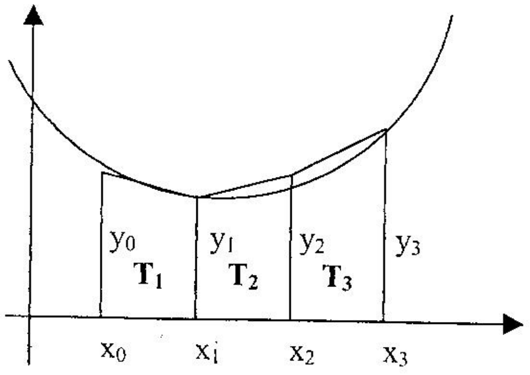
14.9 The trapezium rule

As we have seen previously, the definite integral $\int_a^b f(x)\,dx$ can be evaluated by finding the anti-derivative of f(x). However, there are functions such as e^{x^2}, $\sin(x^3)$ etc whose anti–derivatives cannot be found readily. In such cases, we have to use some methods of approximation. The trapezium rule, one of the methods of approximation, enables us to obtain a good approximation to the value of $\int_a^b f(x)\,dx$.

Let f be function defined on [a,b]. Suppose [a,b] is divided into three equal subintervals each of height h, $h = \frac{b-a}{3}$.



Now y_0, y_1, y_2 and y_3 are called ordinates and each ordinate corresponds to a particular value of x. T_1, T_2 and T_3 are called trapezia or strips.
If we consider the **two** trapezia T_1 and T_2, there are **three** ordinates supporting them. Next if we consider the **three** trapezia T_1, T_2 and T_3, there are **four** ordinates supporting them.
So, we deduce from the above that **Number of trapezia = Number of ordinates – 1**

Now area of $T_1 = \frac{1}{2}(y_0 + y_1) \times h$

area of $T_2 = \frac{1}{2}(y_1 + y_2) \times h$

area of $T_3 = \frac{1}{2}(y_2 + y_3) \times h$

Thus, the total sum of the three areas of trapezia is given by

$$A \approx \frac{1}{2}h\,(y_0 + y_1 + y_1 + y_2 + y_2 + y_3)$$

$$\approx \frac{1}{2}h\,[\,y_0 + 2(\,y_1 + y_2\,) + y_3\,]$$

In general, if [a,b] is divided into n equal trapezia, then

$\int_a^b f(x)\,dx \approx \frac{1}{2}h\,[\,y_0 + 2(\,y_1 + y_2 + \ldots\ldots + y_{n-1}) + y_n\,]$, which is the trapezium rule for approximate evaluation of definite integrals.

Example 1

Use the trapezium rule with intervals of width 0.5 to find an approximation for

$\int_{1}^{2.5} \frac{1}{1+\ln x}\,dx$, giving your answer correct to 2 decimal places. **(C)**

Solution

Here h = 0.5

Let $x_0 = 1$ $\qquad y_0 = \frac{1}{1+\ln 1} = 1$

$x_1 = 1.5$ $\qquad y_1 = \frac{1}{1+\ln 1.5} = 0.71$

$x_2 = 2$ $\qquad y_2 = \frac{1}{1+\ln 2} = 0.59$

$x_3 = 2.5$ $\qquad y_3 = \frac{1}{1+\ln 2.5} = 0.52$

By trapezium rule,

$$\int_{1}^{2.5} \frac{1}{1+\ln x}\,dx \approx \frac{1}{2}h\,[\,1^{st} + \text{last} + 2(\text{rest})]$$

$$\approx \frac{1}{2}(0.5)\,[\,1 + 0.52 + 2(0.71 + 0.59)]$$

$$\approx 1.03 \text{ (2 d.p)}$$

Example 2

Use the trapezium rule with five ordinates to find an approximation for $\int_{2}^{6} e^{x}\,dx$, giving your answer correct to 1 decimal place.

Solution

Here h = $\frac{b-a}{n} = \frac{6-2}{4} = 1$

Let $x_0 = 2$ $\qquad y_0 = e^2$

$x_1 = 3$ $\qquad y_1 = e^3$

$x_2 = 4$ $\qquad y_2 = e^4$

$x_3 = 5$ $\qquad y_3 = e^5$

$x_4 = 6$ $\qquad y_3 = e^6$

By trapezium rule,

$$\int_{2}^{6} e^{x}\,dx \approx \frac{1}{2}h\,[\,1^{st} + \text{last} + 2(\text{rest})]$$

$$\approx \frac{1}{2}(1)\,[e^2 + e^6 + 2(e^3 + e^4 + e^5)]$$

$$\approx 428.5 \text{ } \textbf{(Ans)}$$

Example 3

Estimate to 2 significant figures, the percentage error in evaluating $\int_2^4 \frac{5}{x+2}\,dx$ by using the trapezium rule with 5 ordinates.

Solution

Here $h = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}$

Let $x_0 = 2$ $\quad y_0 = \frac{5}{2+2} = \frac{5}{4}$

$x_1 = \frac{5}{2}$ $\quad y_1 = \frac{5}{\frac{5}{2}+2} = \frac{10}{9}$

$x_2 = 3$ $\quad y_2 = \frac{5}{3+2} = 1$

$x_3 = \frac{7}{2}$ $\quad y_3 = \frac{5}{\frac{7}{2}+2} = \frac{10}{11}$

$x_4 = 4$ $\quad y_4 = \frac{5}{6}$

By trapezium rule,

$$\int_2^4 \frac{5}{x+2}\,dx \approx \frac{1}{2}h\,[\,1^{st} + \text{last} + 2(\text{rest})]$$

$$\approx \frac{1}{2}(1)\,[\frac{5}{4}+\frac{5}{6}+2(\frac{10}{9}+1+\frac{10}{11})]$$

$$\approx \frac{3217}{1584}$$

Now, we find the exact value of the integral $\int_2^4 \frac{5}{x+2}\,dx$

$$\int_2^4 \frac{5}{x+2}\,dx = 5\ln(x+2)\,\Big]_2^4$$

$$= 5\ln 6 - 5\ln 4$$

$$= 5\ln\left(\frac{3}{2}\right)$$

Percentage error = $\dfrac{\text{Exact value} - \text{Approximated value}}{\text{Exact value}}$

$$= \frac{5\ln\left(\frac{3}{2}\right) - \frac{3217}{1584}}{5\ln\left(\frac{3}{2}\right)} = 0.18\%\ \textbf{(Ans)}$$

Exercise 14.9

1 Use the trapezium rule with 5 ordinates to find an approximate value for $\int_{-1}^{1} e^{2x}\,dx$, giving your answer to one decimal place. [Ans : 3.9]

2 Use the trapezium rule with 3 ordinates to find an approximate value for $\int_{1}^{7} \frac{7}{x+3}\,dx$, giving your answer as a fraction. [Ans : $\frac{98}{15}$]

3 Use the trapezium rule with 4 ordinates to find an approximation to $\int_{1}^{4} \ln x\,dx$, giving your answer in exact form. [Ans : ln 12]

4 Use the trapezium rule with 6 ordinates to find an approximation to $\int_{2}^{3} \frac{1}{x+2}\,dx$, giving your answer to two decimal places. [Ans : 0.23]

5 Use the trapezium rule with ordinates at x = 0, x = 1, x = 2, to find an approximation to $\int_{0}^{2} \frac{1}{1+x^3}\,dx$. **(C)** [Ans : 1.06]

6 Use the trapezium rule with ordinates at x = 1, x = 2, x = 3, to estimate the value of $\int_{1}^{3} \sqrt{40-x^3}\,dx$. **(C)** [Ans : 10.6]

7 Use the trapezium rule with ordinates at $x = 0, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi$ to estimate the value of $\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$, giving your answer to 3 significant figures. **(C)** [Ans : 0.713]

8 Estimate to 2 significant figures, the percentage error in evaluating $\int_{0}^{1} \frac{1}{1+x}\,dx$ by using the trapezium rule with 3 ordinates. **(C)** [Ans : 2.2%]

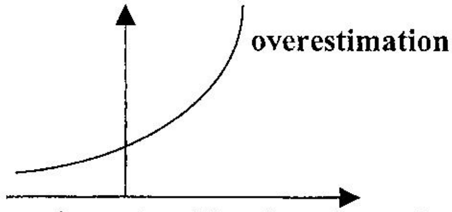
9 Use the trapezium rule with 3 ordinates to find an approximation to $\int_{0}^{1} \frac{3}{x+2}\,dx$, giving your answer in exact form. Find the exact value of the integral. Hence find the percentage error involved. [Ans : $\frac{49}{40}$; 3 ln$\frac{3}{2}$; 0.71%]

10 Use the trapezium rule with subdivisions at x = 3 and x = 5 to obtain an approximation for the integral $\int_{1}^{7} \frac{x^3}{1+x^4}\,dx$, giving your answer to 3 decimal places. By evaluating the integral exactly, show that the error in the approximation is about 4.1%. [Ans : 1.701, 1.773]

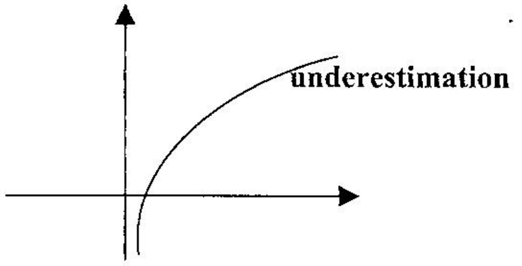
14.10 Use of sketch graphs to show underestimate or overestimate

We have seen in the previous part of the chapter that the trapezium rule gives an approximate value for a particular integral. In some cases, the trapezium rule underestimates the true value of the integral while in other cases the trapezium rule overestimates the true value of the integral. Still there are very few cases where the trapezium rule gives an exact estimate.

If a curve is **convex**, the trapezium rule will overestimate the true value of the integral. An example of a curve that is convex is $y = e^x$. In this case, the trapezia will pass over the curve, thereby overvaluing the true value of the integral.



If a curve is **concave**, the trapezium rule will underestimate the rule value of the integral. An example of a curve that is concave is $y = \ln x$. In this case, the trapezia will pass under the curve, thereby undervaluing the true value of the integral.



Example

Verify that $\int \ln x \; dx = x\ln x - x + c$, where c is an arbitrary constant. Use the trapezium rule with trapezia of unit width to find an estimate for $\int_2^5 \ln x$. Explain with the aid of a sketch why the trapezium rule underestimates the true value of the integral in this case, and calculate, correct to one significant figure, the percentage error involved. **(C)**

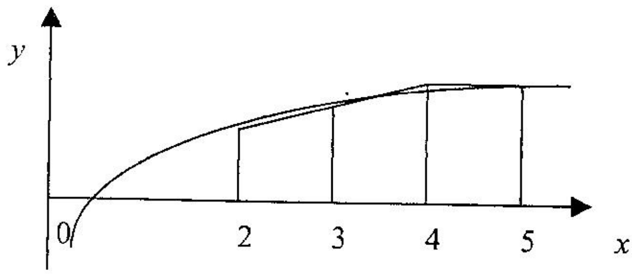
Solution

Consider $\frac{d}{dx}(x\ln x - x + c) = x \cdot \frac{1}{x} + \ln x \cdot 1 - 1 = \ln x$ **(verified)**

Let $x_0 = 2$ $\quad y_0 = \ln 2$
$\quad x_1 = 3$ $\quad y_1 = \ln 3$
$\quad x_2 = 4$ $\quad y_2 = \ln 4$
$\quad x_3 = 5$ $\quad y_3 = \ln 5$

By trapezium rule, $\int_2^5 \ln x \, dx \approx \frac{1}{2} h[1^{st} + \text{last} + 2(\text{rest})]$

$$\approx \frac{1}{2}(1)\,[\,\ln 2 + \ln 5 + 2(\ln 3 + \ln 4)] \approx 3.64 \textbf{ (3 s.f)}$$



Since the shaded part is not included, the trapezium rule underestimates the value of the integral.

Now, we find the exact value of the integral $= \int_2^5 \ln x \, dx$

$$= [x \ln x - x]_2^5$$
$$= [5\ln 5 - 5] - [2\ln 2 - 2]$$
$$= 5\ln 5 - 2\ln 2 - 3$$

Percentage error $= \dfrac{\text{Exact value} - \text{Approximated value}}{\text{Exact value}}$

$$= \frac{(5\ln 5 - 2\ln 2 - 3) - 3.64}{5\ln 5 - 2\ln 2 - 3} = 0.7\% \ \textbf{(Ans)}$$

Exercise 14.10

CLASS ACTIVITY

In questions 1 – 5, explain by means of a graphical argument whether the trapezium rule underestimates or overestimates the true value of the integral.

1 $\int_3^5 \ln(x-2)\, dx$, 3 ordinates

2 $\int_2^3 e^{x+1}\, dx$, 5 ordinates

3 $\int_1^6 \sqrt{x}\, dx$, 6 ordinates

4 $\int_0^{\frac{\pi}{2}} \sin\theta\, dx$, 4 ordinates

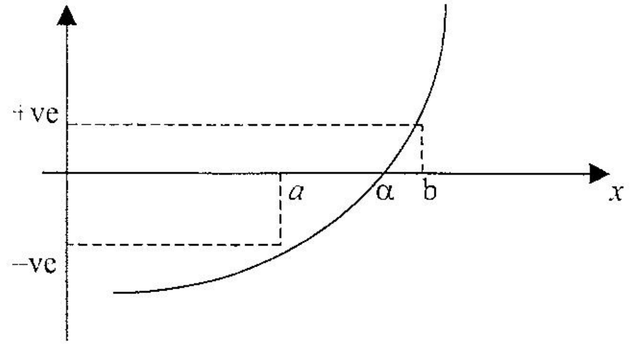
5 $\int_1^5 \frac{7}{x+1}\, dx$, 5 ordinates

CHAPTER 15

NUMERICAL SOLUTION OF EQUATIONS

15.1 Location of roots – searching for sign change

The diagram below shows the graph of $y = f(x)$ crossing the x-axis at α. We say α is a root of the equation $f(x) = 0$.



Now, if lies α between two points a and b, then $f(a) < 0$ (negative) and $f(b) > 0$ (positive). We have $f(a) \times f(b) < 0$.
In general, for a given curve $y = f(x)$, a root α will lie between two points a and b provided $f(a) \times f(b) < 0$. In other words, there must be a **sign change**.

Example 1
Show that $x^3 + x = 5$ has a root between 1 and 2.
Solution
Let $\quad f(x) = x^3 + x - 5$
Now $\quad f(1) = 1^3 + 1 - 5 = -3$ (–ve)
And $\quad f(2) = 2^3 + 2 - 5 = 5$ (+ve)
Since there is a sign change, a root lies between 1 and 2.

