3}{2}$, C = $-\frac{1}{2}$]

9 $\frac{x^2}{(x-1)(x-2)(x-3)}$ [Ans : A = $\frac{1}{2}$, B = -2, C = $\frac{3}{2}$]

10 $\frac{3x+4}{x(x+1)(x-2)}$ [Ans : A = -2, B = $\frac{1}{3}$, C = $\frac{5}{3}$]

10.4.2 Distinct Quadratic factors

Quadratic factors are factors of the type $ax^2 + bx + c$.

Examples of quadratic factors are $x^2 + 1$, $2x^2 - 3$, $x^2 + 3x + 2$ and so on.

To every quadratic factor $ax^2 + bx + c$ in the denominator of a **proper fraction**, there corresponds a partial fraction of the form $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants.

Example

Express $\frac{2+5x+15x^2}{(2-x)(1+2x^2)}$ in partial fractions. (C)

Solution

$$\text{Let } \frac{2+5x+15x^2}{(2-x)(1+2x^2)} = \frac{A}{2-x} + \frac{Bx+C}{1+2x^2}$$

$$\text{Then } 2+5x+15x^2 = A(1+2x^2) + (Bx+C)(2-x)$$

$$\text{Put } x = 2, \quad 2 + 5(2) + 15(2)^2 = A[1+2(2)^2]$$

$$9A = 72 \Rightarrow A = 8$$

Comparing coefficient of x^2 ,

$$15 = 2A - B$$

$$15 = 2(8) - B \Rightarrow B = 1$$

$$\text{Put } x = 0, \quad 2 = A + 2C$$

$$2 = 8 + 2C \Rightarrow C = -3$$

$$\text{Hence } \frac{2+5x+15x^2}{(2-x)(1+2x^2)} = \frac{8}{2-x} + \frac{x-3}{1+2x^2} \quad (\text{Ans})$$

Exercise 10.4.2

Express each of the following in partial fractions

$$1 \quad \frac{x+2}{(x+3)(x^2+1)} \quad [\text{Ans : } A = -\frac{1}{10}, B = \frac{1}{10}, C = \frac{7}{10}]$$

$$2 \quad \frac{3x+5}{(x-4)(x^2+1)} \quad [\text{Ans : } A = 1, B = -1, C = -1]$$

$$3 \quad \frac{2x^2+1}{x(x^2+3)} \quad [\text{Ans : } A = \frac{1}{3}, B = \frac{5}{3}, C = 0]$$

$$4 \quad \frac{2x^2-7}{(x+1)(x^2+6)} \quad [\text{Ans : } A = -\frac{5}{7}, B = \frac{19}{7}, C = -\frac{19}{7}]$$

$$5 \quad \frac{5x^2+x+2}{(x-1)(x^2+3)} \quad [\text{Ans : } A = 2, B = 3, C = 4]$$

$$6 \quad \frac{11x^2+8x-7}{(x-1)(x^2+3)} \quad [\text{Ans : } A = 3, B = 8, C = 16]$$

10.4.2 Repeated linear factors

To every linear factor $(ax + b)$ repeated n times in the denominator of a **proper fraction**, there corresponds the sum of n partial fractions :

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots + \frac{A_n}{(ax+b)^n} \text{ where } A_1, A_2, A_3, \dots, A_n \text{ are constants.}$$

Example

Express $\frac{11+7x}{(1-x)(2+x)^2}$ in partial fractions. (C)

Solution

$$\text{Let } \frac{11+7x}{(1-x)(2+x)^2} = \frac{A}{1-x} + \frac{B}{2+x} + \frac{C}{(2+x)^2}$$

$$\text{Then } 11+7x = A(2+x)^2 + B(1-x)(2+x) + C(1-x)$$

$$\text{Put } x = 1, \quad 11 + 7(1) = A(2+1)^2$$

$$9A = 18 \Rightarrow A = 2$$

$$\text{Put } x = -2 \quad 11 + 7(-2) = C[1 - (-2)]$$

$$3C = -3 \Rightarrow C = -1$$

Comparing coefficient of x^2 ,

$$0 = A - B$$

$$A = B = 2$$

$$\text{Hence } \frac{11+7x}{(1-x)(2+x)^2} = \frac{2}{1-x} + \frac{2}{2+x} - \frac{1}{(2+x)^2} \quad (\text{Ans})$$

Exercise 10.4.3

Express each of the following in partial fractions

$$1 \quad \frac{2x+1}{(x-1)(x-2)^2}$$

$$[\text{Ans : } A = 3, B = -3, C = 5]$$

$$2 \quad \frac{2x+5}{(x-1)(x+2)^2}$$

$$[\text{Ans : } A = \frac{7}{9}, B = -\frac{7}{9}, C = -\frac{1}{3}]$$

$$3 \quad \frac{x^2+6x+9}{(x+5)(x-3)^2}$$

$$[\text{Ans : } A = \frac{1}{16}, B = \frac{15}{16}, C = \frac{9}{2}]$$

$$4 \quad \frac{x^2+5}{(x+4)^2(x-3)}$$

$$[\text{Ans : } A = -3, B = \frac{5}{7}, C = \frac{2}{7}]$$

$$5 \quad \frac{5x-2}{x^2(x+1)}$$

$$[\text{Ans : } A = 7, B = -2, C = -7]$$

$$6 \quad \frac{3x+4}{x^2(x-2)}$$

$$[\text{Ans : } A = -5/2, B = -2, C = 5/2]$$

PARTIAL FRACTIONS – PAST EXAMINATION QUESTIONS

1 Express $\frac{2x^2+1}{x(x-1)^2}$ in partial fractions.

[Ans : A = 1, B = 1, C = 3]

2 Express $\frac{x^2-11}{(x+2)^2(3x-1)}$ in partial fractions.

[Ans : A = 1, B = 1, C = -2]

3 Express $\frac{5x}{(x+2)(x^2+1)}$ in partial fractions.

[Ans : A = -2, B = 2, C = 1]

4 Express in partial fractions $\frac{x-4}{(2x+1)(x-1)^2}$.

[Ans : A = -2, B = 1, C = -1]

5 Express $\frac{2}{x(x+1)(x-1)}$ in partial fractions.

[Ans : A = -2, B = 1, C = 1]

6 Express $\frac{4}{(x-3)(x+1)}$ in partial fractions.

[Ans : A = 1, B = -1]

7 Express in partial fractions $\frac{5x}{(2x+1)(x^2+1)}$.

[Ans : A = -2, B = 1, C = 2]

8 Express in partial fractions $\frac{4x-11}{(x^2+3)(2x+1)}$.

[Ans : A = 2, B = 1, C = -4]

9 Express f(x) in partial fractions, where $f(x) = \frac{3x+8}{(2x+1)(x^2+3)}$.

[Ans : A = 2, B = -1, C = 2]

10 Express $\frac{x^2+x-4}{(2x+1)(x^2+4)}$ in partial fractions.

[Ans : A = -1, B = 1, C = 0]

10.5 Expansion of $(1+x)^n$

We have seen in chapter 7, the expansion of $(a+b)^n$, where n is a positive integer. Now, we are going to extend the idea and consider expansions of the type $(1+x)^n$ where n is a rational number and $|x| < 1$.

The binomial series is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + x^n, \text{ provided } |x| < 1.$$

Example 1

Expand $(1+2x)^{-3}$ as a series of ascending powers of x , where $|x| < \frac{1}{2}$, up to and including the term in x^3 , expressing the coefficients in their simplest form. (C)

Solution

$$(1+2x)^{-3} = 1 + -3(2x) + \frac{-3 \cdot -4}{2!}(2x)^2 + \frac{-3 \cdot -4 \cdot -5}{3!}(2x)^3 = 1 - 6x + 24x^2 - 80x^3 \text{ (Ans)}$$

Example 2

Expand $(1-4x)^{\frac{1}{2}}$ as a series of ascending powers of x , where $|x| < \frac{1}{4}$, up to and including the term in x^3 , expressing the coefficients in their simplest form. (C)

Solution

$$\begin{aligned} (1-4x)^{\frac{1}{2}} &= 1 + \frac{1}{2}(-4x) + \frac{\frac{1}{2} \cdot \frac{1}{2}}{2!}(-4x)^2 + \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2}}{3!}(-4x)^3 \\ &= 1 - 2x - \frac{1}{8}(16x^2) + \frac{1}{16}(-64x^3) = 1 - 2x - 2x^2 - 4x^3 \end{aligned}$$

Example 3

Obtain the first three terms in the expansion, in ascending powers of x , of $(8+3x)^{\frac{2}{3}}$, stating the set of values of x for which the expansion is valid. (C)

Solution

$$\begin{aligned} (8+3x)^{\frac{2}{3}} &= 8^{\frac{2}{3}} \left(1 + \frac{3x}{8}\right)^{\frac{2}{3}} = 4 \left(1 + \frac{3x}{8}\right)^{\frac{2}{3}} \\ &= 4 \left[1 + \frac{\frac{2}{3}}{1} \left(\frac{3x}{8}\right) + \frac{\frac{2}{3} \cdot \frac{2}{3} - 1}{2!} \left(\frac{3x}{8}\right)^2 \right] \\ &= 4 \left[1 + \frac{x}{4} - \frac{1}{9} \left(\frac{9x^2}{64}\right) \right] \\ &= 4 + x - \frac{1}{16}x^2 \text{ (Ans)} \end{aligned}$$

The expansion is valid for $|x| < \frac{8}{3}$.

Exercise 10.5

Write down the first four terms in the expansion of

1 $(1+x)^{-2}$ [Ans : $1 - 2x + 3x^2 - 4x^3$]

2 $(1+2x)^{\frac{5}{2}}$ [Ans : $1 + 5x + \frac{15}{2}x^2 + \frac{5}{2}x^3$]

3 $(1+4x)^{\frac{1}{2}}$ [Ans : $1 + 2x - 2x^2 + 4x^3$]

4 $(1-x)^{-1}$ [Ans : $1 + x + x^2 + x^3$]

5 $(1+3x)^{\frac{1}{3}}$ [Ans : $1 + x - x^2 + \frac{5}{3}x^3$]

6 $(1-2x)^{\frac{1}{4}}$ [Ans : $1 - \frac{1}{2}x - \frac{3}{8}x^2 - \frac{7}{16}x^3$]

Write down the first three terms in the expansion of

7 $(4+3x)^{\frac{1}{2}}$ [Ans : $2 + \frac{3}{4}x - \frac{9}{64}x^2$]

8 $(9+2x)^{\frac{3}{2}}$ [Ans : $27 + 9x + \frac{1}{2}x^2$]

9 $(8+5x)^{\frac{1}{3}}$ [Ans : $2 + \frac{5}{12}x - \frac{25}{288}x^2$]

10 Expand $\frac{1}{\sqrt{1+2x}}$ as a series of ascending powers of x , where $|x| < \frac{1}{2}$, up to and including the term in x^3 , expressing the coefficients in their simplest form. (C)

[Ans : $1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3$]

11 Expand $\frac{1}{(1+2x^2)^2}$ as a series of ascending powers of x , up to and including the term in x^6 , giving the coefficients in their simplest form. (C)

[Ans : $1 - 4x^2 + 12x^4 - 32x^6$]

12 Use the Binomial expansion to find $(1.0006)^{\frac{1}{3}}$ correct to eight decimal places.

[Ans: 1.00019996]

BINOMIAL EXPANSIONS – PAST EXAMINATION QUESTIONS

- 1 Express $\frac{x}{(1+x)(1-2x)}$ in partial fractions. Hence show that, if x is so small that x^4 and higher powers of x can be neglected, $\frac{x}{(1+x)(1-2x)} = x + x^2 + 3x^3$.

- 2 Expand $\sqrt{4+x^2}$ in ascending powers of x up to and including the term in x^4 .

[Ans : $2 + \frac{1}{4}x^2 - \frac{1}{64}x^4$]

- 3 Expand $(4-x)^{-2}$ as a series of ascending powers of x , giving the first three terms and the term in x^n . State for what set of values of x the expansion is valid.

[Ans : $\frac{1}{16} + \frac{x}{32} + \frac{3x^2}{256} + \frac{(n+1)x^n}{4^{n+2}}; |x| < 4$]

- 4 Expand $(1+3x)^{\frac{1}{3}}$ up to and including the term in x^3 . Use your series to obtain a value for $(1.03)^{\frac{1}{3}}$ giving six decimal places in your answer.

[Ans : $1 + x - x^2 + \frac{5}{3}x^3; 1.009902$]

- 5 Write down the first four terms of the series expansion in ascending powers of x of $(1-x)^3$, simplifying the coefficients. By taking $x = 0.1$, use your answer to show that $\sqrt[3]{900} \approx \frac{15641}{1620}$.