Example 2
Show that $e^x = 10 \cos x$ has a root between 1 and 2
Solution
Let $\quad f(x) = e^x - 10 \cos x$
Now $\quad f(1) = e^1 - 10 \cos 1 = -2.68$ (–ve) $\quad$ [**convert calculator to radian mode**]
And $\quad f(2) = e^2 - 10 \cos 2 = 11.6$ (+ve)
Since there is a sign change, a root lies between 1 and 2.

Exercise 15.1

1 Show that $2x^2 - 5x + 1 = 0$ has a root between 2 and 3.

2 Show that $x^3 = x + 12$ has a root between 2 and 3.

3 Show that $e^x = 10x$ has a root between 0 and 1 and another root between 3 and 4.

4 Show that $\ln x = \frac{1}{3}x$ has a root between 1 and 2 and another root between 4 and 5.

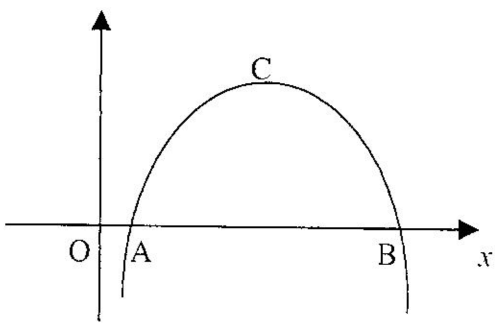
5 Show that $x = \frac{1}{1+\sqrt{x}}$ has a root between 0 and 1. **(C)**

6* Show that $2\sin x = 3 - x$ has a root between 1 and 2.

7 Show that $(x+1)^5 = (x+2)^3 + 4$ has a root between 0.8 and 1.

8 Given that $f(x) = x(x-1)^5 - 500$, show that $f'(x) = (6x-1)(x-1)^4$. Hence show that the equation $f(x) = 0$ has a root in the interval $3 \le x \le 4$. **(C)**

9 The function f is defined by $f(x) = 18\ln x - x^2$, $x > 0$. The diagram shows a sketch of the curve with equation $y = f(x)$.



(a) The point C is the maximum point on the curve. Find the x–coordinate of C.

(b) The curve crosses the x–axis at A and B. Show by calculation that the x–coordinate of A lies between 1.0 and 1.1. **(C)** [Ans : 3]

10 Sketch on the same diagram the graphs of $y = e^{-\frac{1}{2}x}$ and $y = 4 - x^2$. Verify that the negative root α, of the equation $x^2 + e^{-\frac{1}{2}x} = 4$ is such that $-2 < \alpha < -1$, and state the integer which is closest to the positive root, β. [Ans : 2]

15.2 Location of roots – graphical considerations

If there is only one curve, the number of roots is equivalent to the number of times the curve crosses the x–axis. The graph below illustrates this idea.

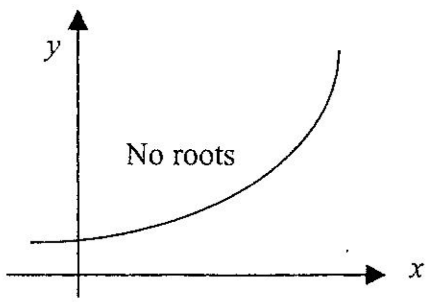


Figure 1

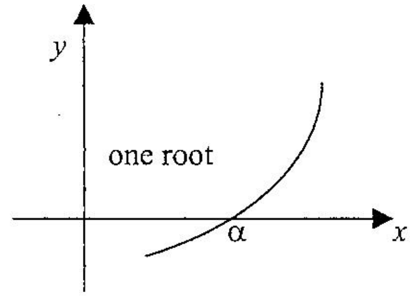


Figure 2

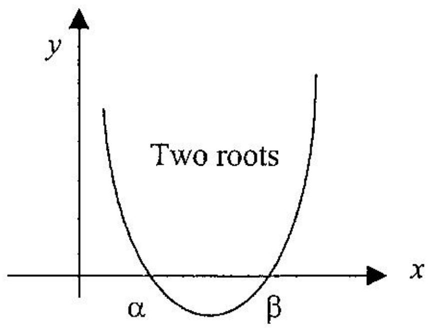


Figure 3

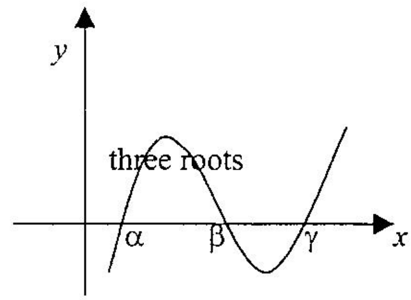


Figure 4

However if there exists two curves on a single diagram, the number of roots is equivalent to the number of points of intersection of the two graphs.

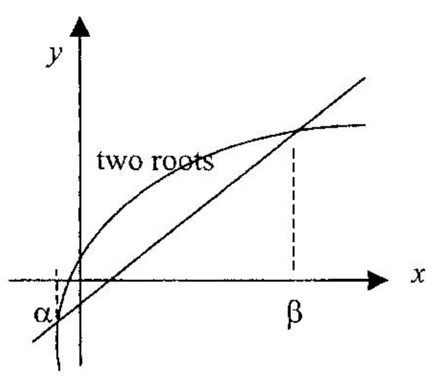


Figure 5

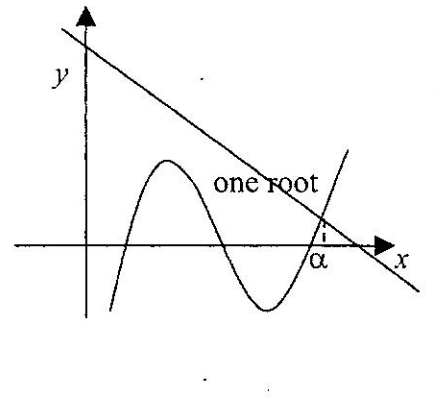


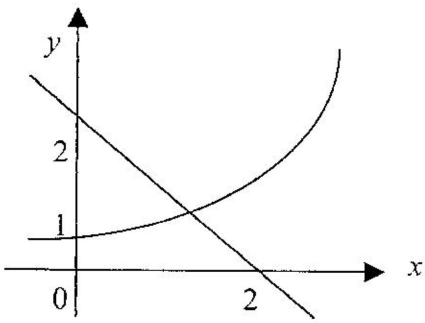
Figure 6

Example 1

Find the number of roots of the equation $e^x = 2 - x$, by means of a graphical argument.

Solution

We sketch the graph of $y = e^x$ and $y = 2 - x$ on a single diagram and we find the number of points of intersection.



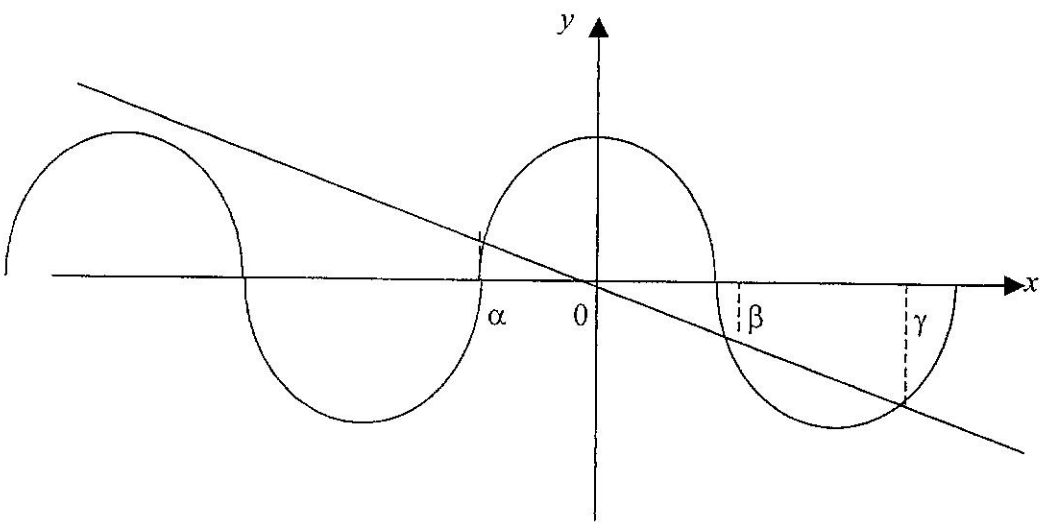
Since the line meets the curve once, there is only one root.

Example 2

By considering the graph of $y = \cos x$ and $y = -\frac{1}{4}x$, or otherwise, show that the equation $x + 4\cos x = 0$ has one negative root and two positive roots. **(C)**

Solution

We have



Clearly the negative root is α and the two positive roots are β and γ.

Exercise 15.2

1 Show graphically that $e^x = 10x$ has exactly two roots.

2 Show graphically that $\ln x = \frac{1}{3}x$ has exactly two roots.

3 Show graphically that $e^x = 3 - x$ has exactly one root.

4 Show graphically that $\ln x + x - 4 = 0$ has exactly one root.

5 Show graphically that $x^3 = 8x + 4$ has exactly one root.

6 Show graphically that $2x^2 = 5x - 1$ has exactly two roots.

7 Show graphically that $x^2 = 2 - x$ has exactly two roots.

8 Show graphically that $e^x + x - 8 = 0$ has exactly one root.

9 By means of a sketch showing the graphs of $y = x^3$ and $y = 4(x+10)$, show that the equation $x^3 = 4(x + 10)$ has exactly one real root. **(C)**

10 By considering the graphs of $y = 1 + x$ and $y = ke^x$, or otherwise, show that the equation $e^{-x}(1 + x) = k$ has exactly two roots for x when k is a constant such that $0 < k < 1$. State the number of real roots for each of the cases $k < 0$ and $k > 1$.**(C)**

[Ans: $k < 0 \Rightarrow 1$ solution , $k > 1 \Rightarrow 0$ solution]

11 Find the coordinates of the turning points on the graph of $y = x^5 - 10x$, and hence show, with the aid of a sketch, that the equation $x^5 - 10x = 5$ has three real roots. State the two consecutive integers between which the positive root of the equation $x^5 - 10x = 5$ lies. **(C)**

[Ans: (–1.19,9.51) ; (1,2)]

12 Show, by sketching the graphs of $y = e^x$ and $y = 6 - 2x$, that the equation $e^x + 2x - 6 = 0$ has exactly one real root and that the root lies between 1 and 2. **(C)**.

13 Show, by means of a graphical argument that the equation $x = 10e^{-\frac{1}{4}x}$ has exactly one real root(denoted by α), and determine the pair of consecutive integers between which α lies. [Ans : (3,4)]

15.3 Iterative methods

An iterative formula is an equation of the form $x_{n+1} = f(x_n)$ which can be used to find a sequence of approximations which converges to a root of an equation.
To obtain an iterative formula from a given equation, use the following steps:
Step 1: Make one of the x's subject of formula.
Step 2 : On the left–hand side replace x by x_{n+1}.
Step 3 : On the right–hand side replace x by x_n.

Example 1
Derive two iterative formulae based on the equation $e^x = 10x$.
Solution
$e^x = 10x$.
$x = \ln(10x)$
$x_{n+1} = \ln(10x_n)$ .... (1)

Also $10x = e^x$

$$x = \frac{e^x}{10}$$

$$x_{n+1} = \frac{e^{x_n}}{10} \quad \text{......... (2)}$$

Hence the two iterative are $x_{n+1} = \ln(10x_n)$ and $x_{n+1} = \dfrac{e^{x_n}}{10}$.

Example 2
Derive three iterative formulae based on the equation $x^3 + x = 5$.
Solution

$$x^3 + x = 5$$
$$x^3 = 5 - x$$
$$x = \sqrt[3]{5-x}$$
$$x_{n+1} = 3\sqrt{5-x_n} \quad \text{........(1)}$$

Also $x = 5 - x^3$

$$x_{n+1} = 5 - x_n^3 \quad \text{.........(2)}$$

And $x(x^2+1)$

$$x = \frac{5}{x^2+1}$$

$$x_{n+1} = \frac{5}{x_n^2+1} \quad \text{........(3)}$$

Hence the three iterative formula are $x_{n+1} = \sqrt[3]{5-x_n}$, $x_{n+1} = 5 - x_n^3$ and $x_{n+1} = \dfrac{5}{x_n^2 - 1}$.

Example 3

Using the iteration $x_{n+1} = \ln(8 - x_n)$, find the root α lying in the interval (1,2), giving your answer to 2 decimal places.

Solution

We have $x_{n+1} = \ln(8 - x_n)$

Since the root lies between 1 and 2,

Let $x_0 = 1.5$

$x_1 = \ln(8 - 1.5) = 1.872$

$x_2 = \ln(8 - 1.872) = 1.813$

$x_3 = \ln(8 - 1.813) = 1.822$

$x_4 = \ln(8 - 1.822) = 1.820$

Hence $\alpha = 1.82$ correct to two decimal places.

The reader has to bear in mind that not all iterative formulae converge to a particular root. While some converges to a root of an equation, others diverge and tend to move away from the root.

The diagrams below illustrate both cases of convergence and divergence.

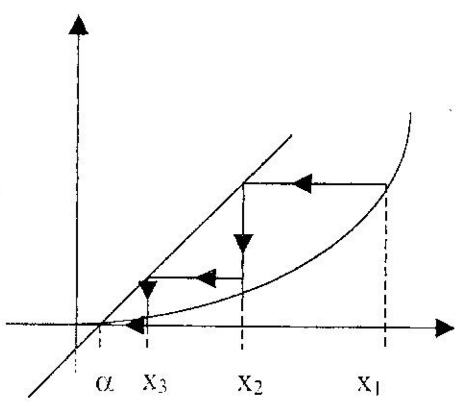


Figure 1 : Convergence

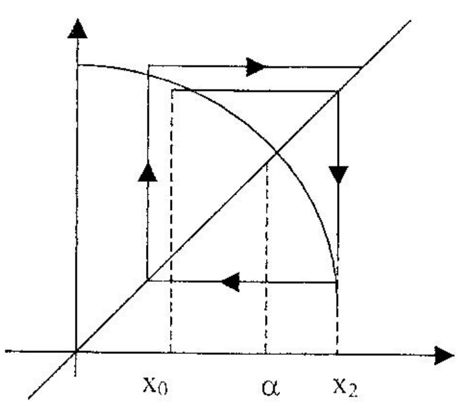


Figure 2 : Divergence

Exercise 15.3

1 Derive two iterative formulae based on the equation $\ln x = \frac{1}{3}x$.

[Ans : $x_{n+1} = e^{\frac{1}{3}x_n}$; $x_{n+1} = 3\ln x_n$]

2 Derive two iterative formulae based on the equation $e^x = 2 - x$.

[Ans : $x_{n+1} = 2 - e^{x_n}$; $x_{n+1} = \ln(2 - x_n)$]

3 Derive three iterative formulae based on the equation $x^3 + 2x^2 = 3$.

[Ans : $x_{n+1} = \sqrt[3]{3 - 2x_n^2}$; $x_{n+1} = \sqrt{\frac{3 - x_n^3}{2}}$; $x_{n+1} = \sqrt{\frac{3}{x_n + 2}}$]

4 Using the iteration $x_{n+1} = \frac{e^{x_n}}{10}$, find the root β, starting with $x_0 = 3$, giving your answer to 2 decimal places. [Ans : 0.11]

5 Using the iteration $x_{n+1} = \frac{1}{5}(1 + 2x_n^2)$, find the root δ, starting with $x_0 = 0.5$, giving your answer to 2 decimal places. [Ans : 0.22]

6 Using the iteration $x_{n+1} = \sqrt[3]{5 - x_n}$, find the root α in the interval (1,2), giving your answer to 3 decimal places. [Ans : 1.516]

7 Using the iteration $x_{n+1} = \sqrt[3]{\frac{9}{(1 + x_n)^2}}$, find the root α, taking $x_0 = 1$ as the first approximation. Give your answer to 2 decimal places. [Ans : 1.22]

8 Using the iteration $x_{n+1} = 3\ln x_n$, find the root β, starting with $x_0 = 4$, giving your answer to 3 significant figures. [Ans : 4.53]

9 Show that $x^3 + 3x^2 = 7$ may be rearranged into the form $x = \sqrt{\frac{a}{x + b}}$, stating the values of a and b. Hence using the iteration formula $x_{n+1} = \sqrt{\frac{a}{x_n + b}}$ with $x_o = 1$, together with the values of a and b, find x_1, x_2, x_3 and x_4.

[Ans: $a = 7$, b = 3 ; 1.32, 1.27, 1.28, 1.28]

10 Show that the equation $x^3 - x^2 - 2 = 0$ may be rearranged in the form $x = \sqrt[3]{f(x)}$, where f(x) is a quadratic function. Use an iteration of the form $x_{n+1} = g(x_n)$, based on this rearrangement and with $x_o = 1.5$ to find x_1 and x_2, giving your answers to 3 decimal places. [Ans : $\sqrt[3]{x^2 + 2}$; 1.620, 1.666]

15.4 Given an iteration formula, how to find the original equation

Very often, we are given an iteration formula of the type $x_{n+1} = f(x_n)$ and we are required to state an equation satisfied by the root α (say). In such a case, we have to make use of the following steps.

Step 1
On the left–hand side replace x_{n+1} by x.
Step 2
On the right–hand side replace x_n by x.
Step 3
Simplify to get the required result

Example 1

The sequence of values given by the iteration formula $x_{n+1} = \frac{2}{x_n^4} + \frac{4x_n}{5}$, converges to α.

State an equation satisfied by α, and hence show that the exact value of α is $\sqrt[5]{10}$.

Solution

The given iteration formula is $x_{n+1} = \frac{2}{x_n^4} + \frac{4x_n}{5}$

We have $x = \frac{2}{x^4} + \frac{4x}{5}$

$$x = \frac{10 + 4x^5}{5x^4}$$

$$5x^5 = 10 + 4x^5$$

$$x^5 = 10$$

$$x = \sqrt[5]{10}$$

Hence $\alpha = \sqrt[5]{10}$ **(shown)**

Exercise 15.4

1. The sequence of values given by the iteration formula $x_{n+1} = \frac{2}{x_n^2} + \frac{x_n}{5}$, converges to α. State an equation satisfied by α, and hence show that the exact value of α is $\sqrt[3]{\frac{5}{2}}$.

   [Ans: $x^3 = \frac{5}{2}$]

2. The sequence of values given by the iteration formula $x_{n+1} = \frac{1}{x_n^4} + \frac{x_n^2}{3}$, converges to α. State an equation satisfied by α, and hence show that the exact value of α is $\sqrt[5]{\frac{3}{2}}$.

   [Ans: $2x^5 = 3$]

3. The sequence of values given by the iteration formula $x_{n+1} = \frac{x_n^2 + 6}{4x_n}$, converges to β. State an equation satisfied by β, and hence show that the exact value of β is $\sqrt{2}$.

   [Ans : $x^2 = 2$]

4. The sequence of values given by the iteration formula $x_{n+1} = \frac{x_n^3 + 1}{3x_n^2}$, converges to α. State an equation satisfied by α, and hence show that the exact value of α is $\sqrt[3]{\frac{1}{2}}$.

   [Ans: $x^3 = ½$]

5. The sequence of values given by the iteration formula $x_{n+1} = (5 - x_n)^{\frac{1}{3}}$, converges to α. State an equation satisfied by α. [Ans: $x^3 + x = 5$]

6. The sequence of values given by the iteration formula $x_{n+1} = \sqrt[3]{\frac{9}{(1 + x_n)^2}}$, converges to α. State an equation satisfied by α. [Ans: $x^5 + 2x^4 + x^3 - 9 = 0$]

7. The sequence of values given by the iteration formula $x_{n+1} = \frac{1}{5}(1 + 2x_n^2)$, converges to α. State an equation satisfied by α. [Ans: $2x^2 - 5x + 1 = 0$]

CHAPTER 16

VECTORS 2

16.1 Vector equation of a line

The vector equation of a line is of the form

$r = a + t\,b$,

where a is a point on the line, b is the direction vector of that line and t is a parameter.
Therefore the equation of the line passing through the points A and B will be

$r = \overrightarrow{OA} + t\overrightarrow{AB}$

Example 1
Write down the vector equation of the line passing through A (2, 4, 3) with direction vector i – j + 3k.
Solution

$$r = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

Example 2
The position vectors of A and B, relative to an origin O, are **i** + 2**j** + 3**k** and 3**i** + 4**j** + 5**k** respectively. Find the vector equation of the line through A and B. **(C)**
Solution

$$OA = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } OB = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

$$AB = OB - OA = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Equation of line is $\quad r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

Or $\quad r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise 16.1

Write down the vector equation of each of the following straight lines

1 passing through P (2,4,3) with direction vector $2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.
[Ans : $r = (2,4,3) + t\,(2,-5,3)$]

3 passing through R (0,–2,1) and parallel to the vector $\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.
[Ans : $r = (0,-2,1) + t\,(1,2,-5)$]

3 passing through A (1,1,3) and parallel to the line joining the points B(0,–2,5) and C(5,–3,4).
[Ans : $r = (1,1,3) + t\,(5,-1,-1)$]

4 The position vectors of A and B are given by

$$\mathbf{OA} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{OB} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k},$$

Where O is the origin. Find a vector equation of the straight line passing through A and B. **(C)**
[Ans : $r = (3,-2,2) + t\,(-4,3,1)$]

5 The position vectors of the points A and B are given by

$$\mathbf{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \qquad \mathbf{OB} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k},$$

Find a vector equation of the line passing through A and B. **(C)**
[Ans : $r = (1,1,1) + t\,(1,2,3)$]

6 Show that the lines with equations

$$\mathbf{r} = 2\mathbf{i} + t(\mathbf{i} + \mathbf{j}),$$
$$\mathbf{r} = 3\mathbf{j} + s(\mathbf{i} - \mathbf{j})$$ are perpendicular. **(C)**

7 The point O is the origin, and points A, B and C have position vectors given by $\mathbf{OA} = 6\mathbf{i}$, $OB = 3\mathbf{j}$ and $\mathbf{OC} = 4\mathbf{k}$. The point P is on the line AB between A and B, and is such that AP = 2 PB. The point Q has position vector given by $\mathbf{OQ} = q\mathbf{i}$, where q is a scalar.

(i) Express, in terms of **i**, **j** and **k**, the vectors **AB**, **AP** and **CP**.

(ii) Show that the line BQ has equation

$$\mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j}), \quad \text{where } t \text{ is a parameter.}$$

Give an equation of the line CP in a similar form. **(C)**
[Ans: (i) $-6i + 3j$, $-4i + 2j$, $2i + 2j - 4k$ (ii) $4k + \beta(2i + 2j - 4k$]

8 The points A, B and C have position vectors

$$\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \quad 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad 2\mathbf{i} + 2\mathbf{j} + \mathbf{k},$$

respectively, with respect to an origin O.
Find an equation of the line BC, in the form $\mathbf{r} = u + t\,v$. **(C)**
[Ans: $r = (2,1,2) + t(0,1,-1)$]

16.2 Parallel, intersecting and skew lines

16.2.1 Parallel lines

Two lines are parallel if they have the same direction vector.

For example, the lines $r = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and the line $r = \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ are parallel as they have the same direction vector.

Example 1

$\boldsymbol{a} = 12i + 8j + k$, $\quad \boldsymbol{b} = 3i + 2j - 5k$, $\quad c = -3i - 2j + pk$,

Find the value of p for which c is parallel to AB.

Solution

$$OA = \begin{pmatrix} 12 \\ 8 \\ 1 \end{pmatrix} \qquad OB = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} \qquad OC = \begin{pmatrix} -3 \\ -2 \\ p \end{pmatrix}$$

Now $AB = OB - OA = \begin{pmatrix} -9 \\ -6 \\ -6 \end{pmatrix}$

Since c is parallel to AB, they must have the same direction vector or AB = k OC

$$\begin{pmatrix} -9 \\ -6 \\ -6 \end{pmatrix} = k \begin{pmatrix} -3 \\ -2 \\ p \end{pmatrix}$$

$$\begin{pmatrix} -9 \\ -6 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} -3 \\ -2 \\ p \end{pmatrix}$$

$3p = -6$

Hence $p = -2$.

16.2.2 Intersecting lines

Two lines intersect if the value of each of the parameters is constant.

Example 2

Determine whether the lines,

$r = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $r = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ where t and s are parameters, intersect.

Solution

We have $r = \begin{pmatrix} 1 \\ 1+t \\ 1+t \end{pmatrix}$ and $r = \begin{pmatrix} s \\ 1 \\ 2-s \end{pmatrix}$

From the 1st row, $1 = s$

From the 2nd row, $1 + t = 1$

$t = 0$

From the 3rd row, $1 + 0 = 2 - 1$, which is true.

Hence the two lines intersect.

Example 3

Determine whether the lines $r = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $r = \begin{pmatrix} 1/2 \\ 3 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ intersect.

Solution

We have $r = \begin{pmatrix} 1+t \\ 4+2t \\ 2-3t \end{pmatrix}$ and $r = \begin{pmatrix} 1/2+s \\ 3+2s \\ 4-4s \end{pmatrix}$

At the point of intersection, $\begin{pmatrix} 1+t \\ 4+2t \\ 2-3t \end{pmatrix} = \begin{pmatrix} 1/2+s \\ 3+2s \\ 4-4s \end{pmatrix}$

From the 1st row, $1 + t = \frac{1}{2} + s$ ... (1)

From the 2nd row, $4 + 2t = 3 + 2s$ ... (2)

Solving (1) and (2), we have no solution.

Since the lines have no common point, they do not intersect.

In fact such type of lines are called **skew lines**.

Exercise 16.2

State or find whether the following lines are **parallel, intersecting** or **skew**.

1 $\quad \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ [Ans : parallel]

2 $\quad \mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ [Ans : intersecting]

3 $\quad \mathbf{r} = \mathbf{i} + \mathbf{j} + t(\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + s(\mathbf{i} + \mathbf{k})$ [Ans : skew]

4 $\quad \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 11 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 12 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 18 \\ 11 \end{pmatrix} + s\begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix}$ [Ans : intersecting]

5 $\quad \mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ [Ans : parallel]

6 $\quad \mathbf{r} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + a\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + b\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ [Ans : parallel]

7 $\quad \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ [Ans : intersecting]

16.3 Cartesian equation of line

The vector equation of a line can be expressed in Cartesian form as under:

If $r = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$ is the vector equation of a line, then the Cartesian equation can be expressed as

$$\frac{x-1}{3} = \frac{y+2}{5} = \frac{z-3}{-2} = t.$$

Exercise 16.3

In questions 1 – 3, find the Cartesian equation of each of the following straight lines

1 passing through 2i – 5j + 3k and 3i – j + 2k.

[Ans : $\frac{x-2}{1} = \frac{y+5}{4} = \frac{z-3}{-1} = t$]

2 passing through i – j + 2k and 4i – 5j + 3k.

[Ans : $\frac{x-1}{3} = \frac{y+1}{-4} = \frac{z-2}{1} = t$]

3 passing through 3i – 4j + 2k and 5i – 7j + k

[Ans : $\frac{x-3}{2} = \frac{y+4}{-3} = \frac{z-2}{-1} = t$]

4 The position vectors of A and B are given by

OA = 3i – 2j + 2k, OB = –i + j +3k,

where O is the origin. Find the Cartesian equation of the line through A and B.

[Ans : $\frac{x-3}{-4} = \frac{y+2}{3} = \frac{z-2}{1} = t$]

5 The position vectors of the points A and B are given by

OA = i + j + k, OB = 3i + 4j – k,

Find the Cartesian equation of the line passing through A and B.

[Ans : $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{-2} = t$]

16.4 Angle between two lines

To find the angle between two lines, we have to find the angle between the direction vectors of the two lines.

Example 1

Find the acute angle between the lines

$$\mathrm{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \text{and } \mathrm{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

Solution

Let θ be the angle between the two given lines

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+1^2+5^2}}$$

$$= \frac{13}{\sqrt{6}\sqrt{30}}$$

$$\therefore \quad \theta = 14.3^{\circ}$$

Example 2

Show that the cosine of the angle between the line $\mathrm{r} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and the line parallel to the vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ is $\frac{1}{\sqrt{28}}$.

Solution

Let θ be the required angle.

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}{\sqrt{1^2+0^2+1^2}\sqrt{2^2+3^3+(-1)^2}}$$

$$= \frac{1}{\sqrt{2}\sqrt{14}} = \frac{1}{\sqrt{28}} \ \textbf{(shown)}$$

Exercise 16.4

1 Find the acute angle between the lines $\mathrm{r} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathrm{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

[Ans : 15.2°]

2 Find the acute angle between the lines $\mathrm{r} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ and $\mathrm{r} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

[Ans : 61.4°]

3 Find, in the form r = a + tb, an equation of the straight line through the points (3,1,–2) and (4,0,2). Find also the acute angle between this line and a line parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. (C) [Ans : r = (3,1,–2) + t(1,-1,4) ; 45°]

4 The equation of a straight line l is $\mathrm{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, where t is a parameter.

The point A on l is given by t = 0, and the origin of position vectors is O. Calculate the acute angle between OA and l, giving your answer to the nearest degree.

[Ans: 51.9°]

5 Find the acute angle between **r** = i + j + t(j + k) and r = j + k + s(i + k) where t and s are parameters. [Ans : 60°]

6 The points A and B have position vectors 3**i** + 2**j** + **k** and **i** + 2**j** + 3**k**, respectively, relative to the origin O. The point C is on the line OA produced and is such that AC = 2 OA. The point D is on OB produced and is such that BD = OB. The point X is such that OCXD is a parallelogram. Show that the line AX is parallel to the vector i + j + k. Find, in the form, r = u + tv, the equations of the lines AX and CD. Find the angle BAX.