[Ans: $1 - \frac{1}{3}x + \frac{1}{9}x^2 - \frac{5}{81}x^3$]

- 6 Expand $(1+2x)^{\frac{1}{3}}$ in a series of ascending powers of x up to and including the term in x^3 , simplifying the coefficients. State the set of the values of x for which the expansion is valid.

[Ans: $1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3; |x| < \frac{1}{2}$]

- 7 Using the binomial theorem, express $\left(1 + \frac{1}{2}x\right)^2(1-2x)^{-\frac{1}{2}}$ as a series of ascending powers of x up to and including the term in x^2 . State the set of values of x for which an expansion in ascending powers of x is valid.

[Ans: $1 + \frac{7}{4}x + \frac{75}{32}x^2; |x| < \frac{1}{2}$]

- 8 Use the binomial theorem to expand $\sqrt{\frac{1+x}{1-x}}$ as a series of ascending powers of x up to and including the term in x^2 , where $|x| < 1$. By putting $x = \frac{1}{10}$ in your result, show that $\sqrt{11}$ is approximately equal to $\frac{663}{200}$.

[Ans: $1 + x + \frac{1}{2}x^2$]

- 9 Find the coefficient of t^3 in the expansion of $(3-2t)^5$.

[Ans: -720]

- 10 Find the expansion in ascending powers of x , up to and including the term in x^6 , of $(1-2x^2)^{-4}$, simplifying the coefficients.

[Ans: $1 + 8x^2 + 40x^4 + 160x^6$]

CHAPTER 11

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

11.1 Laws of logarithms

In this section emphasis would be laid on the various laws of logarithms.

Property 1 $\ln a + \ln b = \ln(ab)$

Example

Write as a single logarithm

(a) $\ln 5 + \ln 4$

(b) $\ln 3 + \ln 4 + \ln 5$

Solution

(a) $\ln 5 + \ln 4 = \ln 20$

(b) $\ln 3 + \ln 4 + \ln 5 = \ln 60$

Property 2 $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

Example

Write as a single logarithm

(a) $\ln 20 - \ln 4$

(b) $\ln 36 + \ln 3 - \ln 9$

Solution

(a) $\ln 20 - \ln 4 = \ln 5$

(b) $\ln 36 + \ln 3 - \ln 9 = \ln\left(\frac{36 \times 3}{9}\right) = \ln 12$

Property 3 $x \ln a = \ln a^x$

Example

Write as a single logarithm

(a) $2 \ln 3 + 3 \ln 5$

(b) $3 \ln 2 + 2 \ln 5 - 4 \ln 3$

Solution

(a) $2 \ln 3 + \ln 5 = \ln 3^2 + \ln 5^3$
 $= \ln 9 + \ln 125 = \ln 1125$

(b) $3 \ln 2 + 2 \ln 5 - 4 \ln 3 = \ln 2^3 + \ln 5^2 - \ln 3^4$
 $= \ln\left(\frac{8 \times 25}{81}\right) = \ln\left(\frac{200}{81}\right)$

Property 4 $\ln\left(\frac{1}{a}\right) = \ln a^{-1} = -\ln a$

Property 5 $\ln e = 1$ and $\ln 1 = 0$

Example 1

Express $\ln(2\sqrt{e}) - \frac{1}{3}\ln\left(\frac{8}{e}\right) - \ln\left(\frac{e}{3}\right)$ in the form $c + \ln d$, where c and d are rational numbers.

(C)

Solution

$$\begin{aligned}\ln(2\sqrt{e}) - \frac{1}{3}\ln\left(\frac{8}{e}\right) - \ln\left(\frac{e}{3}\right) &= \ln 2 + \ln\sqrt{e} - \frac{1}{3}(\ln 8 - \ln e) - (\ln e - \ln 3) \\ &= \ln 2 + \frac{1}{2}\ln e - \frac{1}{3}(\ln 2^3 - 1) - (1 - \ln 3) \\ &= \ln 2 + \frac{1}{2} - \ln 2 - \frac{1}{3} - 1 + \ln 3 \\ &= -\frac{5}{6} + \ln 3 \quad (\text{Ans})\end{aligned}$$

Example 2

It is given that $\ln a = x$ and $\ln b = y$. Express $\ln\left(\frac{a^2b}{e}\right)$ in terms of x and y . (C)

Solution

$$\begin{aligned}\ln\left(\frac{a^2b}{e}\right) &= \ln a^2b - \ln e \\ &= \ln a^2 + \ln b - 1 \\ &= 2 \ln a + \ln b - 1 \\ &= 2x + y - 1 \quad (\text{Ans})\end{aligned}$$

Example 3

Solve the equation $3 \ln(2x) = 1 + \ln a$.

Solution

$$\ln(2x)^3 = \ln e + \ln a$$

$$\ln 8x^3 = \ln ae$$

$$8x^3 = ae$$

$$x^3 = \frac{ae}{8}$$

$$\Rightarrow x = \frac{\sqrt[3]{ae}}{2} \quad (\text{Ans})$$

Exercise 11.1

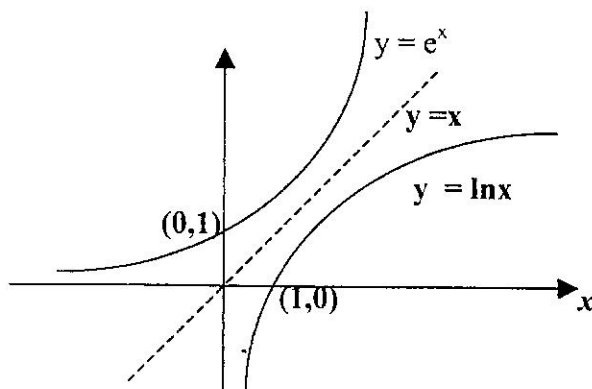
Express the following as single logarithm

- 1 $\ln 2 + \ln 4 + \ln \sqrt{2}$. [Ans: $\frac{7}{2} \ln 2$]
- 2 $\ln 3 + \ln 9 - \ln \sqrt{3}$. [Ans: $\frac{5}{2} \ln 3$]
- 3 $\ln \sqrt{x} + \ln \sqrt[3]{x}$ [Ans: $\ln x^{\frac{5}{6}}$]
- 4 $5 \ln x + 2 \ln x^3 - 3 \ln x^3 + \ln \sqrt{x}$ [Ans: $\frac{5}{2} \ln x$]
- 5 It is given that $\ln a = x$ and $\ln b = y$. Express $\ln \left(\frac{a^3 b^2}{e} \right)$ in terms of x and y .
[Ans: $3x + 2y - 1$]
- 6 It is given that $\ln a = p$ and $\ln b = q$. Express $\ln \left(\frac{a^4}{b} \right)$ in terms of x and y .
[Ans: $4p - q$]
- 7 Express $\frac{1}{2} \ln \left(\frac{4}{e} \right) - \ln \left(\frac{e}{2} \right) + \ln 5$ in the form $c + \ln d$, where c and d are rational numbers.
[Ans: $-\frac{3}{2} + \ln 20$]
- 8 Solve the simultaneous equations $\ln(3x - y) = \ln 2 + \ln 4$,
 $x + y = 4$.
[Ans: $x = 3, y = 1$]
- 9 State the exact value of the constant k such that $3^x = e^{kx}$, for all x . (C) [Ans: $\ln 3$]
- 10 Given the simultaneous equations $2^x = 3^y$, $x + y = 1$, show that $x = \frac{\ln 3}{\ln 6}$. (C)
- 11 By writing $e^x = u$, or otherwise, solve the equation $e^x - 2e^{-x} = 1$. (C)
[Ans: $x = \ln 2$]
- 12 Solve the equation $a^x = e^{3x-2}$, to find x in terms of a . [Ans: $\frac{-2}{\ln a - 3}$]

11.2 Relationship between e^x and $\ln x$ and their graphs

We have already seen in chapter 2 that the graph of $y = e^x$ can be obtained by the graph of $y = \ln x$ by a reflection in the line $y = x$.

If we sketch the graph of f and f^{-1} on a single diagram, we obtain



Example

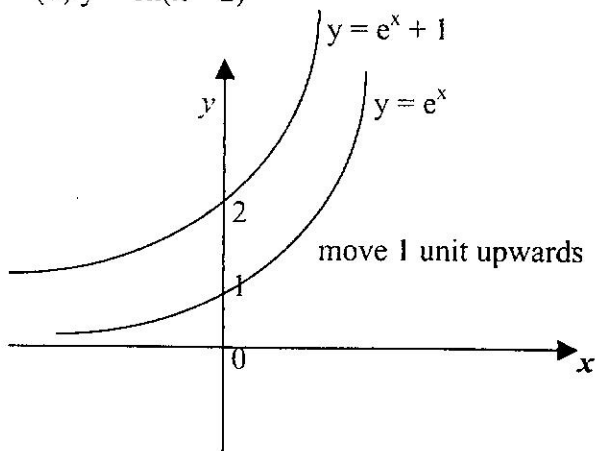
Sketch on separate diagrams

(a) $y = e^x + 1$

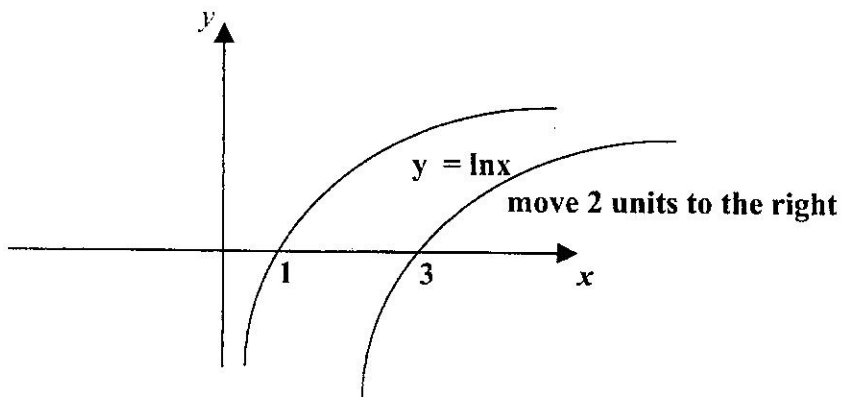
(b) $y = \ln(x - 2)$

Solution

(a)



(b)



Exercise 11.2

Sketch the following graphs

1 $y = e^x + 3, x \in \mathbb{R}$

2 $y = e^x - 1, x \in \mathbb{R}$

3 $y = e^{x-2}, x \in \mathbb{R}$

4 $y = e^{x+3}, x \in \mathbb{R}$

5 $y = 2e^x, x \in \mathbb{R}$

6 $y = e^{\sqrt{x}}, x \geq 0$

7 $y = \frac{1}{2}e^x, x \in \mathbb{R}$

8 $y = \ln x - 2, x > 0$

9 $y = \ln x + 1, x > 0$

10 $y = \ln 2x, x > 0$

11 $y = \ln(x + 1), x > 0$

12 $y = \ln(x - 1) - 1, x > 1$

13 $y = \ln(x + 2) - 3, x > 0$

14 $y = \frac{1}{3} \ln x, x > 0$

15 $y = 2 \ln(x - 1) + 1, x > 1$

11.3 Solving equations of the type $a^x = b$ and similar inequalities

If it is not possible to express an equation of the type $a^x = b$ to the same base, we solve the equation by taking $\ln$ on both sides.

Example 1

Solve the equation $3^{2x} = 4^{2-x}$, giving your answer to three significant figures. (C)

Solution

We have $3^{2x} = 4^{2-x}$

Taking $\ln$ on both sides, we have

$$\ln 3^{2x} = \ln 4^{2-x}$$

$$2x \ln 3 = (2 - x) \ln 4$$

$$2x \ln 3 = 2 \ln 4 - x \ln 4$$

$$\therefore x (2 \ln 3 + \ln 4) = \ln 16$$

$$x = \frac{\ln 16}{\ln 36} = 0.774 \text{ (Ans)}$$

Example 2

Solve $5000(1.06)^n > 20000$

Solution

We have $5000(1.06)^n > 20000$

$$1.06^n > 4$$

Taking $\ln$ on both sides, we have

$$\ln 1.06^n > \ln 4$$

$$n \ln 1.06 > \ln 4$$

$$n > \frac{\ln 4}{\ln 1.06}$$

$$n > 23.8$$

Hence $n = 24$

Example 3

Solve $(0.8)^x > 5$ (C)

Solution

We have $(0.8)^x > 5$

Taking $\ln$ on both sides, we have

$$\ln (0.8)^x > \ln 5$$

$$x \ln 0.8 > \ln 5$$

$$x < \frac{\ln 5}{\ln 0.8} \text{ [change of inequality sign as } \ln 0.8 \text{ is negative]}$$

$$x < -7.21 \text{ (Ans)}$$

Exercise 11.3

Solve the following equations, giving your answers to 3 significant figures, where necessary.