[Ans: r = (3,2,1) + t(1,1,1) ; r = (2,4,6) + s(7,2,–3) ; 90°]

16.5 Finding the point of intersecting of two lines

To find the find of intersect of two lines expressed in vector form, we have to equate both vectors and find the value of the parameters and substitute in either equation to obtain the point of intersection.

Example

Find the point of intersection of the lines

$$r = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}.$$

Solution

We have $r = \begin{pmatrix} \lambda \\ 1+\lambda \\ -2+2\lambda \end{pmatrix}$ and $r = \begin{pmatrix} 1+2\mu \\ -1+\mu \\ 3+5\mu \end{pmatrix}$

From the first row, $\lambda = 1 + 2\mu$ ... (1)

From the second row, $1 + \lambda = -1 + \mu$ ... (2)

Solving (1) and (2) simultaneously, we have

$\mu = -3$ and $\lambda = -5$

We can obtain the point of intersection by replacing $\mu = -3$ in $\begin{pmatrix} 1+2\mu \\ -1+\mu \\ 3+5\mu \end{pmatrix}$

or $\lambda = -5$ in $\begin{pmatrix} \lambda \\ 1+\lambda \\ -2+2\lambda \end{pmatrix}$.

Hence the point of intersection $= \begin{pmatrix} -5 \\ 1+-5 \\ -2+2(-5) \end{pmatrix}$

$= \begin{pmatrix} -5 \\ -4 \\ -12 \end{pmatrix}$ or $(-5, -4, -12)$.

We can check the answer by replacing $\mu =$ 3 in $\begin{pmatrix} 1+2\mu \\ -1+\mu \\ 3+5\mu \end{pmatrix}$. The answer must be the same.

Exercise 16.5

Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$.

[Ans : (1,3,0)]

2 Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

[Ans : (−1,−2,0)]

3 Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix} 5 \\ 13 \\ 11 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 12 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 18 \\ 11 \end{pmatrix} + s\begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix}$.

[Ans : (3,14,23)]

4 Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1/2 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$.

[Ans : (1,4,2)]

5 Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3/2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

[Ans : (1,2,3)]

6 The points A and B have position vectors $2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{i} + \mathbf{j}$ respectively, relative to the origin O. The point P lies on OA produced and is such that OP = 2OA. Q lies on OB produced and is such that OQ = 3 OB. Find the vector equations of the lines AQ and BP and find also the position vector of the point of intersection of these two lines.

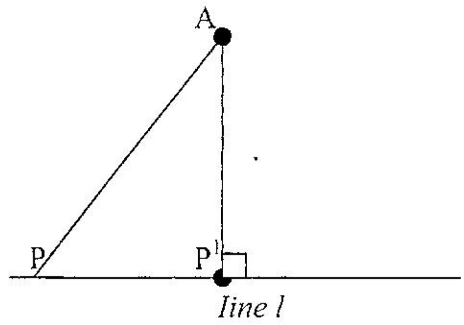
The point Z has position vector $\mathbf{k}$ relative to O. Show that the cosine of the acute angle between ZA and ZB is $\sqrt{6/7}$ and find the area of the triangle ZAB, giving your answer in surd form.

[Ans : AQ = $(2+\lambda)i + 3j$; BP = $i + j + t(3i + 5j)$; $\frac{11}{5}i + 3j$; $\frac{\sqrt{6}}{2}$]

16.6 Finding the perpendicular distance or shortest distance from a point to a line.

Example

Show that the perpendicular distance from a point A with position vector (3, –6, 1) to the line l, $r = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ is $3\sqrt{10}$.



Solution

Let P be any point on the line.

Then AP = OP – OA

$$= \begin{pmatrix} 7-t \\ 1+3t \\ -5+t \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-t \\ 7+3t \\ -6+t \end{pmatrix}$$

When AP' is perpendicular to the line l,

$$\begin{pmatrix} 4-t \\ 7+3t \\ -6+t \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = 0$$

$-4 + t + 21 + 9t - 6 + t = 0$

$11t = -11$

$t = -1$

The foot of the perpendicular form A to l is then,

$$p' = \begin{pmatrix} 7-(-1) \\ 1+3(-1) \\ -5+(-1) \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix}$$

Hence $\quad AP' = \sqrt{(8-3)^2 + (-2-2)^2 + (1--6)^2}$

$$= \sqrt{25+16+49}$$

$$= \sqrt{90}$$

$$= 3\sqrt{10} \text{ (shown)}$$

Exercise 16.6

1 The point A has position vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$, the line l has equation $\mathbf{r}=\begin{pmatrix}1\\4\\-1\end{pmatrix}+t\begin{pmatrix}1\\-1\\1\end{pmatrix}$. Find the length of the perpendicular from A to l. **(C)**

[Ans : $2\sqrt{2}$]

2 The point A has coordinates (3 ,–1, 5) and the line l has equation $\mathbf{r}=\begin{pmatrix}8\\0\\-1\end{pmatrix}+t\begin{pmatrix}-6\\1\\4\end{pmatrix}$.

Find the coordinates of the point B on l such that AB is perpendicular to l. **(C)**

[Ans : (2, 1, 3)]

3 The equation of a straight line l is $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}-1\\1\\1\end{pmatrix}$, where t is a parameter. Find the position vector of P on l such that OP is perpendicular to l. **(C)**

[Ans : $\left(\frac{7}{3},\frac{2}{3},\frac{5}{3}\right)$]

4 The point A has position vector $\begin{pmatrix}3\\0\\-1\end{pmatrix}$, the line l has equation $\mathbf{r}=\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix}$. Find the length of the perpendicular from A to l.

[Ans : $\sqrt{\frac{37}{7}}$]

5 The point A has coordinates (2 ,3, –2) and the line l has equation $\mathbf{r}=\begin{pmatrix}5\\0\\-2\end{pmatrix}+t\begin{pmatrix}-3\\2\\-1\end{pmatrix}$.

Find the coordinates of the point B on l such that AB is perpendicular to l.

[Ans : $(\frac{5}{4},\frac{5}{2},-\frac{13}{4})$]

16.7 Equation of plane

We usually use the symbol π to denote a plane. The equation of a plane can be expressed in Cartesian form : $ax + by + cz = d$ or in vector form $r \bullet \hat{n} = a \bullet \hat{n}$ where $\hat{n}$ is a vector perpendicular to the plane.

It is possible to express the Cartesian form in vector form and vice–versa.

For example, $2x + 3y - 4z = 5$ can be expressed as $r \bullet \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 5$.

Similarly, $r \bullet \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = 3$ can be rewritten as $x + 2y + 5z = 3$.

Exercise 16.7

Express the following Cartesian equations of plane into vector form.

1 $3x - y + 5z = 1$ [Ans : $r \bullet \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 1$]

2 $2x - 3y + z = 4$ [Ans : $r \bullet \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 4$]

3 $5x - 2y + 3z = 2$ [Ans : $r \bullet \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = 2$]

Express the following vector equation of plane to Cartesian form.

4 $r \bullet \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = 5$ [Ans : $2x + 3z = 5$]

5 $r \bullet \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 3$ [Ans : $x + 2y = 3$]

6 $r \bullet \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = -1$ [Ans : $4x + y + 3z + 1 = 0$]

16.8 Using cross–product to find a vector ($\hat{n}$) perpendicular (normal) to a plane

To find a vector normal to a plane, we have to find the direction vectors of two lines and use the method of cross–product. The latter will be explained through an example.

Example 1

Using the cross–product find $\begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 5 & -3 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 5 & -3 & 4 \end{vmatrix} = \mathrm{i}\begin{vmatrix} 1 & 0 \\ -3 & 4 \end{vmatrix} - \mathrm{j}\begin{vmatrix} 2 & 0 \\ 5 & 4 \end{vmatrix} + \mathrm{k}\begin{vmatrix} 2 & 1 \\ 5 & -3 \end{vmatrix}$$

$$= 4\mathbf{i} - 8\mathbf{j} - 11\mathbf{k}.$$

Example 2

Find a vector perpendicular to the plane containing the vectors $\mathbf{AB} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{AC} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

Solution

Using $\mathbf{AB} \times \mathbf{AC}$, we have

$$\hat{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ 2 & -1 & 5 \end{vmatrix} = \mathrm{i}\begin{vmatrix} -1 & 3 \\ -1 & 5 \end{vmatrix} - \mathrm{j}\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} + \mathrm{k}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$$= -2\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Hence a vector perpendicular to the plane containing the vectors **AB** and **AC** is $-2\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Example 3

If $\boldsymbol{a} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\quad \mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\quad \mathrm{c} = 2\mathrm{i} - \mathbf{j} + 5\mathbf{k}$.

Find a vector perpendicular to the plane ABC.

Solution

$$\mathbf{AB} = \mathbf{b} - \boldsymbol{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

Using $\mathbf{AB} \times \mathbf{AC}$, we have

$$\hat{n} = \begin{vmatrix} i & j & k \\ 3 & -2 & 2 \\ 4 & -2 & 4 \end{vmatrix} = \mathrm{i}\begin{vmatrix} -2 & 2 \\ -2 & 4 \end{vmatrix} - \mathrm{j}\begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} + \mathrm{k}\begin{vmatrix} 3 & -2 \\ 4 & -2 \end{vmatrix}$$

$$= -4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}.$$

Hence a vector perpendicular to the plane ABC is $-4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

Exercise 16.8

Perform the following cross–products.

1 $\begin{vmatrix} i & j & k \\ 4 & 0 & 2 \\ 2 & -5 & 5 \end{vmatrix}$ [Ans : 10i – 16j – 20k]

2 $\begin{vmatrix} i & j & k \\ 0 & 5 & 4 \\ -1 & 3 & 6 \end{vmatrix}$ [Ans : 18i + 4j + 5k]

3 $\begin{vmatrix} i & j & k \\ 2 & 5 & 0 \\ -1 & -2 & 3 \end{vmatrix}$ [Ans : 15i – 6j + k]

4 $\begin{vmatrix} i & j & k \\ 4 & 2 & 1 \\ 0 & -3 & 1 \end{vmatrix}$ [Ans : 5i – 4j – 12k]

5 Find a vector normal to a plane containing the vectors $\mathbf{AB} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ and $\mathbf{AC} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. [Ans : –i – 2j – k]

6 If $a = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$.

Find a vector perpendicular to the plane ABC. [Ans : –8i + j – 9k]

7 If $a = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$. Find a vector perpendicular to the plane OAB.

[Ans : 5i + 10j – 25k]

16.9 Using cross–product to find the equation a plane

To find the equation of plane, first we have to find $\hat{n}$, the normal to the plane. Next we have to apply the following formula.

$$\mathbf{r}\bullet\hat{n} = a\bullet\hat{n},$$

Where a is any point on the plane.

Example

If $\boldsymbol{a} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\quad \mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\quad \mathbf{c} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

Find the vector equation of the plane ABC.

Solution

$$\text{AB} = \text{b} - a = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \qquad \text{and}$$

$$\text{AC} = c - a = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$$

Using AB × AC, we have

$$\hat{n} = \begin{vmatrix} i & j & k \\ -1 & -2 & 0 \\ -2 & 2 & 3 \end{vmatrix} = \text{i}\begin{vmatrix} -2 & 0 \\ 2 & 3 \end{vmatrix} - \text{j}\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} + \text{k}\begin{vmatrix} -1 & -2 \\ -2 & 2 \end{vmatrix}$$

$$= -6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}.$$

Using $\mathbf{r}\bullet\hat{n} = a\bullet\hat{n}$, we have

$$\mathbf{r}\bullet\begin{pmatrix} -6 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\bullet\begin{pmatrix} -6 \\ 3 \\ -6 \end{pmatrix}$$

or $\qquad \mathbf{r}\bullet\begin{pmatrix} -6 \\ 3 \\ -6 \end{pmatrix} = -21$

or $\qquad \mathbf{r}\bullet\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 7 \qquad$ [divide by –3 on both sides]

We can also give the answer in Cartesian form as $2x - y + 2z = 7$.

Exercise 16.9

1 Find the equation of the plane containing the vectors,

$AB = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $AC = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, given that A has position vector $\begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}$.

[Ans : $r \cdot \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} = -16$]

2 Show that the equation of the plane containing the vectors $AB = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $AC = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ is $r \bullet \begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix} = 14$, given that the point A has position vector $\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$.

3 If $a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $c = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Show that the vector equation of the plane ABC is $r \bullet \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 4$.

4 If $a = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$, $c = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$.

Show that the vector equation of the plane ABC is $-14x + 9y + 3z + 8 = 0$.

5 The line l has equation $r = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \lambda \, \Re$.

Find the equation of the plane containing O and l. (C)

[Ans : $x - 2y - z = 0$]

16.10 How to show that a point lies on a plane?

If a point satisfies the equation of a plane the point is said to lie on the plane.

Example 1

Show that the point (7, 2, −1) lies on the plane $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 0$. **(C)**

Solution

Consider $\begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 7 - 4 - 3 = 0$ (R.H.S) **(shown)**

Example 2
Show that the point (3, 5, −2) lies on the plane $2x + 3y - 5z = 31$
Solution
Put $x = 3$, $y = 5$ and $z = -2$ in the equation of the plane. We should get 31.
$2(3) + 3(5) - 5(-2) = 6 + 15 + 10 = 31$ **(shown)**

Exercise 16.10

1 Show that the point (3,−2,1) lies on the plane $2x + 4y + 3z = 1$. **(C)**

2 The planes $\prod_1$ and $\prod_2$ have equations
$3x - y - z = 2$ and $x + 5y + z = 14$, respectively.
Show that the point (1,3,−2) lies on both planes. **(C)**

3 Show that the point (5,1,0) lies on the plane $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 9$.

4 Show that the point (2,4,−3) lies on the plane $\mathbf{r} \bullet \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = -2$.

5 Show that the point (3,0,0) lies on the plane $\mathbf{r} \bullet (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$. **(C)**

6 Show that the point (0,2,0) lies on the plane $\mathbf{r} \bullet (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$. **(C)**

7 Show that the point $(3, 1, \frac{11}{4})$ lies on the plane $\mathbf{r} \bullet (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$. **(C)**

16.11 How to show that a line on a plane?

If the equation of a line satisfies the equation of a plane, irrespective of the value of the parameter, the line is said to lie on the plane.

Example

Show that the line l having equation $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ lies on the plane $\mathbf{r} \bullet \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = -5$. (C)

Solution

l can be written as $\mathbf{r} = \begin{pmatrix} 5+2\lambda \\ \lambda \\ 5 \end{pmatrix}$

Substituting in the equation of plane, we have

$$\begin{pmatrix} 5+2\lambda \\ \lambda \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = -5 - 2\lambda + 2\lambda + 0 = -5 \text{ (R.H.S) (shown)}$$

Exercise 16.11

1 Show that the line l having equation $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ lies on the plane $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4$

2 Show that the line l having equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ lies on the plane $\mathbf{r} \bullet \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} = -10$

3 Show that the line l having equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ lies on the plane $\mathbf{r} \bullet \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 4$.

4 The plane p has equation $3x + 2y - z + 1 = 0$ and the line l has Cartesian equations $\frac{x-4}{-1} = \frac{y+3}{2} = \frac{z-7}{1}$. Show that l lies in p. (C)

5 The equation of a plane p is $x - 6y + 2z = -5$, and the line m has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \\ -13 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Show that m lies in p. (C)

16.12 How to show that a line is parallel to a plane?

If a line l is parallel to a plane p, then the direction vector, d, of l and the normal, $\hat{n}$, to the plane p must be perpendicular.

$$d \bullet \hat{n} = 0$$

Example

Show that the line l having equation $r = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ is parallel to the plane $r \bullet \begin{pmatrix}-2\\1\\0\end{pmatrix} = 5$.

Solution

The direction vector of the line is $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\hat{n} = \begin{pmatrix}-2\\1\\0\end{pmatrix}$

Now $d \bullet \hat{n} = \begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}-2\\1\\0\end{pmatrix} = -2 + 2 + 0 = 0$

Since $d \bullet \hat{n} = 0$, l is parallel to p.

Exercise 16.12

1 Show that the line l having equation

$r = \begin{pmatrix}1\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$ is parallel to the plane $r \bullet \begin{pmatrix}-2\\2\\0\end{pmatrix} = 11$.

2 Show that the line l having equation

$r = \begin{pmatrix}3\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\4\end{pmatrix}$ is parallel to the plane $r \bullet \begin{pmatrix}12\\2\\-9\end{pmatrix} = 7$.

3 The straight line l passes through the points A and B with position vectors $7i - 3j + 6k$ and $10i + 3k$ respectively.
The plane p has equation $3x - y + 2z = 8$. Show that l is parallel to p. **(C)**

4 The points A, B and C have position vectors $i + 2j + 2k$, $2i + j + 2k$, $2i + 2j + k$, respectively, with respect to the origin O.

Show that BC is parallel to the plane $\prod$ having equation $r \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 9$.

16.13 Finding the point of intersection of a line and a plane

Example

Find the position vector of the point P where the line l with equation $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ meets the plane $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 1.$

Solution

l can be written as $\mathbf{r} = \begin{pmatrix} 5+2\lambda \\ \lambda \\ 5 \end{pmatrix}$

Substituting in the equation of plane, we have

$$\begin{pmatrix} 5+2\lambda \\ \lambda \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 11$$

$$5 + 2\lambda + 2\lambda + 10 = 11$$

$$4\lambda = -4$$

$$\lambda = -1$$

P has coordinates $(5 + 2(-1), -1, 5)$
$= (3, -1, 5).$

Exercise 16.13

1 The line l has equation $\mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ and the plane π has equation $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1.$

Find the coordinates of the equation of the point A where l meets π. **(C)**

[Ans : (2,–3,1)]

2 The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and the plane p has equation $\mathbf{r} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 2.$

Find the point of intersection of l and p. **(C)**

[Ans : (4,1,2)]

3 Two planes p_1 and p_2 have equation $r \bullet \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 4$ and $r \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1$, respectively.

The line l passes through the point P with position vector $\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$ and is parallel to $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$. Verify that l lies p_1 and find the position vector of the point where l meets p_2.

[Ans : (2,1,0)]

4 The plane p has equation $3x + 2y - z + 1 = 0$ and the line m has equation

$$r = \begin{pmatrix} 0 \\ 10 \\ 7 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

The line m intersects p at the point A. Find the coordinates of A.(**C**)
[Ans (–2,4,3)]

5 The points P and Q have position vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$ respectively, referred to the origin O.
Find the position vector of the point where the line PQ meets the plane $z = 0$. (**C**)

[Ans $(-2, \frac{1}{2}, 0)$

6 The line l has equation $r = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix}$ an the plane π has equation $r \bullet \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 15$.
Find the coordinates of the point C where l meets π. [Ans :(–4,2,7)]

7 The plane p has equation π has equation $3x - y - z = 2$ and the line l_2 has equation $r = \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$. Find the coordinates of the point of intersection of l_2 and π. (**C**)

[Ans (2,–3, 7)]

8 The equation of the plane π_1 is $y + z = 0$ and the equation of the line l is $r = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, where t is a parameter. Find the position vector of the point of intersection of l and π_1. (**C**) [Ans :(1,4,– 4)]

16.14 Finding the perpendicular distance from a point to a plane

Example

Find the perpendicular distance from the point P(2, 1, 4) to the plane $\mathbf{r} \bullet \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6$. (C)

Solution

Equation of the line passing through P and perpendicular to the plane is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+2t \\ 1+3t \\ 4 \end{pmatrix}$$

where the line meets the plane,

$$\begin{pmatrix} 2+2t \\ 1+3t \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6$$

$$4 + 4t + 3 + 9t = -6$$
$$13t = -13$$
$$t = -1$$

The foot of the perpendicular from P to the plane is $\begin{pmatrix} 2+2(-1) \\ 1+3(-1) \\ 4 \end{pmatrix} = (0,-2,4)$

Hence the perpendicular distance from P(2, 1, 4) to (0,–2,4) is

$$\sqrt{(0-2)^2 + (-2-1)^2 + (4-4)^2} = \sqrt{4+9+0} = \sqrt{13}.$$

Alternative

The above question can also be done by using the following formula.

$$p = \left|\frac{ax + by + cz - d}{\sqrt{a^2 + b^2 + c^2}}\right|,$$

where p is the perpendicular distance and $ax + by + cz - d$, the Cartesian equation of the plane.

From the above example, $\quad p = \left|\frac{2x + 3y + 0z + 6}{\sqrt{2^2 + 3^2 + 0^2}}\right|$

At P(2,1, 4), we have $\quad p = \left|\frac{2(2) + 3(1) + 0(4) + 6}{\sqrt{13}}\right|$

$$= \frac{13}{\sqrt{13}} = \sqrt{13}.$$

Exercise 16.14

1 Find the perpendicular distance from the point B (7,0,3) to the plane $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1$.

[Ans : $2\sqrt{6}$]

2 The point A has position vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and the plane p has equation $\mathbf{r} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 2$.

Find the length of the perpendicular from A to P. [Ans : 7/3]

3 The planes Π_1 and Π_2 have equations
$x - 2y + 3z = 0$ and $3x + y + 2z = 0$, respectively.

Find the length of the perpendicular from P(7, 2, −1) to Π_2. **(C)** [Ans : $\frac{3}{2}\sqrt{14}$]

4 Two planes p_1 and p_2 have equations $\mathbf{r} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 4$ and $\mathbf{r} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1$, respectively.

Find the length of the perpendicular from the origin O to p_1. **(C)** [Ans : $\frac{2}{7}\sqrt{14}$

5 The planes Π_1 and Π_2 have equations
$3x - y - z = 2$ and $x + 5y + z = 14$, respectively.

Show that the length of the perpendicular from P(2, −3, 7) to Π_2 is $\frac{20}{3\sqrt{3}}$. **(C)**

6 The equation of a plane p is $x - 6y + 2z = -5$, and the point A has coordinates (3, −12, 1). Show that the perpendicular distance from A to p = $\sqrt{164}$. **(C)**

7 The plane Π has equation $x + 2y + 3z = 6$. Show that the perpendicular distance from the origin to Π is $\sqrt{\frac{6}{7}}$. **(C)**

8 Show that the perpendicular distance from the point the origin O to the plane with equation $\mathbf{r} \bullet (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$ is $\frac{66}{\sqrt{1717}}$. **(C)**

16.15 Angle between two planes

To find the angle between two planes, we have to find the angle between the normals of the two planes.
Thus if $\mathbf{r} \bullet \mathbf{n}_1 = p_1$ and $\mathbf{r} \bullet \mathbf{n}_2 = p_2$ are the equations of the two planes, then the angle θ, between the two planes is given by

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

Example
The planes $\prod_1$ and $\prod_2$ have equations
$x - 2y + 3z = 0$ and $3x + y + 2z = 0$, respectively.
Show that the acute angle between $\prod_1$ and $\prod_2$ is $60°$. **(C)**
Solution
We have

$$\prod_1: \quad \mathbf{r} \bullet \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 0$$

$$\prod_2: \quad \mathbf{r} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 0$$

Let θ be the angle between the two given planes

$$\text{Then} \quad \cos\theta = \frac{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{1^2 + (-2)^2 + 3^2}\sqrt{3^2 + 1^2 + 2^2}}$$

$$= \frac{7}{\sqrt{14}\sqrt{14}}$$

$$\therefore \quad \theta = 60° \text{ (shown)}$$

Exercise 16.15

1 Two planes p_1 and p_2 have equation $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 4$ and $\mathbf{r} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1$, respectively.

Find the acute angle between p_1 and p_2. **(C)** [Ans : 79.1°]

2 The planes Π_1 and Π_2 have equations
$3x - y - z = 2$ and $x + 5y + z = 14$, respectively.
Show that the acute angle between Π_1 and Π_2 is 80.0°.

3 Two planes p_1 and p_2 have equations $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 15$ and $\mathbf{r} \bullet \begin{pmatrix} 10 \\ 16 \\ 11 \end{pmatrix} = 5$, respectively.

Show that the acute angle between p_1 and p_2 is 68.1°.

4 The plane Π has equation $x + 2y + 3z = 6$. Calculate the acute angle between Π and the plane $z = 0$. **(C)** [Ans : 36.7°]

5 Two planes Π_1 and Π_2 have equations $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2$ and $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} = 3$, respectively.

Show that the acute angle between p_1 and p_2 is 68.0°.

6 The equation of the plane π_1 is $y + z = 0$ and the equation of the plane π_2 is

$$\mathbf{r} \bullet \begin{pmatrix} -8 \\ 11 \\ 9 \end{pmatrix} = 2$$

Show that the acute angle between Π_1 and Π_2 is 29.9°. **(C)**

7 Two planes p_1 and p_2 have equations $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 2$ and $\mathbf{r} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 4$, respectively.

Show that the acute angle between p_1 and p_2 is 50.8°.

16.16 Angle between a line and a plane

To find the angle between a line and a plane, first we have to find the angle between the normal of the plane and the direction vector of the line.

$\cos\theta = \dfrac{\hat{n}\cdot d}{|\hat{n}||d|}$

Now the angle between the line and the plane is given by $\alpha = 90^\circ - \theta$.

Note : An alternative way is to find the sine of the angle between the normal to the plane and the direction vector of the line.

Example

Find the acute angle between the line l $\quad r = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

and the plane p $\quad r \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 2.$ **(C)**

Solution

Let θ be the angle between the normal to the plane and the direction vector of the line

Then $\cos\theta = \dfrac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\bullet\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}}{\sqrt{1^2+(-1)^2+1^2}\sqrt{(-1)^2+2^2+2^2}}$

$= \dfrac{-1}{\sqrt{3}\sqrt{9}}$

$\therefore$ acute angle $\theta = 78.9$

Hence the angle between the line and the plane is $90^\circ - 78.9^\circ = 11.1^\circ$.

Exercise 16.16

1 Find the acute angle between the line l with equation $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and the plane $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 11$. [Ans : 36.6°]

2 The line l has equation $\mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ and the plane π has equation $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1$.

Find the acute angle between l and π. [Ans : 52.6°]

3 The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the plane p has equation $\mathbf{r} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 2$.

Find the acute angle between l and p. [Ans : 11.1°]

4 The plane p_1 has equation $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$. The line l passes through the point A with position vector $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ and is parallel to $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find the acute angle between l and p_1.

[Ans : 28.6°]

5 The plane p has equation $2x + 3y - 4z + 5 = 0$ and the line l has equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

Find the acute angle between l and p_1. [Ans : 39.1°]

6 The points P and Q have position vectors $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ respectively, referred to the origin O. Find the acute angle between the line PQ and the plane $x + y = 0$.

[Ans : 90°]

16.17 How to find the equation of the line of intersection of two planes

Use the following steps to find the required equation:
Step 1 : Express the given equation in Cartesian form
Step 2 : Express x and y in terms of z, **or** x and z in terms of y **or** y and z in terms of x.

Example 1
Two planes have equations $3x - y - z = 2$ and $x + 5y + z = 14$. The planes intersect in a straight line l. Find the vector equation of the line l.
Solution
The given equations are
$3x - y - z = 2$ ....(1)
$x + 5y + z = 14$ ....(2)
Adding , we have
$4x + 4y = 16$
$x + y = 4$
$x = 4 - y$
From (2), we have $4 - y + y + z = 14$
$z = 10 - 4y$
Hence equation of line of intersection is
$r = (4 - y, y, 10 - 4y)$
$= (4, 0, 10) + y(-1, 1, -4)$
or $r = (4, 0, 10) + t(1, -1, 4)$

Example 2
Two planes have equations $x + 2y - 2z = 2$ and $2x - 3y + 6z = 3$. The planes intersect in a straight line l. Find the vector equation of the line l. **(June 2003/Q9)**
Solution
The given equations are
$x + 2y - 2z = 2$ ....(1)
$2x - 3y + 6z = 3$ ....(2)
Multiplying (1) by 2, we have $2x + 4y - 4z = 4$
Solving simultaneously with (2), we obtain
$2x + 4y - 4z = 4$
$\underline{2x - 3y + 6z = 3}$
$7y - 10z = 1$
$y = \frac{1+10z}{7}$

From (1), we have $x + 2(\frac{1+10z}{7}) - 2z = 2$
$7x + 2 + 20z - 14z = 14$
$7x = 12 - 6z$
$x = \frac{12-6z}{7}$

Hence equation of line of intersection is

$$r = \left(\frac{12-6z}{7}, \frac{1+10z}{7}, z\right)$$
$$= \left(\frac{12}{7}, \frac{1}{7}, 0\right) + z\left(-\frac{6}{7}, \frac{10}{7}, 1\right)$$
$$= \left(\frac{12}{7}, \frac{1}{7}, 0\right) + t(-6, 10, 7) \quad \textbf{(Ans)}$$

Exercise 16.17

1 Two planes p_1 and p_2 have equation $r \bullet \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 4$ and $r \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1$, respectively.

The planes intersect in a straight line l. Find the vector equation of the line l.
[Ans : r = (2, 1, 0) + t(5, –1, 1)]

2 Two planes p_1 and p_2 have equations $r \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 2$ and $r \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 4$, respectively.

The planes intersect in a straight line l. Find the vector equation of the line l.
[Ans : r = (–6, 0, 4) + t(2, 1, –1)]

3 Two planes p_1 and p_2 have equations $r \bullet \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 4$ and $r \bullet \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 5$, respectively.

The planes intersect in a straight line l. Find the vector equation of the line l.
[Ans : r = (0, 23, 9) + t(–1, 8, 3)]

4 Two planes $\prod_1$ and $\prod_2$ have equations $r \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2$ and $r \bullet \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} = 3$, respectively.