1 $3^x = 10$ [Ans : 2.10]

2 $2^{x-2} = 15$ [Ans : 5.91]

3 $3^{x+1} = 2^{x+2}$ [Ans : 0.709]

4 $7^{x+1} = 2^{3x-2}$ [Ans : 24.4]

5 $2^{x-3} = 5^{2x-4}$ [Ans : 1.73]

Solve the following inequalities, giving your answer correct to the nearest whole number.

6 $(1.06)^{n-1} > 45$ [Ans : 67]

7 $2^{x-2} < 35$ [Ans : 7]

8 $(0.88)^{n+1} < 2$ [Ans : -6]

9 $(0.96)^{n-1} < 58$ [Ans : -98]

10 $(1.02)^{n-1} > 32$ [Ans : 177]

11 $25\{4 - 3 \times (0.85)^n\} > 50$ [Ans : 3]

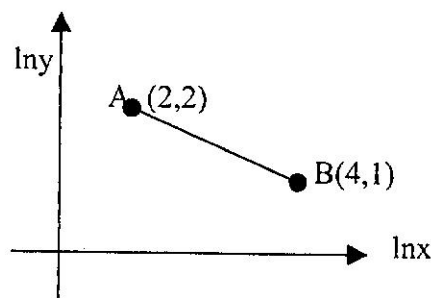
12 $6\{5 - 4 \times (0.75)^n\} < 18$ [Ans : 2]

11.4 Linear law

In this part of the chapter, we are going to use logarithms to transform a given relationship to linear form $y = mx + c$.

Example 1

The diagram shows part of the straight line drawn to represent the equation $y = ax^b$, where a and b are constants. Find the value of a and b .



Solution

The equation of the line is of the form

$$\ln y = m(\ln x) + c$$

Gradient of the line is, $m = \frac{1-2}{4-2} = -\frac{1}{2}$

Consider $y = ax^b$

Taking $\ln$ on both sides, we have

$$\ln y = \ln ax^b$$

$$\ln y = \ln a + \ln x^b$$

$$\ln y = \ln a + b \ln x$$

Since (2, 2) lies on the line,

$$2 = \ln a - \frac{1}{2}(2)$$

$$\ln a = 3$$

$$a = e^3$$

Hence $a = e^3$ and $b = -\frac{1}{2}$.

Example 2

Two variables x and y are related by the equation $y = a(b^x)$, where a and b are constants. When a graph is plotted showing values of $\ln y$ on the vertical axis and values of x on the horizontal axis, the point lies on a straight line having gradient 1.8 and crossing the vertical axis at the point (0, 4.1). Find the values of a and b .

Solution

Consider $y = a(b^x)$

Taking $\ln$ on both sides, we have

$$\ln y = \ln a + \ln b^x$$

$$\ln y = \ln a + x \ln b$$

Since gradient = 1.8, $\ln b = 1.8$

$$\Rightarrow b = e^{1.8} = 6.05$$

Also

$$\ln a = 4.1$$

$$\Rightarrow a = e^{4.1} = 60.3 \text{ (Ans)}$$

Exercise 11.4

- 1 The variables x and y satisfy the equation $\frac{1}{3}y = 3^x$. Express this equation in a form giving $\ln y$ in terms of x . When corresponding values of x and $\ln y$ are plotted on graph paper, with x on the horizontal axis and $\ln y$ on the vertical axis, the points lie on a straight line. For this straight line, write down
- (a) the gradient,
- (b) the coordinates of the point where it crosses the vertical axis. (C)
- [Ans : $\ln y = \ln 4 + (\ln 3)x$; (a) $\ln 3$ (b) $(0, \ln 4)$]

- 2 The graph of y against x is a straight line which has gradient 3 and passes through the point $(0, -1)$. Express y in terms of x .
- Given also that $y = \frac{t}{s^3}$ and $x = \frac{1}{s}$, express t in terms of s . (C)
- [Ans : $y = 3x - 1$, $t = 3s^2 - s^3$]

- 3 The graph of $\ln y$ against $\ln x$ is a straight line with gradient 3 and intercept 2 on the $\ln y$ axis. Express y in the form ax^n , where the constant a and n are to be found.
- [Ans : $e^2 x^3$]

- 4 The variables x and y are connected by the equation $y = ax + bx^2$, where a and b are constants. Four pairs of values of x and y , measured experimentally, are shown in the table.

x	1.1	2.0	3.4	3.9
y	7.5	10.1	7.5	4.7

By plotting the graph of $\frac{y}{x}$ against x , determine the values of a and b . (C)

[Ans : $a = 9$, $b = -2$]

- 5 The relation between the variable quantities x and y is $y = a + \frac{b}{x}$, where a and b are constants. Four pairs of corresponding values of x and y , measured experimentally, are shown in the following table.

x	1.0	1.5	2.6	4.3
y	4.9	4.3	3.8	3.5

By plotting a graph of y against $\frac{1}{x}$, determine the values of a and b . (C)

[Ans : $a = 3.1$, $b = 1.818$]

CHAPTER 12

TRIGONOMETRY 2

12.1 Compound angle formulae

In this section, emphasis shall be laid on compound angle formulae which are provided in the list of formula:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Consider $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

If we replace $B = A$, we have

$$\sin(A + A) = \sin A \cos A + \cos A \sin A$$

$$\sin 2A = 2 \sin A \cos A \quad \dots(1)$$

Consider $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

If we replace $B = A$, we have

$$\cos(A + A) = \cos A \cos A - \sin A \sin A$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad \dots(2)$$

$$\text{But } \sin^2 A = 1 - \cos^2 A,$$

$$\text{So we have, } \cos^2 A = 2\cos^2 A - 1 \quad \dots(3)$$

$$\text{Again } \cos^2 A = 1 - \sin^2 A,$$

$$\text{So we have, } \cos 2A = 1 - 2 \sin^2 A \quad \dots(4)$$

$$\text{Consider } \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

If we replace $B = A$, we have

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A} \quad \dots(5)$$

Example 1

The angle x , measured in degrees, satisfies the equation $\sin(x + 30^\circ) = 2\cos(x + 60^\circ)$. By expanding each side, show that this equation may be simplified to $\cos x = (3\sqrt{3})\sin x$.

Hence find

- the two possible values of x lying between 0° and 360° .
- the exact value of $\cos^2 x$, giving your answer as a fraction. (C)

Solution

Given

$$\sin(x + 30^\circ) = 2\cos(x + 60^\circ)$$

$$\sin x \cos 30^\circ + \cos x \sin 30^\circ = 2[\cos x \cos 60^\circ - \sin x \sin 60^\circ]$$

$$\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x = 2\left[\frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x\right]$$

$$\sqrt{3} \sin x + \cos x = 2 \cos x - 2\sqrt{3} \sin x$$

$$\cos x = 3\sqrt{3} \sin x \quad (\text{shown}) \quad \dots\dots (1)$$

- Dividing by $\cos x$ in (1), we have

$$\tan x = \frac{1}{3\sqrt{3}} \Rightarrow x = 10.9^\circ, 190.9^\circ$$

- Squaring (1) on both sides, we have

$$\cos^2 x = 27 \sin^2 x$$

$$\cos^2 x = 27(1 - \cos^2 x)$$

$$\cos^2 x = 27 - 27\cos^2 x$$

$$28 \cos^2 x = 27$$

Hence $\cos^2 x = \frac{27}{28}$ (Ans)

Example 2

Given that $\tan A = t$, express in terms of t

- $\sin A$
- $\sec A$
- $\sin 2A$

Solution

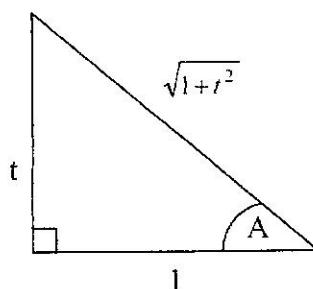
From the triangle,

$$(a) \sin A = \frac{t}{\sqrt{1+t^2}}$$

$$(b) \cos A = \frac{1}{\sqrt{1+t^2}}$$

$$\sec A = \frac{1}{\cos A} = \sqrt{1+t^2}$$

$$\begin{aligned} (c) \sin 2A &= 2 \sin A \cos A \\ &= 2 \cdot \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} \\ &= \frac{2t}{1+t^2} \end{aligned}$$



Exercise 12.1

- 1 Given that $\tan(A+B) = 5$ and that $\tan B = \frac{1}{2}$, find the value of $\tan A$.
[Ans : 9/7]
- 2 Given that $\sin A = \frac{5}{13}$, where $0^\circ < A < 90^\circ$, and that $\cos B = -\frac{3}{5}$, where $180^\circ < B < 270^\circ$, find without using a calculator the value of $\sin(A-B)$.
[Ans : 33/65]
- 3 Given that $\frac{\sin(A+B)}{\sin(A-B)} = \frac{2}{3}$, find the value of $\tan A \cot B$.
[Ans : -5]
- 4 Given that $\tan(A-B) = \frac{25}{24}$ and that $\tan A = \frac{3}{4}$, find the value of $\tan B$.
[Ans : -28/171]
- 5 If $\sin(\theta - \alpha) = \cos(\theta - \beta)$, express $\tan \theta$ in terms of α and β .
[Ans : $\frac{\sin \alpha + \cos \beta}{\cos \alpha - \sin \beta}$]
- 6 Write down the value of $\tan 2x$ in terms of x , and hence show that
$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$
- 7 By first expanding $\sin(2x + x)$, prove that $\sin 3x = 3 \sin x - 4 \sin^3 x$. (C)
- 8 By first expanding $\cos(2x + x)$, prove that $\cos 3x = 4 \cos^3 x - 3 \cos x$. (C)
- 9 Given that $A + B = 45^\circ$ and that $\tan A = 2$, find, without using a calculator, the value of $\tan B$.
[Ans: -1/3]
- 10 It is given that $\tan A = 2$. Find, without using trigonometrical tables or a calculator,
(a) the exact value of $\tan \theta$, given that $\tan(\theta + A) = 4$,
(b) the exact value of $\tan \phi$, given that $\sin(\phi + A) = 3 \cos(\phi - A)$.
[Ans : (a) 2/9 (b) -1/5]

12.2 Expressing $a\cos\theta + b\sin\theta$ in the form $R\cos(\theta - \alpha)$

In this section we are going to introduce to the reader expression such as $a\cos\theta + b\sin\theta$ and the ways in which it can be expressed in the form $R\cos(\theta - \alpha)$.

$$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$$

$$a\cos\theta - b\sin\theta = R\cos(\theta + \alpha)$$

$$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha), \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \alpha = \tan^{-1} \left(\frac{b}{a} \right).$$

Example 1

Express $5\cos\theta - 12\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R \geq 0$ and $0 \leq \alpha < 360^\circ$, stating the value of R and giving the value α of correct to 0.1° . (C)

Solution

$$R = \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \quad \text{and} \quad \alpha = \tan^{-1} \left(\frac{12}{5} \right) = 67.4^\circ$$

$$\text{Hence } 5\cos\theta - 12\sin\theta = 13\cos(\theta + 67.4^\circ).$$

Example 2

Solve the equation $\sin x + \cos x = 1$ for $0^\circ < x < 180^\circ$, giving your answer correct to 1 d.p.

Solution

$$\text{Let } \sin x + \cos x = R\sin(x + \alpha)$$

$$R = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \text{and} \quad \alpha = \tan^{-1} \left(\frac{1}{1} \right) = 45^\circ$$

$$\therefore \sin x + \cos x = \sqrt{2} \sin(x + 45^\circ).$$

$$\text{Now } \sqrt{2} \sin(x + 45^\circ) = 1$$

$$\text{Or } \sin(x + 45^\circ) = \frac{1}{\sqrt{2}}$$

$$(x + 45^\circ) = 45^\circ, 135^\circ$$

$$x = 0^\circ, 90^\circ$$

Example 2

Given that $3\cos x - 4\sin x = R\cos(x + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, find the values of R and α , giving the value of α correct to two decimal places. Hence solve the equation $3\cos 2\theta - 4\sin 2\theta = 2$, for $0^\circ < \theta < 360^\circ$, giving your answer correct to 1 decimal place. (C)

Solution

$$R = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \quad \text{and} \quad \alpha = \tan^{-1} \left(\frac{4}{3} \right) = 53.13^\circ$$

$$\therefore 3\cos x - 4\sin x = 5\cos(x + 53.13^\circ).$$

$$\text{Now } 3\cos 2\theta - 4\sin 2\theta = 5\cos(2\theta + 53.13^\circ)$$

$$\text{Or } 5\cos(2\theta + 53.13^\circ) = 2$$

$$\cos(2\theta + 53.13^\circ) = \frac{2}{5}$$

$$2\theta + 53.13^\circ = 66.42^\circ, 293.58^\circ, 426.42^\circ, 653.58^\circ$$

$$2\theta = 13.29^\circ, 240.45^\circ, 373.42^\circ, 653.58^\circ$$

Exercise 12.2

Express the following in the form $R\cos(x \pm \alpha)$ and $R\sin(x \pm \alpha)$ as appropriate.

- 1 $6 \cos x + 8 \sin x$ [Ans : $10 \cos(x - 53.1^\circ)$]
- 2 $3 \sin x + 4 \cos x$ [Ans : $5 \sin(x + 53.1^\circ)$]
- 3 $\cos x + \sqrt{3} \sin x$ [Ans : $2\cos(x - 60^\circ)$]
- 4 $2 \cos x - 5 \sin x$ [Ans : $\sqrt{29} \cos(x + 68.2^\circ)$]
- 5 $\cos x + \sin x$ [Ans : $\sqrt{2} \cos(x - 45^\circ)$]
- 6 $\sqrt{3} \sin x - \cos x$ [Ans : $2 \sin(x - 30^\circ)$]
- 7 $7\cos x - 24 \sin x$ [Ans : $25 \cos(x + 73.7^\circ)$]
- 8 $12 \sin x + 5 \cos x$ [Ans : $13 \sin(x + 22.6^\circ)$]

Find all the angles between 0° and 360° inclusive which satisfy the following equations:

- 9 $4 \cos x - 3 \sin x = 1$ [Ans : $41.6^\circ, 244.7^\circ$]
- 10 $8 \sin x + 15\cos x = 10$ [Ans : $82.0^\circ, 334.1^\circ$]
- 11 $4 \sin x + 2 \cos x = 3$ [Ans : $15.6^\circ, 111.3^\circ$]
- 12 $2 \cos 2x - \sin 2x = 1$ [Ans : $18.4^\circ, 135^\circ, 198.4^\circ, 315^\circ$]
- 13 Solve the equation $4 \sin \theta - 7\cos \theta + 3 = 0$, giving all values of θ between 0° and 360° correct to the nearest 0.1° .
[Ans : $38.4^\circ, 262.1^\circ$]
- 14 Solve for θ the equation $3 \cos 2\theta - \sin 2\theta = 2$, giving all solutions between 0° and 360° correct to the nearest degree.
[Ans : $16^\circ, 145^\circ, 196^\circ, 325^\circ$]
- 15 Find all the values of θ between 0° and 360° which satisfy the equation $6 \cos \theta + 7\sin \theta = 4$.
[Ans : $113.7^\circ, 345.1^\circ$]

TRIGONOMETRY II – PAST EXAMINATION QUESTIONS

- 1 Find all solutions between 0° and 360° of the equation $4 \tan x - 2 \sec x = 5$, giving to the nearest 0.1° . [Ans : $69.5^\circ, 213.1^\circ$]

- 2 Given that $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{12}{13}$, prove that one possible value of $(\alpha + \beta)$ is $\frac{33}{65}$, and find all the other possible values. [Ans : $-\frac{33}{65}, \frac{63}{65}, -\frac{63}{65}$]

- 3 Solve for θ the equation $3 \cos 2\theta - \sin 2\theta = 2$, giving all solutions between 0° and 360° , correct to the nearest degree. [Ans : $16^\circ, 145^\circ, 196^\circ, 325^\circ$]

- 4 Solve for x , where $-180^\circ < x \leq 180^\circ$, the equations $5 \cos x = 2 + 3 \sin x$, giving your answer to the nearest 0.1° . [Ans : $39.0^\circ, -100.9^\circ$]

- 5 Given that θ and ϕ are acute angles such that $\sin \theta = \frac{24}{25}$ and $\cos \phi = \frac{12}{13}$, find, without using tables or a calculation, the exact values of $\sin (\theta + \phi)$ and $\cos (\theta + \phi)$. Deduce that $\frac{1}{2} \pi < \theta + \phi < \pi$. [Ans : $\frac{323}{325}, -\frac{36}{325}$]

- 6 Solve the equation $4 \sin \theta - 7 \cos \theta + 3 = 0$, giving all values of θ between 0° and 360° correct to the nearest 0.1° . [Ans : $38.4^\circ, 262.1^\circ$]

- 7 Find all values of θ , where $0^\circ < \theta < 360^\circ$, satisfying $\cos 2\theta = 2 \cos \theta$, giving your answers correct to the nearest 0.1° . [Ans : $111.5^\circ, 248.5^\circ$]

- 8 Solve the equation $3 \cos x - 4 \sin x = 2$, where $0^\circ < x < 360^\circ$, giving your answer correct to the nearest 0.1° . [Ans : $13.3^\circ, 240.5^\circ$]

- 9 Express $5 \cos \theta - 12 \sin \theta$ in the form $R \cos (\theta + \alpha)$, where $R \geq 0$ and $0 \leq \alpha < 360^\circ$, stating the value of R and giving the value of α correct to 0.1° . [Ans : $13, 67.4^\circ$]

- 10 Find the values of θ such that $0^\circ < \theta < 360^\circ$ for which $2 \cos 2\theta = 3 - 2 \cos \theta$, giving your answer correct to 0.1° . [Ans : $26.4^\circ, 333.6^\circ$]

- 11 It is given that $\cos A = \frac{5}{13}$, where $-90^\circ < A < 0^\circ$, and that $\tan B = \frac{4}{3}$, where $180^\circ < B < 270^\circ$. Without using a calculator or mathematical tables, find the exact value of $\sin (A + B)$ [Ans : $\frac{16}{65}$]

- 12 (i) Write down an expression for $\tan 2x$ in terms of $\tan x$, and hence show that $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$.
 (ii) Solve the equation $\tan 3x = 11 \tan x$, giving all solutions such that $0 \leq x \leq \pi$.

$$[\text{Ans : (i) } \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \quad \text{(ii) } 0, 0.464, 2.68, \pi]$$

- 13 Prove the identity $\cos 2\theta + \tan \theta \sin 2\theta \equiv 1$.

By letting $\theta = 15^\circ$, find the exact value of $\tan 15^\circ$.

$$[\text{Ans : } 2 - \sqrt{3}]$$

- 14 Prove the identity $\tan A + \cot A \equiv \frac{2}{\sin 2A}$.

Hence, or otherwise, solve the equation $\tan (\theta + 45^\circ) + \cot (\theta + 45^\circ) = 4$, giving all the solutions in the interval $0^\circ < \theta < 360^\circ$.

$$[\text{Ans : } 30^\circ, 150^\circ, 210^\circ, 330^\circ]$$

CHAPTER 13

DIFFERENTIATION 2

13.1 Derivative of exponential functions

Let $y = e^{f(x)}$, where $f(x)$ is a function of x .

$$\ln y = f(x)$$

differentiating both sides w.r.t. x , we have

$$\frac{d}{dx} (e^{f(x)}) = f'(x) \cdot y$$

$$\text{Hence } \frac{d}{dx} (e^{f(x)}) = f'(x) \cdot e^{f(x)}$$

So, to differentiate an exponential function, we have to rewrite the function as it is and then multiply by the derivative of the power.

Example

Differentiate the following w.r.t. x .

(a) e^{3x}

(b) e^{-2x}

(c) $e^{3\sin x}$

(d) $e^{3x} + \frac{1}{e^{4x}}$

Solution

(a) $\frac{d}{dx} (e^{3x}) = 3e^{3x}$

(b) $\frac{d}{dx} (e^{-2x}) = -2e^{-2x}$

(c) $\frac{d}{dx} (e^{3\sin x}) = 3\cos x \cdot e^{3\sin x}$

(d) $e^{3x} + \frac{1}{e^{4x}} = e^{3x} + e^{-4x}$

$$\begin{aligned} \frac{d}{dx} (e^{3x} + e^{-4x}) &= 3e^{3x} + 4e^{-4x} \\ &= 3e^{3x} + \frac{4}{e^{4x}} \end{aligned}$$

Exercise 13.1

Differentiate the following exponential functions w.r.t.x.

1 e^{4x} [Ans : $4e^{4x}$]

2 $e^{\frac{1}{2}x}$ [Ans : $\frac{1}{2}e^{\frac{1}{2}x}$]

3 e^{-2x} [Ans : $-2e^{-2x}$]

4 $e^{\cos x}$ [Ans : $-\sin x e^{\cos x}$]

5 e^{3x^2} [Ans : $6xe^{3x^2}$]

6 e^{x^2-3x} [Ans : $(2x-3)e^{x^2-3x}$]

7 $e^{\sqrt{x}}$ [Ans : $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$]

8 $e^{\frac{1}{x}}$ [Ans : $-\frac{1}{x^2}e^{\frac{1}{x}}$]

9 $\frac{1}{e^{1-3x}}$ [Ans : $3e^{3x-1}$]

10 $e^{2x} + \frac{1}{e^{3x}}$ [Ans : $2e^{2x} - 3e^{-3x}$]

11 $e^{4x} + \frac{1}{e^{2x}}$ [Ans : $4e^{4x} - 2e^{-2x}$]

12 $e^{\tan x}$ [Ans : $\sec^2 x e^{\tan x}$]

13.2 Derivative of logarithmic functions

Consider $y = \ln f(x)$, where $f(x)$ is a function of x .

Let $y = \ln u$ and $u = f(x)$

Then $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = f'(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} f'(x) = \frac{f'(x)}{f(x)}$$

Hence $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$ and in particular $\frac{d}{dx}[\ln(ax+b)] = \frac{a}{ax+b}$.

Example

Differentiate the following w.r.t. x

(a) $\ln(2-5x)$ (b) $\ln(x^3+2x)$ (c) $\ln\left(\frac{3x^2-1}{x^3+2}\right)$

Solution

(a) $\frac{d}{dx} \ln(2-5x) = \frac{-5}{2-5x}$

(b) $\frac{d}{dx} \ln(x^3+2x) = \frac{3x^2+2}{x^3+2x}$

(c) $\frac{d}{dx} \ln\left(\frac{3x^2-1}{x^3+2}\right) = \frac{d}{dx} [\ln(3x^2-1) - \ln(x^3+2)]$
 $= \frac{6x}{3x^2-1} - \frac{3x^2}{x^3+2}$

Exercise 13.2

Differentiate the following w.r.t. x .

1 $\ln(2+3x)$ [Ans: $\frac{3}{2+3x}$]

6 $\ln\sqrt{1+x}$ [Ans: $\frac{1}{2(1+x)}$]

2 $\ln(2x^2+1)$ [Ans: $\frac{4x}{2x^2+1}$]

7 $\ln\left(\frac{x^2-2}{3x+1}\right)$ [Ans: $\frac{2x}{x^2-2} - \frac{3}{3x+1}$]

3 $\ln \sin x$ [Ans: $\cot x$]

8 $\ln\sqrt{x^2+5}$ [Ans: $\frac{x}{x^2+5}$]

4 $\ln\frac{2}{x}$ [Ans: $\frac{-1}{x}$]

9 $\ln\sqrt{\frac{2+x}{2-x}}$ [Ans: $\frac{2}{4-x^2}$]

5 $\ln \cos x$ [Ans: $-\tan x$]

10 $\ln\sqrt{x}$ [Ans: $\frac{1}{2x}$]

13.3 Derivative of trigonometric functions

We expect the reader to be familiar with the following

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\cos x) &= -\sin x \\ \frac{d}{dx}(\tan x) &= \sec^2 x\end{aligned}$$

Example 1

Differentiate the following w.r.t.x.

(a) $\sin 2x$

(b) $\cos(1-x^2)$

(c) $\tan(3x+5)^2$

Solution

(a) $\frac{d}{dx}(\sin 2x) = \cos 2x \cdot \frac{d}{dx}(2x) = 2 \cos 2x$

(b) $\begin{aligned}\frac{d}{dx}[\cos(1-x^2)] &= -\sin(1-x^2) \cdot \frac{d}{dx}(1-x^2) \\ &= -2x \cdot -\sin(1-x^2) \\ &= 2x \sin(1-x^2)\end{aligned}$

(c) $\begin{aligned}\frac{d}{dx}[\tan(3x+5)^2] &= \sec^2(3x+5)^2 \cdot \frac{d}{dx}(3x+5)^2 \\ &= 2(3x+5) \times 3 \times \sec^2(3x+5)^2 \\ &= 6(3x+5) \sec^2(3x+5)^2\end{aligned}$

Example 2

Differentiate the following w.r.t.x.