The planes intersect in a straight line l. Find the vector equation of the line l.
[Ans : r = (0, 0, 1) + t(–9, 1, 4)]

5 The equation of the plane π_1 is y + z = 0 and the equation of the plane π_2 is
$r \bullet \begin{pmatrix} -8 \\ 11 \\ 9 \end{pmatrix} = 2$

The planes intersect in a straight line l. Find the vector equation of the line l.
[Ans : r = (–¼, 0, 0) + t(1, 4, –4)]

CHAPTER 17

DIFFERENTIAL EQUATIONS

A differential equation is an equation which involves at least one derivative of an unknown function. The following are some examples of equations:

$$\frac{dy}{dx} = \cos x$$

$$x\frac{dy}{dx} = y^2 - 1$$

17.1 Formulating a simple statement involving a rate of change as a differential equation

Example 1

It is known that the rate of decay of a radioactive substance is proportional to the amount present. Express this in the form of a differential equation.

Solution

Let y be the amount of radioactive substance present at time t.

Then $\quad -\frac{dy}{dt}$ is the rate of change. There is a minus sign because there is **decay**.

Given $\quad -\frac{dy}{dt} \propto y$

$-\frac{dy}{dt} = ky$, where k is the constant of proportionality.

Example 2

It is known that the rate of increase of the population of bacteria is proportional to the number of bacteria. Express this statement as differential equation.

Solution

Let p be population of bacteria present at time t.

Then $\frac{dp}{dt}$ is the rate of change

Given $\quad \frac{dp}{dt} \propto p$

$\frac{dp}{dt} = kp$, where k is the constant of proportionality.

Exercise 17.1

Express the following statements as differential equations.

1 The rate of increase of the length, l, of a shadow is assumed to be constant.

[Ans : $\frac{dl}{dt} = k$]

2 The rate of decay of a radioactive material is proportional to the mass, m, of the material.

[Ans : $-\frac{dm}{dt} = km$]

3 The deceleration of a moving body is proportional to the square root of the velocity, v.

[Ans : $-\frac{dv}{dt} = k\sqrt{v}$]

4 The gradient of a curve at any point is proportional to the square of the x–coordinates at that point.

[Ans : $\frac{dy}{dx} = kx^2$]

5 The surface of a pond is partially covered with weed. The weed is increasing in area at a rate proportional to its area, A, at that instant.

[Ans : $\frac{dA}{dt} = kA$]

6 A vessel contains liquid which flows out from a small hole at a point O in the base of a vessel. At time t, in minutes, the height of the liquid surface above O is y and the rate at which y is decreasing is inversely proportional to $y^{\frac{3}{2}}$. Express the statement in the form of a differential equation relating to y and t.

[Ans : $-\frac{dy}{dt} = \frac{k}{y^{\frac{3}{2}}}$]

7 A circular patch of oil on the surface of water has radius, r metres at time t minutes. The rate of increase of r is taken to be a constant. Express the statement as a differential equation.

[Ans : $\frac{dr}{dt} = k$]

8 Newton's law of cooling states that the rate of change of temperature in a cooling body is proportional to the difference in temperature between the body and its surroundings. Using t for the time in minutes, x for temperature of the cooling body in °C and x_0 for the temperature of the surroundings (assumed to be constant), express the law in the form of a differential equation.

[Ans : $\frac{dx}{dt} = k($

9 Newton's law of gravitation states that the acceleration of a particle is inversely proportional to the square of the distance between the particle and the centre of the Earth. Using x for that variable distance and t for time, express the law in the form of a differential equation relating to x and t. [Ans : [illegible]

17.2 First order differential equations – variable Separable

A first order differential equation of the form

$$\frac{dy}{dx} = f(x)\, g(y),$$

where f is a function of x alone and g is a function of y alone, is called a separable equation. The general solution of such a separable equation is obtained by direct integration

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Example 1

Find y in terms of x given that $\quad x\frac{dy}{dx} = 1 - 2y$. **(C)**

Solution

We have $\displaystyle\int \frac{1}{1-2y}\,dy = \int \frac{1}{x}\,dx$

$$-\frac{1}{2}\ln(1-2y) = \ln x + c$$

$$-\ln(1-2y) = 2\ln x + 2c$$

$$\ln(1-2y)^{-1} = \ln x^2 + \ln A$$

$$\ln\frac{1}{1-2y} = \ln Ax^2$$

Removing $\ln$, we have

$$\frac{1}{1-2y} = Ax^2$$

$$1 - 2y = \frac{1}{Ax^2}$$

$$-2y = \frac{1}{Ax^2} - 1$$

$$2y = 1 - \frac{1}{Ax^2}$$

$$y = \frac{1}{2}\left(1 - \frac{1}{Ax^2}\right),$$

which is the general solution.

Exercise 17.3

Find the particular solution for each of the following differential equation.

1 $y\dfrac{dy}{dx} = x^3$; $y = 4$ when $x = 2$ [Ans : $2y^2 = x^4 + 16$]

2 $\dfrac{dy}{dx} = \dfrac{y}{x}$; $y = e^2$ when $x = e$ [Ans : $y = ex$]

3 $\cos y \dfrac{dy}{dx} = \dfrac{1}{x}$; $y = \dfrac{\pi}{2}$ when $x = 1$ [Ans : $\sin y = \ln x + 1$]

4 $\sec^2\theta \dfrac{d\theta}{dt} = \dfrac{2t}{t^2+1}$; $\theta = \dfrac{\pi}{4}$ when $t = 0$ [Ans : $\tan\theta = \ln(1+t^2) + 1$]

5 $r\dfrac{dr}{dt} = 4$; $r = 2$ when $t = 1$ [Ans : $r^2 = 8t - 4$]

6 $\dfrac{dr}{dt} = \dfrac{1}{r}$; $r = 4$ when $t = 0$ [Ans : $r^2 = 2t + 16$]

7 Solve the differential equation $\dfrac{dy}{dx} = 1 - y$ given that $y < 1$ and that $y = 0$ when $x = 0$. (C) [Ans : $y = 1 - e^{-x}$]

8 Find y in terms of x given that $\dfrac{dy}{dx} = (x + 1)(y + 1)$, $(y > -1)$, and that $y = 0$ when $x = 0$. (C) [Ans : $y = e^{\frac{x^2}{2}+x} - 1$]

9 Find the general solution of the differential equation $\cos^2 x \dfrac{dy}{dx} = 1 - y^2$, where $0 < y < 1$. (C) [Ans : $\dfrac{1}{2}\ln\left(\dfrac{1+y}{1-y}\right) = \tan x + c$]

10 Solve the differential equation $\dfrac{dy}{dx} = y \cot x$ $(0 < x < \pi, y > 0)$ given that $y = 2$ when $x = \dfrac{\pi}{6}$. (C) [Ans : $y = 4 \sin x$]

11 Solve the differential equation $\dfrac{dy}{dx} = \tan y \cos x$ given that $x = \dfrac{\pi}{4}, y = \dfrac{\pi}{4}$. (C) [Ans : $\ln \sin y = \sin x - 1.05$]

12 Solve the differential equation $\dfrac{dy}{dx} = xy$ given that, $y = 2$ when $x = 0$. (C) [Ans : $y = 2e^{\frac{x^2}{2}}$]

17.4 Progressive refinement of a model

In this section emphasis would be laid on how to interpret the solution of a differential equation in the context of a problem modelled by the equation.

Example 1

A vessel contains liquid which flows out from a small hole at a point O in the base of the vessel. At time t, in minutes, the height of the liquid surface above O is y and the rate at which y is decreasing is inversely proportional to $y^{\frac{3}{2}}$. Write down a differential equation which expresses $\frac{dy}{dt}$ in terms of y.

Given that $y = 25$ when t = 0 and that $y = 16$ when t = 30, find the value of t when $y = 0$.

Solution

The rate of change of y, $\frac{dy}{dt}$, is negative since the liquid is flowing out

Given $\quad -\frac{dy}{dt} \propto \frac{1}{y^{\frac{3}{2}}}$

$$-\frac{dy}{dt} = \frac{k}{y^{\frac{3}{2}}}$$

We have $\quad \int -y^{\frac{3}{2}}\,dy = \int k\,dt$

$$-\frac{2}{5}y^{\frac{5}{2}} = kt + c$$

Given $y = 25$ when t = 0,

$$-\frac{2}{5}(25)^{\frac{5}{2}} = k(0) + c$$

$$c = -1250$$

Given $y = 16$ when t = 30,

$$-\frac{2}{5}(16)^{\frac{5}{2}} = k(30) - 1250$$

$$30k = 845.2$$

$$k = 28$$

The particular solution is $\quad -\frac{2}{5}y^{\frac{5}{2}} = 28t - 1250$

Now when $y = 0$, $\quad 0 = 28t - 1250$

$$t = \frac{1250}{28}$$

$$= 44.4 \text{ minutes}$$

4 A circular patch of oil on the surface of water has radius r metres at time t minutes. When t = 0, r = 1 and when t = 10, r = 2. It is desired to predict the value T of t, when r = 4.

(i) In a simple model the rate of increase of r is taken to be a constant. Find T for this model.

(ii) In a more refined model, the rate of increase of r is taken to be proportional to $\frac{1}{r}$.

Express this statement as a differential equation, and find the general solution. Find T for this second model. **(C)**

[Ans : (i) T = 30 (ii) $\frac{1}{2}r^2 = kt + c$; T = 50]

5 During a spell of freezing weather, the ice on a pond has thickness x mm at time t hours after the start of freezing. At 3.00 p.m., after one hour of freezing weather, the ice is 2 mm thick and it is desired to predict when it will be 4 mm thick.

(i) In a simple model, the rate of increase of x is assumed to be constant. For this model, express x in terms of t and hence determine when the ice will be 4 mm thick.

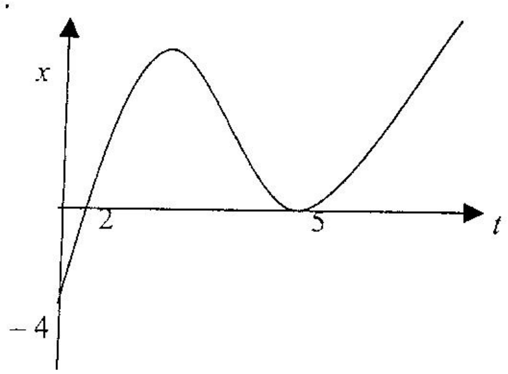
(ii) In a more refined model, the rate of increase of x taken to be proportional to $\frac{1}{x}$.

Set up a differential equation and hence show that the thickness of the ice is proportional to the square root of the time elapsed from the start of freezing. Determine the time at which the second model predicts that the ice will be 4 mm thick.

(iii) What assumption about the weather underlies both models? **(C)**

[Ans : (i) x = 2t ; 5 p.m. (ii) $\frac{dx}{dt} = \frac{k}{x}$; $x = 2\sqrt{t}$; 7 p.m. (iii) constant temperature]

6 The displacement from the origin, s cm, of a particle at time t seconds is shown by the graph below.



It is proposed to model the motion by expressing s as a polynomial in t.

(a) In a simple model the order of the polynomial is taken as 3.

(i) Give a reason why this is a sensible assumption.

(ii) The model to be used is of the form $s = a(t - \alpha)(t - \beta)^2$.
Write down the points where the curve meets the coordinates axes and use them to find values for α, β and a.

(iii) Calculate the values obtained from the model when t = 1 and t = 3.

(iv) The true values when t = 1 and t = 3 are −1.5 and 0.4. What are the percentage errors in the values predicted by the model?

(b) A refined model for the data is $s = a(t-\alpha)(t-\beta)^2(1-\gamma t)$, where a, α and β take the same values as before and γ is a small positive constant.

(i) If γ is chosen so that the model and the graph are in agreement when t = 3, find the value of γ.

(ii) Using this value of γ, find the percentage error in the value predicted by the model when t = 1.

(iii) Comment on the relative merits of the two models. **(C)**

[Ans : (a) (i) 2 turning points ; (ii) 2, 5 , $\frac{2}{25}$ (iii) −1.28, 0.32 (iv) 14.7%, 20%

(b) (i) $\gamma = -\frac{1}{12}$ (ii) −1.39, 7.3%]

7(a) In a simple model of a certain process the rate of production of yeast is kx grams per minute, where x grams is the amount already produced.
Write down a differential equation relating x and the time t, measured in minutes.
Show that, if $k = 0.003$, then the amount of yeast is doubled in about 230 minutes.

(b) In a refined model of the process, yeast is removed at a constant rate of m grams per minute, so that the differential equation becomes $\frac{dx}{dt} = kx - m$.
Find an expression for the amount of yeast at time t minutes, given that at t = 0 there were p grams.
Deduce that, if $m \le kp$, the supply of yeast is never exhausted.
Find the value of m to 3 significant figures if $k = 0.003$, $p = 20\ 000$ and the supply is exhausted in 100 minutes. **(C)**

[Ans : (a) $\frac{dx}{dt} = kx$ b(i) $x = \frac{m}{k} + \left(p - \frac{m}{k}\right)e^{kt}$ (iii) m = 231]

8(a) Show that, if x>1, the sum of the infinite series $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots$ is $\frac{x}{x-1}$.

(b) A doctor prescribes a dose of 10 mg of a drug to be taken at the same time each day. In a simple model, the amount, x mg, of the drug retained t hours after the dose is given by $x = 10e^{-\frac{t}{24}}$.

(i) Show that the amount of the drug remaining in the body immediately after the second dose is $10 + \frac{10}{e}$ mg.

(ii) Show that the amount of the drug remaining in the body immediately after the third dose is $10 + \frac{10}{e} + \frac{10}{e^2}$ mg.

(iii) State the amount of drug remaining in the body immediately after the fourth dose.

(iv) Sketch a graph of x against t for the first three days of the treatment.

(v) Show that, if the course of treatment is continued indefinitely, the amount of the drug in the body after each dose approaches $\frac{10e}{e-1}$.

(c) The model is refined, so that the doctor gives an initial dose of $\frac{10e}{e-1}$ mg, followed by doses of 10 mg.

(i) Show that, immediately before the second dose, the amount of the drug remaining in the body is $\frac{10e}{e-1}$ mg.

(ii) Show that, immediately after the second dose, the amount of the drug remaining in the body is $\frac{10e}{e-1}$ mg.

(iii) Give a reason why the second model may be preferable to the first model in some circumstances. **(C)**

[Ans : (b) (iii) $10+\frac{10}{e}+\frac{10}{e^2}+\frac{10}{e^3}$ mg]

9 The number of television sets, n, sold each month by a company depends on the profit £p, that it makes on each set. When p = 20, n = 60 and when p = 50, n = 0.

(a) In a simple model n is taken to be linearly related to p.

(i) Find an expression for n in terms of p.