(a) $\sin^5(3x-1)$

(b) $\cos^7(x^2)$

(c) $\tan^3(2x+1)$

Solution

(a) $\begin{aligned}\frac{d}{dx}[\sin^5(3x-1)] &= 5 \sin^4(3x-1) \frac{d}{dx} \sin(3x-1) \\ &= 5 \sin^4(3x-1) \times 3 \cos(3x-1) \\ &= 15 \cos(3x-1) \sin^4(3x-1)\end{aligned}$

(b) $\begin{aligned}\frac{d}{dx}[\cos^7(x^2)] &= 7 \cos^6(x^2) \cdot \frac{d}{dx} \cos(x^2) \\ &= 7 \cos^6(x^2) \cdot -2x \sin(x^2) \\ &= -14x \cos^6(x^2) \sin(x^2)\end{aligned}$

(c) $\begin{aligned}\frac{d}{dx}[\tan^3(2x+1)] &= 3 \tan^2(2x+1) \frac{d}{dx} \tan(2x+1) \\ &= 3 \tan^2(2x+1) \times 2 \sec^2(2x+1) \\ &= 6 \tan^2(2x+1) \sec^2(2x+1)\end{aligned}$

Exercise 13.3

Differentiate the following w.r.t.x.

- | | | |
|----|------------------|--|
| 1 | $\sin 3x$ | [Ans: $3 \cos 3x$] |
| 2 | $\sin(2x - 1)$ | [Ans: $2 \cos(2x - 1)$] |
| 3 | $\cos(x^2)$ | [Ans: $-2x \sin(x^2)$] |
| 4 | $\tan(3x+4)$ | [Ans: $3 \sec^2(3x+4)$] |
| 5 | $\cos(2x + \pi)$ | [Ans: $-2 \sin(2x+\pi)$] |
| 6 | $\tan^2 x$ | [Ans: $2 \tan x \sec^2 x$] |
| 7 | $\sin^2(3x)$ | [Ans: $6 \sin 3x \cos 3x$] |
| 8 | $\cos^5(x^2)$ | [Ans: $-10x \cos^4(x^2) \sin(x^2)$] |
| 9 | $\tan^3 2x$ | [Ans: $6 \tan^2 2x \sec^2 2x$] |
| 10 | $\sin^4(2x+1)$ | [Ans: $8 \sin^3(2x+1) \cos(2x+1)$] |
| 11 | $\cos^3(1-2x)$ | [Ans: $6 \cos^2(1-2x) \sin(1-2x)$] |
| 12 | $\tan^5(3x+\pi)$ | [Ans: $15 \tan^4(3x+\pi) \sec^2(3x+\pi)$] |
| 13 | $\sin(\sqrt{x})$ | [Ans: $\frac{1}{2\sqrt{x}} \cos(\sqrt{x})$] |
| 14 | $\cos(e^x)$ | [Ans: $-e^x \sin(e^x)$] |
| 15 | $\tan(x^3)$ | [Ans: $3x^2 \sec^2(x^3)$] |

13.4 The product rule

If $y = uv$, where u and v are functions of x , then

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

Example 1

If $y = (2x+3)(x+1)^5$, find $\frac{dy}{dx}$.

Solution

$$\begin{aligned} \text{Let } u &= (2x+3) & \text{and } v &= (x+1)^5 \\ \frac{du}{dx} &= 2 & \frac{dv}{dx} &= 5(x+1)^4 \end{aligned}$$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$, we have

$$\begin{aligned} \frac{dy}{dx} &= (2x+3) \times 5(x+1)^4 + (x+1)^5 \times 2 \\ &= (x+1)^4 \{5(2x+3) + 2(x+1)\} \\ &= (x+1)^4 \{10x+15+2x+2\} \\ &= (x+1)^4 (12x+17) \end{aligned}$$

Example 2

Given that $x = e^{3t} \cos 3t$, $y = e^{3t} \sin 3t$, show that $\frac{dy}{dx} = \tan \left(3t + \frac{1}{4}\pi\right)$. (C)

Solution

$$\begin{aligned} x &= e^{3t} \cos 3t & y &= e^{3t} \sin 3t \\ \frac{dx}{dt} &= e^{3t} \times -3 \sin 3t + 3 e^{3t} \cos 3t & \frac{dy}{dt} &= e^{3t} 3 \cos 3t + 3 e^{3t} \sin 3t \\ &= 3e^{3t} (-\sin 3t + \cos 3t) & &= 3e^{3t} (\cos 3t + \sin 3t) \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3e^{3t} (\cos 3t + \sin 3t)}{3e^{3t} (\cos 3t - \sin 3t)} \\ &= \frac{1 + \tan 3t}{1 - \tan 3t} \quad [\text{divide by } \cos 3t] \\ &= \frac{\tan \frac{\pi}{4} + \tan 3t}{1 - \tan \frac{\pi}{4} \tan 3t} \\ &= \tan \left(3t + \frac{1}{4}\pi\right) \text{ (shown)} \end{aligned}$$

Exercise 13.4

Differentiate the following w.r.t.x.

- 1 $e^x \sin x$ [Ans: $e^x (\sin x + \cos x)$]
- 2 $x^2 \tan x$ [Ans: $x (x \sec^2 x + 2 \tan x)$]
- 3 $x \ln x$ [Ans: $1 + \ln x$]
- 4 $e^{2x} \sin 3x$ [Ans: $e^{2x} (3 \cos 3x + 2 \sin 3x)$]
- 5 $x^3 \tan(x^2)$ [Ans: $x^2 [2x^2 \sec^2(x^2) + 3 \tan(x^2)]$]
- 6 $(x+2) \ln (3x-1)$ [Ans: $\frac{3x+6}{3x-1} + \ln(3x-1)$]
- 7 $\sqrt{x} (2x+5)$ [Ans: $\frac{6x+5}{2\sqrt{x}}$]
- 8 $x \sqrt{x+1}$ [Ans: $\frac{3x+2}{2\sqrt{x+1}}$]
- 9 $\ln x \tan x$ [Ans: $\ln x \sec^2 x + \frac{\tan x}{x}$]
- 10 $(2x+1)(x+1)^3$ [Ans: $(x+1)^2(8x+5)$]
- 11 $(4x+3)^2 (x+4)^3$ [Ans: $(4x+3) (x+4)^2 (20x+41)$]
- 12 If $y = 2bx(a^2 + x^2)^{15}$, show that $\frac{dy}{dx} = 2b(a^2 + x^2)^{14} (31x^2 + a^2)$

13.5 The Quotient Rule

If $y = \frac{u}{v}$, where u and v are two functions of x , then

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Example 1

If $y = \frac{x^2}{3x-1}$, find $\frac{dy}{dx}$.

Solution

Let $u = x^2$ and $v = 3x - 1$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x-1) \cdot 2x - x^2 \cdot 3}{(3x-1)^2} \\ &= \frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \\ &= \frac{3x^2 - 2x}{(3x-1)^2}\end{aligned}$$

Example 2

If $y = \frac{\ln x}{x^2}$, find the value of $\frac{dy}{dx}$ when $x = e$. (C)

Solution

Let $u = \ln x$ and $v = x^2$

Then $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 2x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2 \cdot \frac{1}{x} - 2x \cdot \ln x}{(x^2)^2} \\ &= \frac{x - 2x \ln x}{x^4} \\ &= \frac{1 - 2 \ln x}{x^3}\end{aligned}$$

When $x = e$, $\frac{dy}{dx} = \frac{1 - 2 \ln e}{e^3} = -\frac{1}{e^3}$ ($\because \ln e = 1$)

Exercise 13.5

Differentiate the following w.r.t. x or t .

1 $\frac{x}{2x+1}$ [Ans: $\frac{1}{(2x+1)^2}$]

2 $\frac{x^2}{2x+5}$ [Ans: $\frac{2x(x+5)}{(2x+5)^2}$]

3 $\frac{\ln 2x}{x}$ [Ans: $\frac{1 - \ln 2x}{x^2}$]

4 $\frac{\sin x}{\cos x}$ [Ans: $\sec^2 x$]

5 $\frac{\sin x}{2 - \cos x}$ [Ans: $\frac{2 \cos x - 1}{(2 - \cos x)^2}$]

6 If $y = \frac{x}{\sqrt{1-3x^2}}$, show that $\frac{dy}{dx} = \frac{1}{(1-3x^2)^{3/2}}$.

7 $\frac{e^{2x}}{\cos x}$ [Ans: $\frac{e^{2x}(2 \cos x + \sin x)}{\cos^2 x}$]

8 Show that the gradient of the curve $y = \frac{3x^2 - 8}{5 - 2x}$ at the point (2,4) is 20.

9 Use differentiation to find the x -coordinates of the points at which the graph of $y = \frac{x^2}{2x-1}$ has either a maximum or a minimum and distinguish between them.
[Ans: $x = 0$ (max) ; $x = 1$ (min)]

10 If $y = \frac{3t}{t^2 + 1}$, show that $\frac{dy}{dt} = \frac{3(1-t^2)}{(1+t^2)^2}$.

13.6 Differentiation of implicit functions

When a variable y is expressed directly in terms of x , y is said to be an explicit function of x . For example $y = 3x + 5$ and $y = 2x^2 - 1$ are explicit functions of x .

When both x and y occur on one or both sides of an equation, then either x or y is said to be an implicit function of the other. For example $2xy^3 = 3x^2 + 5y$ is an implicit function.

Example 1

A curve is defined by the equation $x^3 + y^3 + 3xy - 1 = 0$. Find the gradient of the curve at the point $(2, -1)$. (C)

Solution

Differentiating w. r. t. x , we have

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) + \frac{d}{dx}(3xy) - \frac{d}{dx}(1) = 0$$

$$3x^2 + 3y^2 \frac{dy}{dx} + [3x \cdot \frac{dy}{dx} + 3y] = 0 \quad [\text{Use product rule to differentiate } 3xy]$$

$$x^2 + y^2 \frac{dy}{dx} + x \frac{dy}{dx} + y = 0 \quad [\text{divide by } 3]$$

$$(y^2 + x) \frac{dy}{dx} = -(x^2 + y)$$

$$\frac{dy}{dx} = -\frac{x^2 + y}{y^2 + x}$$

$$\text{At the point } (2, -1), \quad \frac{dy}{dx} = -\frac{2^2 - 1}{(-1)^2 + 2} = -1 \quad (\text{Ans})$$

Example 2

Differentiate w.r.t. x $y^2 = e^{2x}y + xe^x$ to find $\frac{dy}{dx}$ in terms of x and y . (C)

Solution

Differentiating w. r. t. x , we have

$$\frac{d}{dx}(y^2) = y \frac{d}{dx}(e^{2x}) + e^{2x} \frac{d}{dx}(y) + x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x) = 0$$

$$2y \frac{dy}{dx} = y \cdot 2e^{2x} + e^{2x} \cdot \frac{dy}{dx} + x e^x + e^x$$

$$(2y - e^{2x}) \frac{dy}{dx} = 2ye^{2x} + x e^x + e^x$$

$$\frac{dy}{dx} = \frac{2ye^{2x} + (x+1)e^x}{2y - e^{2x}} \quad (\text{Ans})$$

Exercise 13.6

Differentiate the following w.r.t. x .

- 1 $3x^2 + y^2 = 1$ [Ans: $-\frac{3x}{y}$]

- 2 $2x^2 - 5y^2 - 3x = 5$ [Ans: $\frac{4x-3}{10y}$]

- 3 $x^2 + 3y^2 - 3x - 2y = 7$ [Ans: $\frac{3-2x}{6y-2}$]

- 4 $3x^2 + y^2 - 7x - 4y = 1$ [Ans: $\frac{3-2x}{6y-2}$]

- 5 $x^2 + xy + y^3 = 1$ [Ans: $-\frac{y+2x}{3y^2+x}$]

- 6 $x^2 + 3xy - 2x = 5$ [Ans: $\frac{2-2x-3y}{3x}$]

- 7 Find the gradient of the curve $x^3 + y^3 = 9$ at the point $(1,2)$. (C) [Ans: $-\frac{1}{4}$]

- 8 The equation of a curve is $y^4 + x^2y^2 = 4a^3(x+4a)$, where a is a constant. Find the gradient of the curve at the point $(a,2a)$. (C) [Ans: $-\frac{1}{9}$]

- 9 Find the equation of the normal to the curve $3y^2 - x^2 = 3$ at the point $(3,2)$. (C) [Ans: $2x + y = 8$]

- 10 Find the equation of the normal to the curve $y - x + 3e^{-y} = 0$ at the point $(0,0)$. (C). [Ans: $y = 2x$]

- 11 Given that $\tan x + \tan y = 3$, find the value of $\frac{dy}{dx}$ when $x = \frac{1}{4}\pi$. (C) [Ans: $-\frac{2}{5}$]

- 12 The equation of a curve is $(y-x)^2 = 2a(y+x)$, where a is a constant. Find an expression for $\frac{dy}{dx}$ in terms of x , y and a . [Ans: $\frac{y-x+a}{y-x-a}$]

13.7 Differentiation of parametric equations

For the parametric equation with parameter t ,

$$x = f(t), y = g(t)$$

$$\text{Then } \frac{dx}{dt} = f'(t), \quad \frac{dy}{dt} = g'(t)$$

By using the rule of a function of a function

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\text{or } \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

Example 1

If $x = t^2$ and $y = 3t + 1$, find $\frac{dy}{dx}$ in terms of t .