(ii) Explain why the total profit for the month, £T, is given by $T = 100p - 2p^2$.

(iii) Sketch a graph of T against p.

(iv) What value of p will result in the maximum profit for the company?

(v) For what values of p is the model valid?

(b) In a refined model, the company uses the relationship $n = k(p^2 - 2500)$, where k is a constant.

(i) What value of k will ensure that the refined model agrees with the original model when p = 0?

(ii) Find an expression for T in terms of p.

(iii) What value of p will maximize T? Give your answer correct to the nearest £.

[Ans: (a) (i) n = 100 – 2p (ii) T = np ⇒ T = (100 – 2p) p
(iv) p = 25 (v) $0 \le p \le 25$ b(i) $\frac{1}{25}$ (ii) $T = \frac{1}{25}(p^3 - 2500p)$ (iii) £29]

CHAPTER 18

COMPLEX NUMBERS

18.1 The complex number system

Within the real number system, we can solve equations of the form $ax^2 + bx + c = 0$ where a, b and $c \in \Re$ and $b^2 - 4ac \geq 0$; but the quadratic equation $x^2 + 1 = 0$ has no real number x such that $x^2 = -1$. However, if we define a number i such that $i^2 = -1$ (or $i = \sqrt{-1}$), the solution to $x^2 + 1 = 0$ is then $x = i$ or $x = -i$. Thus it is necessary to extend the real number system, and the complex number was developed for this purpose.

Consider the equation $x^2 + 4x + 20 = 0$.
Here $a = 1$, b = 4 and c = 20

Using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we have

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 20}}{2}$$

$$x = \frac{-8 \pm \sqrt{-64}}{2}$$

$$x = \frac{-8 \pm 8i}{2}$$

either $x = -4 + 4i$ or $x = -4 - 4i$

The numbers $-4 + 4i$ and $-4 - 4i$ are called complex numbers.
In general any number of the form $x + yi$ where $x, y \in \Re$ is called a complex number. We usually use the letter z to denote a complex number. Thus if $z = x + yi$ is a complex number, then x is termed as the real part of z, denoted by $x = \text{Re}(z)$, and y is termed as the imaginary part of z, denoted by $y = \text{Im}(z)$.

For example, if $z = 3 + 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = 4$
if $z = -2 + i$, then $\text{Re}(z) = -2$ and $\text{Im}(z) = 1$
if $z = 5i$, then $\text{Re}(z) = 0$ and $\text{Im}(z) = 5$ and so on.

We usually use the symbol C to denote the set of complex numbers.
That is

$$C = \text{the set of complex numbers}$$
$$C = \{z : z = x + yi,\ x, y \in \Re\}$$

18.2 Addition of complex numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then

$$z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$$

We add the real part to the real part and the imaginary part to the imaginary part.

Example 1

If $z_1 = 3 + 4i$ and $z_2 = 7 + 5i$, find $z_1 + z_2$.

Solution

$z_1 + z_2 = (3 + 4i) + (7 + 5i) = 10 + 9i$.

Example 2

If $z_1 = 4 + i$, $z_2 = 3 - 2i$, and $z_3 = 2 + 5i$, find $z_1 + z_2 + z_3$.

Solution

$z_1 + z_2 + z_3 \quad = (4 + i) + (3 - 2i) + (2 + 5i) = 9 + 4i$

Exercise 18.2

Express in the form $x + yi$, the following complex numbers.

1 $(2 + 4i) + (1 + 5i)$ [Ans : $3 + 9i$]

2 $(5 + i) + (3 + 2i)$ [Ans : $8 + 3i$]

3 $(3 + 2i) + (-1 + 4i)$ [Ans : $2 + 6i$]

4 $(5 + 2i) + (6 + i)$ [Ans : $11 + 3i$]

5 $(3 - 5i) + (-2 + 3i)$ [Ans : $1 - 2i$]

6 $(-2 + 5i) + (6 - 4i)$ [Ans : $4 + i$]

7 $(4 + 2i) + (3 - 5i) + (1 + 4i)$ [Ans : $8 + i$]

8 $(2 + 5i) + (1 - 3i) + (8 + 7i)$ [Ans : $11 + 9i$]

9 $(-1 + 3i) + (4 - i) + (6 - 5i)$ [Ans : $9 - 3i$]

10 $(3 + 4i) + (2 + 5i) + (5 + 5i)$ [Ans : $10 + 14i$]

18.3 Subtraction of Complex numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_{2i}$, then

$$z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 - x_2) + (y_1 - y_2)i$$

We subtract the real part from the real part and the imaginary part from the imaginary part.

Example 1
If $z_1 = 8 + 4i$ and $z_2 = 7 + 5i$, find $z_1 - z_2$.
Solution
$z_1 + z_2 = (8 + 4i) - (7 + 5i) = 1 - i$.

Example 2
If $z_1 = 4 + i$, $z_2 = 3 - 2i$, and $z_3 = 2 + 5i$, find $z_1 + z_2 - z_3$.
Solution

$$\begin{aligned} z_1 + z_2 - z_3 &= (4 + i) + (3 - 2i) - (2 + 5i) \\ &= (7 - i) - (2 + 5i) \\ &= 5 - 6i. \end{aligned}$$

Exercise 18.3

Express in the form $x + yi$, the following complex numbers.

1	$(2 + 4i) - (1 + 3i)$	[Ans : $1 + i$]
2	$(7 + i) - (3 + 4i)$	[Ans : $4 - 3i$]
3	$(3 + 2i) - (-2 + 4i)$	[Ans : $5 - 2i$]
4	$(6 + i) - (3 + 2i)$	[Ans : $3 - i$]
5	$(1 - 4i) - (-1 + 5i)$	[Ans : $2 - 9i$]
6	$(-6 + 8i) - (5 - 7i)$	[Ans : $-11 + 15i$]
7	$(5 + 3i) + (4 - i) - (8 + 9i)$	[Ans : $1 - 7i$]
8	$(3 + 2i) + (5 - 4i) - (4 + 2i)$	[Ans : $4 - 4i$]
9	$(-5 + 2i) - (3 - 2i) + (9 - 3i)$	[Ans : $1 + i$]
10	$(7 + 11i) - (10 + 13i) + (6 + i)$	[Ans : $3 - i$]

18.4 Multiplication of a complex number by a real number

For any $k \in \Re$, $kz = k(x + yi) = kx + kyi$

Example 1
If $z = 5 + 3i$, express $4z$ in the form $x + yi$.
Solution
$4z = 4(5 + 3i) = 20 + 12i$.

Example 2
If $z_1 = 8 + 4i$ and $z_2 = 7 + 5i$, express $3z_1 + 2z_2$ in the form $x + yi$.
Solution

$$\begin{aligned} 3z_1 + 2z_2 &= 3(8 + 4i) + 2(7 + 5i) \\ &= 24 + 12i + 14 + 10i \\ &= 38 + 22i \end{aligned}$$

Example 3
If $z_1 = 3 + 4i$ and $z_2 = 7 + 5i$, express $5z_1 - 3z_2$ in the form $x + yi$.
Solution

$$\begin{aligned} 5z_1 - 3z_2 &= 5(3 + 4i) - 3(7 + 5i) \\ &= 15 + 20i - 21 - 15i \\ &= -6 + 5i. \end{aligned}$$

Exercise 18.4

1 If $z = 6 + 5i$, express $3z$ in the form $x + yi$. [Ans : $18 + 15i$]

2 If $z_1 = 7 + 3i$ and $z_2 = 5 + 2i$, express $4z_1 + 3z_2$ in the form $x + yi$.
[Ans : $43 + 18i$]

3 If $z_1 = -2 + i$ and $z_2 = 3 + 4i$, find express $2z_1 - 3z_2$ in the form $x + yi$.
[Ans : $-13 - 10i$]

4 If $z_1 = 10 + 13i$ and $z_2 = 9 + 8i$, express $2z_1 + 3z_2$ in the form $x + yi$.
[Ans : $47 + 50i$]

5 If $z_1 = -12 + 11i$ and $z_2 = 15 + 14i$, find express $4z_1 - 5z_2$ in the form $x + yi$.
[Ans : $-123 - 26i$]

6 If $z_1 = 4 + i$, $z_2 = 2 - 2i$ and $z_3 = 2 + 5i$ express $4z_1 + 5z_2 - 6z_3$ in the form $x + yi$.
[Ans : $14 - 36i$]

18.5 Multiplication of two complex numbers

If $z_1 = x_1 + y_1i$ and $z_2 = x_2 + y_2i$, then

$$\begin{aligned} z_1z_2 &= (x_1 + y_1i)(x_2 + y_2i) \\ &= x_1(x_2 + y_2i) + y_1i(x_2 + y_2i) \\ &= x_1x_2 + xy_2i + x_2y_1i + y_1y_2i^2 \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i, \qquad \text{since } i^2 = -1. \end{aligned}$$

Example 1

If $z_1 = 3 + 4i$ and $z_2 = 7 + 5i$, express z_1z_2 in the form $x + yi$.

Solution

$$\begin{aligned} z_1z_2 &= (3 + 4i)(7 + 5i) \\ &= 3(7 + 5i) + 4i(7 + 5i) \\ &= 21 + 15i + 28i + 20i^2 \\ &= 21 - 20 + (15 + 28)i \qquad \text{since } i^2 = -1. \\ &= 1 + 43i. \end{aligned}$$

Example 2

If $z_1 = 4 + i$, $z_2 = 3 - 2i$ and $z_3 = 2 + 5i$ find $z_1(z_2z_3)$.

Solution

$$\begin{aligned} z_2z_3 &= (3 - 2i)(2 + 5i) \\ &= 6 + 15i - 4i - 10i^2 \\ &= 6 + 10 + (15 - 4)i \qquad \text{since } i^2 = -1. \\ &= 16 + 11i. \\ z_1(z_2z_3) &= (4 + i)(16 + 11i) \\ &= 64 + 44i + 16i + 11i^2 \\ &= (64 - 11) + (44 + 16)i \\ &= 53 + 60i. \end{aligned}$$

Exercise 18.5

Express in the form $x + yi$, the following complex numbers.

1	$(2 + 4i)(1 + 5i)$	[Ans : $-18 + 14i$]
2	$(5 + i)(3 + 2i)$	[Ans : $13 + 13i$]
3	$(3 + 2i)(-1 + 4i)$	[Ans : $-11 + 10i$]
4	$(5 + 2i)(6 + i)$	[Ans : $28 + 17i$]
5	$(3 - 5i)(-2 + 3i)$	[Ans : $9 + 19i$]
6	$(-2 + 5i)(6 - 4i)$	[Ans : $8 + 38i$]
7	$(4 + 2i)(3 - 5i)$	[Ans : $22 - 14i$]

18.6 Conjugate of a complex number

If $z = x + yi$, then we define the conjugate of z, denoted by $\bar{z}$ or z^* to be $\bar{z} = x - yi$.
For example, if $z = 3 + 4i$, then $\bar{z} = 3 - 4i$.
if $z = 5 - 6i$, then $\bar{z} = 5 + 6i$.
The conjugate pairs z and $\bar{z}$ have the following properties:

Property 1

$$\begin{aligned} z + \bar{z} &= (x + yi) + (x - yi) \\ &= 2x \\ &= 2\mathrm{Re}(z) \end{aligned}$$

For example, $2 + 3i + 2 - 3i = 4$
Hence the sum of a conjugate pair is a real number.

Property 2

$$\begin{aligned} z - \bar{z} &= (x + yi) - (x - yi) \\ &= 2yi \\ &= 2\mathrm{Im}(z) \end{aligned}$$

For example, $(2 + 3i) - (2 - 3i) = 6i$
Hence the difference of a conjugate pair is a number of the form ki, $k \in \Re$.

Property 3

$$\begin{aligned} z\bar{z} &= (x + yi)\ (x - yi) \\ &= x^2 - xyi + xyi - y^2i^2 \\ &= x^2 + y^2 \end{aligned}$$

For example, $(2 + 3i)(2 - 3i)$
$= 4 - 6i + 6i + 9$
$= 13$
Hence the product of a conjugate pair is a real number.

Property 4

$$\bar{\bar{z}} = \overline{\overline{x + iy}} = \overline{x - iy} = x + yi = z$$

For example, $\overline{\overline{3 - 5i}} = \overline{3 + 5i} = 3 - 5i$.
Hence the conjugate of the conjugate of a complex number z is the number z itself.

Example

If $z_1 = 3 + 4i$ and $z_2 = 2 - i$, express in the form $x + yi$,
(a) $z_1 + \overline{z_2}$ **(b)** $\overline{z_1 z_2}$

Solution

(a) $z_1 + \overline{z_2} = 3 + 4i + 2 + i = 5 + 5i$.

(b) $z_1 z_2 = (3 + 4i)(2 - i) = 10 + 5i$
$\overline{z_1 z_2} = 10 - 5i$.

Exercise 18.6

Express the conjugate of the following in the form $x + yi$.

1 If $z_1 = 5 + 3i$ and $z_2 = 5 - 4i$, express $z_1 + \overline{z_2}$ in the form $x + yi$,

[Ans : $10 + 7i$]

2 If $z_1 = 2 - 4i$ and $z_2 = 7 - i$, express $\overline{z_1 z_2}$ in the form $x + yi$,

[Ans : $10 + 30i$]

3 If $z_1 = 2 + 8i$ and $z_2 = 15 - 11i$, express $\overline{z_1 + z_2}$ in the form $x + yi$,

[Ans : $17 + 3i$]

4 If $z_1 = 9 - 7i$ and $z_2 = 10 - 5i$, express in the form $x + yi$,

(a) $\overline{z_1} + \overline{z_2}$ (b) $\overline{z_1 + z_2}$

What do you notice?

[Ans : (a) $19 + 12i$; (b) $19 + 12i$]

5 If $z_1 = 8 - 3i$ and $z_2 = 12 - 10i$, express in the form $x + yi$,

(a) $\overline{z_1} - \overline{z_2}$ (b) $\overline{z_1 - z_2}$

What do you notice?

[Ans : (a) $-4 - 7i$; (b) $-4 - 7i$]

6 If $z_1 = 4 + 11i$ and $z_2 = 5 - 12i$, express $\overline{z_1}\, z_2$ in the form $x + yi$,

[Ans : $-112 - 103i$]

7 If $z_1 = -5 - 4i$ and $z_2 = -1 - 8i$, express in the form $x + yi$,

(a) $\overline{z_2} + z_1$ (b) $\overline{z_1}\, z_2$

[Ans : (a) $-6 + 4i$; (b) $37 + 36i$]

8 If $z_1 = 1 - 4i$, $z_2 = 2 + i$ and $z_3 = 5 - 8i$, express $\overline{z_1 + z_2 + z_3}$ in the form $x + yi$,

[Ans : $8 + 11i$]

18.7 Division of two complex numbers

To divide two complex numbers, we have to multiply the numerator and the denominator by the conjugate of the complex number found in the denominator. The procedure used here is similar to that for rationalizing of surds.