Solution

$$x = t^2 \qquad y = 3t + 1$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3$$

$$\therefore \frac{dy}{dx} = \frac{3}{2t}$$

Example 2

The parametric equation of a curve C are $x = t + e^t$, $y = t + e^{-t}$. Find $\frac{dy}{dx}$ in terms of t , and hence find the coordinates of the stationary point C . (C)

Solution

$$\begin{aligned} \text{We have } x &= t + e^t & y &= t + e^{-t} \\ \frac{dx}{dt} &= 1 + e^t & \frac{dy}{dt} &= 1 - e^{-t} \end{aligned}$$

$$\text{Using } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1 - e^{-t}}{1 + e^t}$$

At stationary point, $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{1 - e^{-t}}{1 + e^t} = 0$$

$$\Rightarrow 1 - e^{-t} = 0$$

$$\Rightarrow e^{-t} = 1$$

$$\Rightarrow -t = \ln 1 = 0$$

$$\text{When } t = 0, \quad x = 0 + e^0 = 1, \quad y = 0 + e^0 = 1$$

Hence the required point is $(1, 1)$.

Exercise 13.7

Differentiate the following with respect to x . (a is a constant)

1 $x = t^3, y = 3t + 1$ [Ans: $\frac{1}{t^2}$]

2 $x = 2t + 5, y = 3t + 4$ [Ans: $\frac{3}{2}$]

3 $x = a \cos t, y = a \sin t$ [Ans: $-\cot t$]

4 $x = a(1 + \sin \theta), y = a(1 - \cos \theta)$ [Ans: $\tan \theta$]

5 $x = \frac{2}{t}, y = \frac{3}{t^2}$ [Ans: $\frac{3}{t}$]

6 $x = \frac{t}{1+t}, y = \frac{3t}{1+t}$ [Ans: 3]

7 $x = \frac{t^2}{t-1}, y = \frac{2t}{t-1}$ [Ans: $\frac{-2}{t^2 - 2t}$]

8 A curve is defined parametrically by $x = t^2 - 4, y = 3t^4 + 8t^3$. Find the equation of the tangent to the curve at the point where $t = -1$. (C)
[Ans: $6x + y + 23 = 0$]

9 the parametric equation of a curve are $x = a\left(1 + \frac{1}{t}\right), y = a\left(t - \frac{1}{t}\right)$,

where a is a constant.

Find the equation of the normal to the curve at the point where $t = 2$. (C)
[Ans: $4x - 20y + 29a = 0$]

10 A curve has parametric equations $x = 2t, y = \frac{1}{t^2}$, where t is a non-zero parameter. Sketch the curve. Find the equations of the tangent and the normal to the curve at the $\left(2t, \frac{1}{t^2}\right)$. The tangent at the point $P\left(4, \frac{1}{4}\right)$ meets the y -axis at T . The normal at P meets the y -axis at N . Show that the area of ΔPTN is 65.

[Ans: $x + t^3y = 3t; t^3x + \frac{1}{t^2} = y + 2t^4$]

DIFFERENTIATION 2 – PAST EXAMINATION QUESTIONS

1 Differentiate with respect to x

(a) $e^{2x} \sin 3x$,

(b) $\ln(1+x^2)$.

[Ans: (a) $e^{2x}(2\sin 3x + 3\cos 3x)$ (b) $\frac{2x}{1+x^2}$]

2 Differentiate with respect to x

(a) $\frac{1}{\sqrt{3x^2-1}}$

(b) $x^2 \ln x$,

(c) $\tan^2(5x)$;

[Ans: (a) $\frac{-3x}{(3x^2-1)^{\frac{3}{2}}}$ (b) $x(2\ln x + 1)$ (c) $10 \tan 5x \sec^2 5x$]

3 (a) Differentiate, with respect to x

(i) $\sec^3(2x)$,

(ii) $\ln(\sin x)$.

(b) Find $\frac{dy}{dx}$ in terms of x and y if $x^3 + y^3 = 3xy$.

[Ans (a) (i) $6 \tan 2x \sec^3 2x$ (ii) $\cot x$ (b) $\frac{y-x^2}{y^2-x}$]

4(a) Differentiate xe^{-x^2} with respect to x :

(b) Find the gradient of the curve $x^3 + xy^2 = 5$ at the point $(1, 2)$.

[Ans: (a) $(1-2x^2)e^{-x^2}$ (b) $-\frac{7}{4}$]

5 If $y = ae^{\sin x}$, where a is a constant, prove that $\frac{d^2y}{dx^2} = \frac{dy}{dx} (\cos x - \tan x)$.

6 Differentiate with respect to x :

(a) $\frac{e^{-x}}{x}$

(b) $\ln(\sin 2x)$.

[Ans: (a) $\frac{-e^{-x}(x+1)}{x^2}$ (b) $2 \cot 2x$]

7 Find $\frac{dy}{dx}$ when

(a) $y = e^{\sqrt{x}}$,

(b) $y = \frac{\tan x}{2x+1}$.

[Ans: (a) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ (b) $\frac{(2x+1)\sec^2 x - 2 \tan x}{(2x+1)^2}$]

8 Find, in terms of x , both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each of the following cases:

(a) $y = \sqrt{x^2 + 1}$;

(b) $y = e^{-\frac{1}{2}x^2}$

[Ans: (a) $\frac{x}{\sqrt{x^2+1}}, \frac{1}{(x^2+1)^{\frac{3}{2}}}$ (b) $-xe^{-\frac{1}{2}x^2}, (x^2-1)e^{-\frac{1}{2}x^2}$]

- 9 Differentiate with respect to x :
 (a) $x^2 \cos x$, (b) $\frac{\ln(2x)}{x}$, (c) $\sqrt{x + \frac{1}{x}}$.

$$[\text{Ans: (a) } 2x \cos x - x^2 \sin x \quad (b) \frac{1 - \ln(2x)}{x^2} \quad (c) \frac{1 - \sqrt{x}}{2\sqrt{x + \frac{1}{x}}}]$$

- 10 Find, in terms of x and y , the gradient at the point (x, y) on the curve with equation $x^3 + 2y^3 = 3x^2y^2$. Hence find the point (x, y) , other than the origin, at which $\frac{dy}{dx} = 0$.

$$[\text{Ans: } \frac{2xy^2 - x^2}{2y^2 - 2x^2y}, \left(2^{\frac{1}{3}}, 0.5^{\frac{1}{3}}\right)]$$

- 11 Differentiate with respect to x (a) $\ln(1 + \cos x)$, (b) $\frac{x^2 - 9x}{2x + 3}$.

$$[\text{Ans: (a) } -\frac{\sin x}{1 + \cos x} \quad (b) \frac{2x^2 + 6x - 27}{(2x + 3)^2}]$$

- 12 A curve has parametric equations $x = 2t - \ln(2t)$, $y = t^2 - \ln(t^2)$, where $t > 0$. Find the value of t at the point on the curve at which the gradient is 2. [Ans: 2]

- 13 Differentiate with respect to x (a) $\frac{1}{(3-4x)^2}$, (b) $\cos(\sqrt{x})$.

$$[\text{Ans: (a) } \frac{8}{(3-4x)^3} \quad (b) -\frac{\sin \sqrt{x}}{2\sqrt{x}}]$$

- 14 Differentiate with respect to t (a) $\frac{1}{(3t^2+5)^2}$, (b) $\ln\left(\frac{4}{t^2}\right)$.

$$[\text{Ans: (a) } -\frac{12t}{(3t^2+5)^3} \quad (b) -\frac{2}{t}]$$

- 15 Differentiate $3u^2 \sin 5u$ with respect to u . [Ans: $15u^2 \cos 5u + 6u \sin 5u$]

- 16 Given that $y = \cos^2 x$, find $\frac{dy}{dx}$. [Ans: $-2 \cos x \sin x$ or $-\sin 2x$]

- 17 Given that $x = \frac{1}{1+t^2}$ and $y = \frac{t}{1+t^2}$, where t is a parameter, find $\frac{dy}{dx}$ in terms of t .

$$[\text{Ans: } \frac{1}{2}t - \frac{1}{2t}]$$

- 18 A curve is given parametrically by $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, where $t \neq 0$. Find the coordinates of the points on the curve where the gradient is zero, and find the equation of the tangent at the point where $t = 2$.
[Ans : (0,2) or (0,-2) ; $3x - 5y + 8 = 0$]
- 19 Find the equation of the tangent to the curve given by $x = 1 - \cos t$, $y = t + \sin t$, at the point where $t = \frac{1}{2}\pi$.
[Ans : $y = x + \frac{1}{2}\pi$]
- 20 The equation of a curve is $2x^2 + 8xy + 5y^2 = -3$. Find the equations of the two tangents which are parallel to the x -axis. [Ans : $y = 1$ and $y = -1$]
- 21 A curve has equation $x^2 + xy + y^2 = 3$.
(i) Show that $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$.
(ii) Find the coordinates of the points on this curve where the gradient is zero. [Ans : (ii) (1,-2), (-1,2)]
- 22 Obtain the coordinates of the turning point on the curve with equation $y = x^2 \ln x$, where $x > 0$, and find whether it is a maximum point or a minimum point.
[Ans : $\left(e^{-\frac{1}{2}}, -\frac{1}{2e}\right)$]
- 23 Find the coordinates of the turning points on the curve with equation $y = \frac{x^2}{1+x^4}$. Sketch the curve. [Ans : (0,0), (1, $\frac{1}{2}$), (-1, $\frac{1}{2}$)]
- 24 Show that there is one stationary point on the curve $y = e^x + 2e^{-x}$, and that the stationary value of y is $2\sqrt{2}$.
- 25 A cylinder of radius x is inscribed in a fixed sphere of internal radius R , the circumferences of the circular ends of the cylinder being in contact with the inner surface of the sphere. Show that $A^2 = 16\pi^2 x^2 (R^2 - x^2)$, where A is the curved surface area of the cylinder. Prove that, if x is made to vary, the maximum value of A is obtained when the height of the cylinder is equal to its diameter.
- 26 A piece of wire of length 8 cm is cut into two pieces, one of length x cm, the other of length $(8 - x)$ cm. The piece of length x cm is bent to form a circle with circumference x cm. The other piece is bent to form a square with perimeter $(8 - x)$ cm. Show that, as x varies, the sum of the areas enclosed by these two pieces of wire is a minimum when the radius of the circle is $\frac{4}{4+\pi}$ cm.

CHAPTER 14

INTEGRATION 2

14.1 Integration of exponential functions

Exponential functions can be integrated provided the power is **linear**.

To differentiate exponential functions

step 1 : rewrite the function as it is

step 2 : differentiate the power and divide.

Example

Integrate the following w.r.t.x.

(a) e^{4x} (b) e^{1-3x} (c) $\frac{1}{e^{1-2x}}$

Solution

(a) $\int e^{4x} dx = \frac{e^{4x}}{4} + c$ (b) $\int e^{1-3x} dx = \frac{e^{1-3x}}{-3} + c$

(c) $\int \frac{1}{e^{1-2x}} dx = \int e^{2x-1} dx = \frac{e^{2x-1}}{2} + c$

Exercise 14.1

Integrate the following w.r.t.x.

1 e^{3x} [Ans: $\frac{e^{3x}}{3} + c$] 6 $e^x(e^x + 2)$ [Ans: $\frac{e^{2x}}{2} + 2e^x + c$]

2 e^{2x+5} [Ans: $\frac{e^{2x+5}}{2} + c$] 7 $e^x(e^{2x} + 1)$ [Ans: $\frac{e^{3x}}{3} + e^x + c$]

3 $e^{\frac{1}{2}x}$ [Ans: $2e^{\frac{1}{2}x} + c$] 8 $(e^x + 2)^2$ [Ans: $\frac{e^{2x}}{2} + 4e^x + 4x + c$]

4 $e^{\frac{5}{3}x}$ [Ans: $\frac{3}{5}e^{\frac{5}{3}x} + c$] 9 $\frac{e^{3x} - 2}{e^x}$ [Ans: $\frac{e^{2x}}{2} + 2e^{-x} + c$]

5 $\frac{1}{2}e^{4x-1}$ [Ans: $\frac{1}{8}e^{4x-1} + c$] 10 $3e^{-3x}$ [Ans: $-e^{-3x} + c$]

14.2 Integration of the type $\frac{1}{ax+b}$

We have seen that for $x > 0$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Thus $\int \ln x dx = \frac{1}{x} + c$

Also $\frac{d}{dx}[\ln(ax+b)] = \frac{a}{ax+b}$

Therefore $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(ax+b) + c$

Example 1

(a) $\int \frac{1}{5x+2} dx$

(b) $\int \frac{7}{3x+1} dx$

(c) $\int \frac{a}{b-cx} dx$

Solution

(a) $\int \frac{1}{5x+2} dx = \frac{1}{5} \ln(5x+2) + c$

(b) $\int \frac{7}{3x+1} dx = \frac{7}{3} \ln(3x+1) + c$

(c) $\int \frac{a}{b-cx} dx = -\frac{a}{c} \ln(b-cx) + k$

Example 2

(a) $\int_1^5 \frac{5}{x} dx$

(b) $\int_1^2 \frac{5}{3x+2} dx$

Solution

(a) $\int_1^5 \frac{5}{x} dx = [5 \ln x]_1^5 = 5 \ln 5 - 5 \ln 1 = 5 \ln 5 = \ln 3125$

(b) $\int_1^2 \frac{5}{3x+2} dx = \left[\frac{5}{3} \ln(3x+2) \right]_1^2$
 $= \left[\frac{5}{3} \ln[3(2)+2] \right] - \left[\frac{5}{3} \ln[3(1)+2] \right]$
 $= \frac{5}{3} (\ln 8 - \ln 5)$
 $= \frac{5}{3} \ln \frac{8}{5}$

Exercise 14.2

Integrate the following indefinite integrals:

1 $\int \frac{1}{3x+4} dx$ [Ans : $\frac{1}{3} \ln(3x+4) + c$]

2 $\int \frac{1}{2x-3} dx$ [Ans : $\frac{1}{2} \ln(2x-3) + c$]

3 $\int \frac{4}{5x-1} dx$ [Ans : $\frac{4}{5} \ln(5x-1) + c$]

4 $\int \frac{2}{4-3x} dx$ [Ans : $-\frac{2}{3} \ln(4-3x) + c$]

5* $\int \frac{5}{2x} dx$ [Ans : $\frac{5}{2} \ln x + c$]

6 $\int 3 - \frac{4}{x} dx$ [Ans : $3x - 4 \ln x + c$]

7 $\int \frac{p}{q-2r} dx$ [Ans : $-\frac{p}{2} \ln(q-2r) + c$]

Integrate the following definite integrals giving your answer in an exact form:

8 $\int_0^1 \frac{1}{3x+1} dx$ [Ans : $\frac{1}{3} \ln 4$]

9 $\int_3^5 \frac{3}{x-2} dx$ [Ans : $\ln 27$]

10 $\int_1^8 \frac{5+x}{x} dx$ [Ans : $5 \ln 8 + 7$]

11 $\int_1^5 \frac{x^4 + 3x^2}{x^3} dx$ [Ans : $12 + \ln 125$]

12 $\int_{-1}^0 \frac{3}{4-2x} dx$ [Ans : $\frac{3}{2} \ln \frac{3}{2}$]

14.3 Integration of trigonometric functions

$$\text{If } y = \sin x, \quad \frac{dy}{dx} = \cos x$$

$$\text{If } y = \cos x, \quad \frac{dy}{dx} = -\sin x$$

$$\text{If } y = \tan x, \quad \frac{dy}{dx} = \sec^2 x$$

$$\text{Hence} \quad \int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

Example 1

$$(a) \int 3 + \cos x \, dx$$

$$(b) \int 2 + \tan^2 x \, dx$$

Solution

$$(a) \int 3 + \cos x \, dx = 3x + \sin x + c$$

$$\begin{aligned}(b) \int 2 + \tan^2 x \, dx &= \int 2 + \sec^2 x - 1 \, dx \\ &= \int 1 + \sec^2 x \, dx \\ &= x + \tan x + c\end{aligned}$$

Example 2

$$(a) \int \cos(5x+3) \, dx$$

$$(b) \int \sin\left(\frac{2}{3}x+1\right) \, dx$$

$$(c) \int \sec^2(3-4x) \, dx$$

Solution

$$(a) \int \cos(5x+3) \, dx = \frac{1}{5} \sin(5x+3) + c$$

$$(b) \int \sin\left(\frac{2}{3}x+1\right) \, dx = -\frac{3}{2} \cos\left(\frac{2}{3}x+1\right) + c$$

$$(c) \int \sec^2(3-4x) \, dx = -\frac{1}{4} \tan(3-4x) + c$$

Exercise 14.3

Integrate the following with respect to x :

1 $2 + \cos x$ [Ans : $2x + \sin x + c$]

2 $3 - \sin x$ [Ans : $3x + \cos x + c$]

3 $\cos 3x$ [Ans : $\frac{1}{3} \sin 3x + c$]

4 $\sin(2x+1)$ [Ans : $-\frac{1}{2} \cos(2x+1) + c$]

5 $\sin(1-3x)$ [Ans : $\frac{1}{3} \cos(1-3x) + c$]

6 $\cos(5x-3)$ [Ans : $\frac{1}{5} \sin(5x-3) + c$]

7 $\tan^2 x$ [Ans : $\tan x - x + c$]

8 $3 \sec^2 5x$ [Ans : $\frac{3}{5} \tan 5x + c$]

9 $2 \cos 2x - \sin 4x$ [Ans : $\sin 2x + \frac{1}{4} \cos 4x + c$]

10 $\sin x + \sin 2x - \sin 5x$ [Ans : $-\cos x - \frac{1}{2} \cos 2x + \frac{1}{5} \cos 5x + c$]

11 $\sec(\sqrt{3}x)$ [Ans : $\frac{1}{\sqrt{3}} \tan(\sqrt{3}x) + c$]

12 $\cos 2x + \frac{1}{\cos^2 3x}$ [Ans : $\frac{1}{2} \sin 2x + \frac{1}{3} \tan 3x + c$]

13 $\frac{1}{x^3} + \sec^2 2x$ [Ans : $-\frac{1}{2x^2} - \frac{1}{2} \tan 2x + c$]

14.4 Use of double angle formulae to integrate $\cos^2 x$ and $\sin^2 x$

We have seen earlier that

$$\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2 \cos^2 x - 1 \\ 1 - 2 \sin^2 x \end{cases}$$

$$\cos 2x = 2 \cos^2 x - 1$$

$$2 \cos^2 x = 1 + \cos 2x$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

So if we have to integrate $\cos^2 x$, we integrate $\frac{1}{2}(1 + \cos 2x)$.

$$\begin{aligned} \text{Hence } \int \cos^2 x \, dx &= \int \frac{1}{2}(1 + \cos 2x) \, dx \\ &= \frac{1}{2} \int (1 + \cos 2x) \, dx \\ &= \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right] + c \end{aligned}$$

Similarly $\cos 2x = 1 - 2 \sin^2 x$

$$2 \sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

So if we have to integrate $\sin^2 x$, we integrate $\frac{1}{2}(1 - \cos 2x)$.

$$\begin{aligned} \text{Hence } \int \sin^2 x \, dx &= \int \frac{1}{2}(1 - \cos 2x) \, dx \\ &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + c \end{aligned}$$

Example

(a) $\int \cos^2 2x \, dx$

(b) $\int \sin^2 \frac{3}{2}x \, dx$

Solution

(a) $\int \cos^2 2x \, dx = \int \frac{1}{2}(1 + \cos 2(2x)) \, dx$

$$= \frac{1}{2} \int (1 + \cos 4x) \, dx$$

$$= \frac{1}{2} \left[x + \frac{\sin 4x}{4} \right] + c$$

(b) $\int \sin^2 \frac{3}{2}x \, dx = \int \frac{1}{2} \left(1 - \cos 2 \left(\frac{3}{2}x \right) \right) \, dx$

$$= \frac{1}{2} \int (1 - \cos 3x) \, dx$$

$$= \frac{1}{2} \left[x - \frac{\sin 3x}{3} \right] + c$$

Exercise 14.4

Integrate the following with respect to x :

- 1 $\cos^2 3x$ [Ans : $\frac{1}{2}\left[x + \frac{\sin 6x}{6}\right] + c$]
- 2 $\cos^2 5x$ [Ans : $\frac{1}{2}\left[x + \frac{\sin 10x}{10}\right] + c$]
- 3 $\cos^2 \frac{3}{2}x$ [Ans : $\frac{1}{2}\left[x + \frac{\sin 3x}{3}\right] + c$]
- 4 $\cos^2 nx$ [Ans : $\frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right] + c$]
- 5 $(2\cos x)^2$ [Ans : $2\left[x + \frac{\sin 2x}{2}\right] + c$]
- 6 $(\cos x + \sin x)^2$ [Ans : $\left[x - \frac{\cos 2x}{2}\right] + c$]
- 7 $\sin^2 2x$ [Ans : $\frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] + c$]
- 8 $\sin^2 \frac{5}{2}x$ [Ans : $\frac{1}{2}\left[x - \frac{\sin 5x}{5}\right] + c$]
- 9 $\sin^2 \frac{2}{3}x$ [Ans : $\frac{1}{2}\left[x - \frac{3}{4}\sin \frac{4x}{3}\right] + c$]
- 10 $\sin^2 nx$ [Ans : $\frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right] + c$]
- 11 $(3\sin x)^2$ [Ans : $\frac{9}{2}\left[x - \frac{\sin 2x}{2}\right] + c$]
- 12 $\sin^2 \frac{1}{2}x$ [Ans : $\frac{1}{2}[x - \sin x] + c$]

14.5 Integration using partial fractions

We have already come across partial fractions in chapter 10. In this section we are going to integrate rational functions by means of decomposition into partial fractions.

Example 1

Show that $\int \frac{1}{(x-1)(2-x)} dx$

Solution

$$\text{Let } \frac{1}{(x-1)(2-x)} \equiv \frac{A}{(x-1)} + \frac{B}{(2-x)}$$

$$1 \equiv A(2-x) + B(x-1)$$

$$\text{When } x = 1, \quad 1 = A$$

$$\text{When } x = 2, \quad 1 = B$$

$$\begin{aligned}\therefore \int \frac{1}{(x-1)(2-x)} &= \int \frac{1}{x-1} + \frac{1}{2-x} dx \\ &= \ln(x-1) - \ln(2-x) + \ln A \\ &= \ln A \left(\frac{x-1}{2-x} \right) \quad (\text{Ans})\end{aligned}$$

Example 2

Show that $\int_1^2 \frac{6x+7}{(2x-1)(x+2)} dx = \ln(12)$ (C)

Solution

$$\text{Let } \frac{6x+7}{(2x-1)(x+2)} \equiv \frac{A}{(2x-1)} + \frac{B}{(x+2)}$$

$$6x+7 \equiv A(x+2) + B(2x-1)$$

$$\text{When } x = -2, \quad 6(-2) + 7 = -5B$$

$$B = 1$$

$$\text{When } x = \frac{1}{2}, \quad 6\left(\frac{1}{2}\right) + 7 = \frac{5}{2}A$$

$$A = 4$$

$$\therefore \frac{6x+7}{(2x-1)(x+2)} = \frac{4}{(2x-1)} + \frac{1}{(x+2)}$$

$$\begin{aligned}\text{Now } \int_1^2 \frac{6x+7}{(2x-1)(x+2)} dx &= \int_1^2 \frac{4}{(2x-1)} + \frac{1}{(x+2)} dx \\ &= [2 \ln(2x-1) + \ln(x+2)]_1^2 \\ &= [2 \ln 3 + \ln 4] - [2 \ln 1 + \ln 3] \\ &= \ln 9 + \ln 4 - 0 - \ln 3 \\ &= \ln \left(\frac{9 \times 4}{3} \right) \\ &= \ln(12) \quad (\text{shown})\end{aligned}$$

Exercise 14.5

Integrate the following with respect to x :

1 $\int \frac{3}{(x-1)(x+2)} dx$ [Ans : $\ln\left(\frac{x-1}{x+2}\right) + c$]

2 $\int \frac{3x+5}{(x-3)(x+4)} dx$ [Ans : $2\ln(x-3) + \ln(x+4) + c$]

3 $\int \frac{1}{(x-3)(x+3)} dx$ [Ans : $\frac{1}{6} \ln\left(\frac{x-3}{x+3}\right) + c$]

4 $\int \frac{1}{(x+1)(x+3)} dx$ (C) [Ans : $\frac{1}{2} \ln\left(\frac{x+1}{x+3}\right) + c$]

5 $\int \frac{1}{x(x+1)} dx$ (C) [Ans : $\ln\left(\frac{Ax}{x+1}\right)$]

6 $\int \frac{1}{(x+1)(2-x)} dx$ (C) [Ans : $\frac{1}{3} \ln\left(\frac{x+1}{2-x}\right) + c$]

7 $\int \frac{1}{x^2+3x+2} dx$ (C) [Ans : $\ln\left(\frac{x+1}{x+2}\right) + c$]

8 $\int \frac{1}{x^2-2x-3} dx$ (C) [Ans : $\frac{3}{4} \ln(x-3) + \frac{1}{4} \ln(x+1) + c$]

9 Show that $\int_7^{10} \frac{7}{x^2-5x-6} dx = \ln\left(\frac{32}{11}\right)$. (C)

10 Express $\frac{3}{(2x+1)(x-1)}$ in partial fractions. Hence find the exact value of

$\int_2^3 \frac{3}{(2x+1)(x-1)} dx$, giving your answer as a single logarithm. (C)

[Ans : $\ln\left(\frac{10}{7}\right)$]

14.6 Integration of the form $\frac{kf'(x)}{f(x)}$

We know that $\frac{d}{dx} [\ln f(x)] = \frac{f'(x)}{f(x)}$,

Thus $\int \frac{f'(x)}{f(x)} = [\ln f(x)] + c$

Example 1

(a) $\int \frac{x}{x^2+1} dx$

(b) $\int_0^1 \frac{1-4x}{3+x-2x^2} dx \quad (C)$

(c) evaluate $\int_0^{\frac{\pi}{4}} \tan x dx$, giving your answer to 3 significant figures.

Solution

(a) $\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx$
 $= \frac{1}{2} \ln(x^2+1) + c$

(b) Since $\frac{d}{dx}(3+x-2x^2) = 1-4x$

$$\begin{aligned} \int_0^1 \frac{1-4x}{3+x-2x^2} dx &= [\ln(3+x-2x^2)]_0^1 \\ &= [\ln(3+1-2(1)^2)] - [\ln(3+0-2(0)^2)] \\ &= \ln 2 - \ln 3 \\ &= \ln \frac{2}{3} \end{aligned}$$

(c) $\int_0^{\frac{\pi}{4}} \tan x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx$
 $= - \int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} dx$
 $= [-\ln \cos x]_0^{\frac{\pi}{4}}$
 $= \left[-\ln \cos\left(\frac{\pi}{4}\right) \right] - [-\ln \cos(0)]$
 $= 0.347 \text{ (3 s.f.)}$

Exercise 14.6

Integrate the following indefinite integrals with respect to x :

1 $\int \frac{x^2}{x^3 + 1} dx$ [Ans : $\frac{1}{3} \ln(x^3 + 1) + c$]

2 $\int \frac{x}{2x^2 + 1} dx$ [Ans : $\frac{1}{4} \ln(2x^2 + 1) + c$]

3 $\int \frac{x^2}{2x^3 + 5} dx$ [Ans : $\frac{1}{6} \ln(2x^3 + 5) + c$]

4 $\int \frac{3x}{2x^2 + 4} dx$ [Ans : $\frac{3}{4} \ln(2x^2 + 4) + c$]

5 $\int \frac{5x^3}{3x^4 + 2} dx$ [Ans : $\frac{5}{12} \ln(3x^4 + 2) + c$]

6 $\int \cot x dx$ [Ans : $\ln \sin x + c$]

7 $\int \frac{x+1}{x^2 + 2x + 1} dx$ [Ans : $\frac{1}{2} \ln(x^2 + 2x + 1) + c$]

8 $\int \frac{x-2}{3x^2 - 12x + 5} dx$ [Ans : $\frac{1}{6} \ln(3x^2 - 12x + 5) + c$]

9 Show that $\int \tan 3x dx = \frac{1}{3} \ln \sec 3x + c$.

Integrate the following definite integrals with respect to x :

10 $\int_0^3 \frac{x}{1+x^2} dx$ [Ans: $\frac{1}{2} \ln 10$]

11* $\int_e^{e^2} \frac{1}{x \ln x} dx$ (C) [Ans: $\ln 2$]

12 $\int_0^{\frac{\pi}{2}} \frac{\cos x}{3 + \sin x} dx$ [Ans: $\ln \frac{4}{3}$]

14.7 Integration by parts

Recall $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Integrating both sides, we have

$$\int \frac{d}{dx}(uv) = \int u \frac{dv}{dx} + v \frac{du}{dx} dx$$

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

Hence

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Example 1

$$\int x \sin 2x \, dx$$

Solution

$$\text{Let } u = x \quad \text{and} \quad \frac{dv}{dx} = \sin 2x$$

$$\frac{du}{dx} = 1 \quad \text{and} \quad v = -\frac{\cos 2x}{2}$$

$$\text{Using } \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx, \text{ we have}$$

$$\int x \sin 2x \, dx = -x \cdot \frac{\cos 2x}{2} - \int -\frac{\cos 2x}{2} \cdot 1 \, dx$$

$$\text{Hence } \int x \sin 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{\sin 2x}{4} + c$$

Example 2

$$\int \ln x \, dx$$

Solution

$$\text{Let } u = \ln x \quad \text{and} \quad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad v = x$$

$$\text{Using } \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx, \text{ we have}$$

$$\int \ln x \, dx = x \cdot \ln x - \int x \times \frac{1}{x} \, dx$$

$$\text{Hence } \int \ln x \, dx = x \ln x - x + c.$$

Now we are going to take an example where it is necessary to perform the procedure of integration by parts more than once to obtain the final answer.

Example 3

$$\int x^2 e^x \, dx$$

Solution

$$\text{Let } u = x^2 \quad \text{and} \quad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = 2x \quad \text{and} \quad v = e^x$$

$$\begin{aligned} \text{Using} \quad \int u \frac{dv}{dx} \, dx &= uv - \int v \frac{du}{dx} \, dx, \text{ we have} \\ \int x^2 e^x \, dx &= x^2 \cdot e^x - \int e^x \cdot 2x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx \dots\dots (1) \end{aligned}$$

Now consider $\int x e^x \, dx$. We have to use integration by parts once more.

$$\text{Let } u = x \quad \text{and} \quad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = 1 \quad \text{and} \quad v = e^x$$

$$\begin{aligned} \text{Using} \quad \int u \frac{dv}{dx} \, dx &= uv - \int v \frac{du}{dx} \, dx, \text{ we have} \\ \int x e^x \, dx &= x \cdot e^x - \int e^x \cdot 1 \, dx \\ &= x e^x - e^x + c \quad \dots\dots (2) \end{aligned}$$

Replacing (2) in (1), we have

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 [x e^x - e^x] + c \\ &= x^2 e^x - 2x e^x + 2e^x + c \\ &= e^x (x^2 - 2x + 2) + c \end{aligned}$$

Exercise 14.7

Integrate the following with respect to x .

1 $\int x e^x dx$ [Ans : $x e^x - e^x + c$]

2 $\int x \cos x dx$ [Ans : $x \sin x + \cos x + c$]

3 $\int x e^{4x} dx$ [Ans : $\frac{1}{4} x e^{4x} - \frac{1}{16} e^{4x} + c$]

4 $\int x \sin x dx$ [Ans : $-x \cos x + \sin x + c$]

5 $\int x \ln x dx$ [Ans : $\frac{x^2}{2} \ln x - \frac{x^2}{4} + c$]

6 $\int x \sec^2 x dx$ (C) [Ans : $x \tan x + \ln \cos x + c$]

7 $\int x^2 \ln x dx$ (C) [Ans : $\frac{x^3}{3} \ln x - \frac{x^3}{9} + c$]

8 $\int x^2 \cos x dx$ [Ans : $x^2 \sin x + 2x \cos x - 2 \sin x + c$]

9 $\int x^2 e^{3x} dx$ [Ans : $\frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + c$]

10 $\int (x+1)e^{-x} dx$ (C) [Ans : $-(x+2)e^{-x} + c$]

11 Show that $\int_0^2 x e^{2x} dx = \frac{1}{4}(3e^4 + 1)$

12 Show that $\int_2^4 x \ln x dx = 14 \ln 2 - 3$

13 Evaluate the integral $\int_0^1 x e^{-2x} dx$, leaving your answer in terms of e .
[Ans : $\frac{1}{4} - \frac{3}{4} e^{-2}$]

14.8 Integration using substitution

Consider the integral $\int_0^1 x(5x-3)^8 dx$. To evaluate such an integral a substitution $u = 5x-3$ will be very helpful.

The following steps will help the reader to evaluate a particular definite integral.

- Step 1 : Differentiate the given substitution
- Step 2 : Make dx the subject of formula
- Step 3 : Change the x limits to u limits
- Step 4 : Replace the variable x by the variable u .

Example 1

Evaluate $\int_0^1 x(5x-3)^8 dx$, using the substitution $u = 5x-3$.

Solution

$$u = 5x-3 \qquad \Rightarrow \quad x = \frac{u+3}{5}$$

$$\frac{du}{dx} = 5$$

$$dx = \frac{1}{5} du$$

For lower limits, $x = 0, u = 5(0) - 3 = -3$
 For upper limits, $x = 1, u = 5(1) - 3 = 2$

The integral $\int_0^1 x(5x-3)^8 dx$ becomes $\int_{-3}^2 \left(\frac{u+3}{5}\right) u^8 \frac{1}{5} du$

$$\begin{aligned} \text{Now} \quad \int_{-3}^2 \left(\frac{u+3}{5}\right) u^8 \frac{1}{5} du &= \frac{1}{25} \int_{-3}^2 u^9 + 3u^8 du \\ &= \frac{1}{25} \left[\frac{u^{10}}{10} + \frac{u^9}{3} \right]_{-3}^2 \\ &= \frac{1}{25} \left[\frac{2^{10}}{10} + \frac{2^9}{3} \right] - \frac{1}{25} \left[\frac{(-3)^{10}}{10} + \frac{(-3)^9}{3} \right] \\ &= 37.2 \text{ (3 s.f)} \end{aligned}$$

Note that if there are no limits, remember to change back to the original variable at the end.

Example 2

Using the substitution $x = u^2 + 2$, find $\int x\sqrt{x-2} \, dx$. (C)

Solution

$$x = u^2 + 2$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u \, du$$

$$\begin{aligned} \int x\sqrt{x-2} \, dx &= \int (u^2 + 2)\sqrt{u^2 + 2 - 2} \cdot 2u \, du \\ &= \int (u^2 + 2)\sqrt{u^2} \cdot 2u \, du \\ &= \int 2u^2(u^2 + 2) \, du \\ &= \int 2u^4 + 4u^2 \, du \\ &= \frac{2u^5}{5} + \frac{4u^3}{3} + c \\ &= \frac{2(\sqrt{x-2})^5}{5} + \frac{4(\sqrt{x-2})^3}{3} + c \end{aligned}$$

Example 3

Using the substitution $x = \sin\theta$, show that $\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2}$.

Solution

$$x = \sin\theta,$$

$$\frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta \, d\theta$$

For lower limits, $x = 0$, $\sin\theta = 0 \Rightarrow \theta = 0$

For upper limits, $x = 1$, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\begin{aligned} \therefore \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta} \cos\theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} 1 \, d\theta = [\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \text{ (shown)} \end{aligned}$$

Exercise 14.8

- 1 Using the substitution $u = x^2$, find $\int x e^{x^2} dx$. [Ans : $\frac{1}{2} e^{x^2} + c$]
- 2 Using the substitution $u = x^3$, find $\int 2x^2 e^{x^3} dx$. [Ans : $\frac{2}{3} e^{x^3} + c$]
- 3 Using the substitution $u = \sin x$, find $\int \sin^3 x \cos x dx$. [Ans : $\frac{1}{4} \sin^4 x + c$]
- 4 Using the substitution $u = \cos x$, find $\int \cos^5 x \sin x dx$. [Ans : $-\frac{1}{6} \cos^6 x + c$]
- 5 Using the substitution $u = \tan x$, find $\int \tan x \sec^2 x dx$. [Ans : $\frac{1}{2} \tan^2 x + c$]
- 6 Using the substitution $u = \ln x$, find $\int \frac{\ln x}{x} dx$. [Ans : $\frac{1}{2} (\ln x)^2 + c$]
- 7 Using the substitution $u = 1 + 3x^2$, find $\int \frac{5x}{1+3x^2} dx$. [Ans : $\frac{5}{6} \ln(1+3x^2) + c$]
- 8 Using the substitution $u = 2x + 3$ to find $\int \frac{x}{(2x+3)^3} dx$. (C)

[Ans : $-\frac{1}{4(2x+3)} + \frac{3}{8(2x+3)^2} + c$]
- 9 Using the substitution $x = \tan \theta$, show that $\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$.
- 10 By means of the substitution $x = 2 \tan \theta$, prove that $\int_0^2 \frac{1}{(x^2+4)^{3/2}} dx = \frac{1}{8} \sqrt{2}$. (C)
- 11 By using the substitution $x = 2 \sin \theta$, or otherwise, evaluate the integral $\int_0^{\sqrt{2}} \sqrt{4-y^2} dy$. (C)

[Ans : $\frac{\pi}{2} + 1$]
- 12 The value of $\int_1^4 \frac{1}{x(1+\sqrt{x})} dx$ is denoted by I. Use the substitution $u = \sqrt{x}$ to show that $I = \int_1^2 \frac{2}{u(1+u)} du$. Hence, by using partial fractions or otherwise, show that $I = \ln\left(\frac{16}{9}\right)$. (C)