Example 1

Express $\frac{5+2i}{3-i}$ in the form a + bi, where a, b $\in \Re$.

Solution

$$\begin{aligned}\frac{5+2i}{3-i} &= \frac{5+2i}{3-i}\cdot\frac{3+i}{3+i}\\ &= \frac{15+5i+6i-2}{9+1}\\ &= \frac{13+11i}{10} = \frac{13}{10}+\frac{11}{10}i\end{aligned}$$

Example 2

If $z = \frac{2+i}{1-i}$, find the real and imaginary parts of

(a) z^2 (b) $z + \frac{1}{z}$

Solution

(a) Consider $$\begin{aligned}\frac{2+i}{1-i} &= \frac{2+i}{1-i}\cdot\frac{1+i}{1+i}\\ &= \frac{2+3i-1}{1+1} = \frac{1}{2}+\frac{3}{2}i\end{aligned}$$

Now $$\begin{aligned}z^2 &= \left(\frac{2+i}{1-i}\right)^2 = \left(\frac{1}{2}+\frac{3}{2}i\right)^2\\ &= (\frac{1}{2}+\frac{3}{2}i)\cdot(\frac{1}{2}+\frac{3}{2}i)\\ &= \frac{1}{4}+\frac{3}{4}i+\frac{3}{4}i-\frac{9}{4} = -2+\frac{3}{2}i\end{aligned}$$

(b) $$\begin{aligned}\frac{1}{z} &= \frac{1-i}{2+i}\\ &= \frac{1-i}{2+i}\cdot\frac{2-i}{2-i}\\ &= \frac{2-i-2i-1}{4+1} = \frac{1}{5}-\frac{3}{5}i\end{aligned}$$

$$\begin{aligned}\therefore\ z+\frac{1}{z} &= \frac{1}{2}+\frac{3}{2}i+\frac{1}{5}-\frac{3}{5}i\\ &= \frac{7}{10}+\frac{9}{10}i\end{aligned}$$

Exercise 18.7

Express in the form $a + bi$, the following complex numbers.

1 $\dfrac{4+i}{3-i}$ [Ans: $\dfrac{11}{10}+\dfrac{7}{10}i$]

2 $\dfrac{1}{1+3i}$ [Ans: $\dfrac{1}{10}-\dfrac{3}{10}i$]

3 $\dfrac{7+5i}{4-3i}$ [Ans: $\dfrac{13}{25}+\dfrac{41}{25}i$]

4 $\dfrac{4+2i}{5-i}$ [Ans: $\dfrac{9}{13}+\dfrac{7}{13}i$]

5 $\dfrac{2i}{4-5i}$ [Ans: $-\dfrac{10}{41}+\dfrac{8}{41}i$]

6 $\dfrac{3+4i}{2-i}$ [Ans: $\dfrac{2}{5}+\dfrac{11}{5}i$]

7 $\dfrac{1+i}{2-3i}$ [Ans: $-\dfrac{1}{13}+\dfrac{5}{13}i$]

8 $\dfrac{2+3i}{1+5i}$ [Ans: $\dfrac{17}{26}-\dfrac{7}{26}i$]

9 $\dfrac{8i}{2+5i}+\dfrac{3}{2-5i}$ [Ans: $\dfrac{46}{29}+\dfrac{31}{29}i$]

10 $\dfrac{1+\sqrt{3i}}{2}+\dfrac{2}{1+\sqrt{3i}}$ [Ans: $1 + 0\,i$]

18.8 Equal complex numbers

Two complex numbers are said to be equal if both the real and imaginary parts are equal.

For example, if $\quad x + yi = 3 + 5i$
Then $\quad x = 2$ and $y = 5$

Example 1
Find the real numbers x and y such that
$$(x + yi)(2 - 3i) = -13i.$$
Solution
$$\begin{aligned}(x + yi)(2 - 3i) &= 2x - 3xi + 2yi + 3y \\ &= (2x + 3y) + (-3x + 2y)i\end{aligned}$$
But $\quad (2x + 3y) + (-3x + 2y)i = -13i$
Comparing real parts, $\quad 2x + 3y = 0 \quad \ldots\ (1)$
Comparing imaginary parts, $\quad -3x + 2y = -13 \quad \ldots\ (2)$

Solving (1) and (2) simultaneously, we have
$$6x + 9y = 0$$
$$-6x + 4y = -26$$
Adding, we get
$$13y = -26$$
$$y = -2$$
From (1), $\quad 2x - 6 = 0$
$$x = 3$$
Hence $x = 3$ and $y = -2$.

Exercise 18.8

Find the real numbers x and y such that

1 $\quad (x + yi)(3 - 5i) = 34 \qquad$ [Ans : $x = 3, y = 5$]

2 $\quad (x + yi)(5 + 7i) = 71 - 19i \qquad$ [Ans : $x = 3, y = -8$]

3 $\quad (x + yi)(6 - 8i) = 2 - 36i \qquad$ [Ans : $x = 3, y = -2$]

4 $\quad (x + yi)(1 + 4i) = 6 + 7i \qquad$ [Ans : $x = 2, y = -1$]

5 $\quad 2(x + yi) - 1 = (4 - i)^2 \qquad$ [Ans : $x = 8, y = -4$]

6 $\quad (1 + i)x + (2 - 3i)y = 10 \qquad$ [Ans : $x = 6, y = 2$]

18.9 Solving polynomial equations having non–real roots

All polynomial equations with real coefficients, any non–real roots occur in conjugate pairs

Example 1

Solve for z in the equation $z^2 + 9 = 0$

Solution

We have $z^2 = -9$

$z = \sqrt{-9}$

Hence $z = \pm\ 3\,i.$

Example 2

Solve for z in the equation $z^2 - 2z + 3 = 0$

Solution

Completing the square, we have

$$z^2 - 2z + 1 + 3 - 1 = 0$$
$$(z-1)^2 + 2 = 0$$
$$(z-1)^2 = -2$$
$$z - 1 = \sqrt{-2}$$

Hence $z = 1 \pm \sqrt{2}\,i.$

Exercise 18.9

Solve for z the following equations:

1 $z^2 + 4 = 0$ [Ans : $\pm 2i$]

2 $z^2 + 25 = 0$ [Ans : $\pm 5i$]

3 $z^2 + 36 = 0$ [Ans : $\pm 6i$]

4 $z^2 + 2z + 17 = 0$ [Ans : $-1 \pm 4i$]

5 $z^2 + 4z + 20 = 0$ [Ans : $-2 \pm 4i$]

6 $z^2 + z + 1 = 0$ [Ans : $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$]

7 $z^2 - 6z + 25 = 0$ [Ans : $3 \pm 4i$]

8 $9z^2 + 6z + 5 = 0$ [Ans : $-\frac{1}{3} \pm \frac{2}{3} i$]

18.10 Square roots of a complex number

Within the complex number system, the roots of the equation $z^2 = x + yi$, where $x, y \in \Re$ are called the square roots of $x + yi$.

Example

Find the square roots of $-5 + 12i$. **(C)**

Solution

Let $\quad z^2 = -5 + 12i$, where $z = x + yi$, $x, y \in \Re$.

Then $\quad (x + yi)^2 = -5 + 12i$

$$x^2 - y^2 + 2xyi = -5 + 12i$$

Equating real and imaginary parts, we have

$$x^2 + y^2 = -5 \qquad \ldots\ldots (1)$$

$$2xy = 12 \Rightarrow xy = 6 \qquad \ldots\ldots (2)$$

From (2) $\qquad y = \frac{6}{x}$

From (1), we have $\qquad x^2 - \left(\frac{6}{x}\right)^2 = -5$

$$x^2 - \frac{36}{x^2} = -5$$

$$x^4 + 5x^2 - 36 = 0$$

$$(x^2 + 9)(x^2 - 4) = 0$$

Since $x^2 \neq -9$, $x^2 - 4 = 0$

$$x^2 = \pm 2$$

When $x = 2$, $y = 3$

When $x = -2$, $y = -3$

Hence the square roots of $-5 + 12i$ are $2 + 3i$ and $-2 - 3i$.

Exercise 18.10

Find the square roots of each of the following complex numbers.

1 $\quad 3 + 4i \qquad$ [Ans : $2 + i$, $\ 2 \quad i$]

2 $\quad 2i \qquad$ [Ans : $1 + i$, $\ 1 \quad i$]

3 $\quad -8 + 6i \qquad$ [Ans : $1 + 3i$, $-1 - 3i$]

4 $\quad 5 - 12i \qquad$ [Ans : $3 - 2i$, $-3 + 2i$]

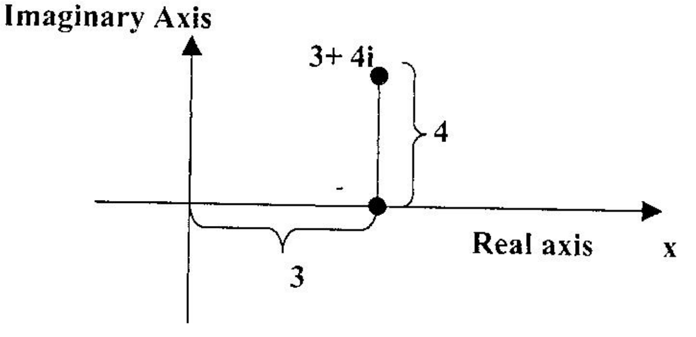
5 $\quad 4i \quad$ **(C)** $\qquad$ [Ans : $\pm\sqrt{2}\,(1 + i)$]

6 $\quad 35 + 12i \qquad$ [Ans : $6 + i$, $-6 - i$]

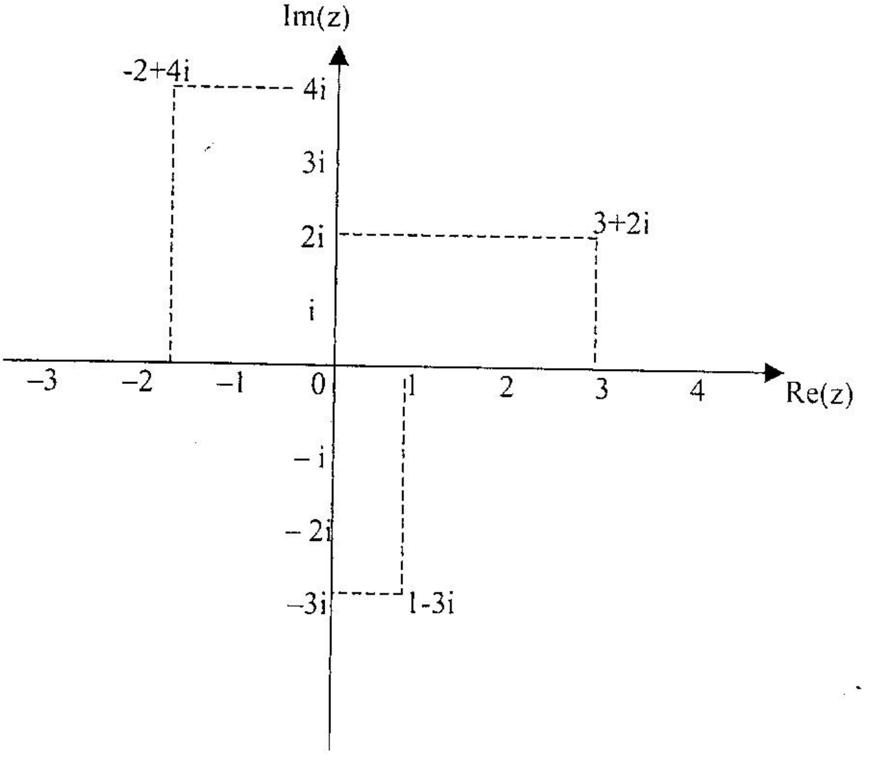
7 $\quad 24 - 10i \qquad$ [Ans : $5 - i$, $-5 + i$]

18.11 The Argand diagram

An Argand diagram or z–plane is similar to the Cartesian plane.
The x–axis represents the real axis whereas the y–axis represents the imaginary axis.
Thus the point (3,4) on the Argand diagram can represent a complex number $(3 + 4i)$.



Example
Show on the z–plane the following complex numbers.
$-2 + 4i$, $3 + 2i$ and $1 - 3i$.
Solution



Exercise 18.11

1 Show the following complex numbers on an Argand diagram.
(a) $-3 + 5i$ (b) $4 - i$ (c) $2 - 3i$
(d) $2(1 + 2i)$ (e) $3(1 + i)$ (f) $-2(3 + i)$

2 If $a = 2 + i$ and $b = 2 + 3i$, represent each of the following on an Argand diagram.
(i) a (ii) $\bar{a}$ (ii) $a + b$ (iv) $\frac{1}{a}$

3 If $a = 1 + i$ and $b = 3 - 2i$, represent each of the following on an Argand diagram.
(i) $a - b$ (ii) $2a + b$ (iii) ib

4 Given $z = 3 + 4i$, represent each of the following on an Argand diagram.
(a) $-z$ (b) $\overline{2z}$ (c) $\frac{5}{z}$

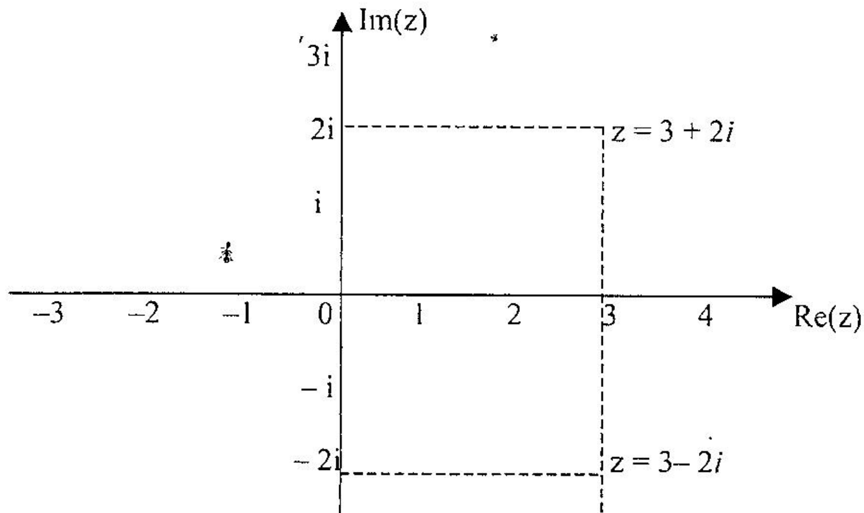
5 Given $z = 2 + i$, represent each of the following on an Argand diagram.
(a) $z + \frac{1}{z}$ (b) $z\bar{z}$

6 Given $z = 5 + 4i$, represent each of the following on an Argand diagram.
(a) $z - \frac{1}{z}$ (b) z^2

18.12 Geometrical properties

18.12.1 Geometrical interpretation of $\bar{z}$.

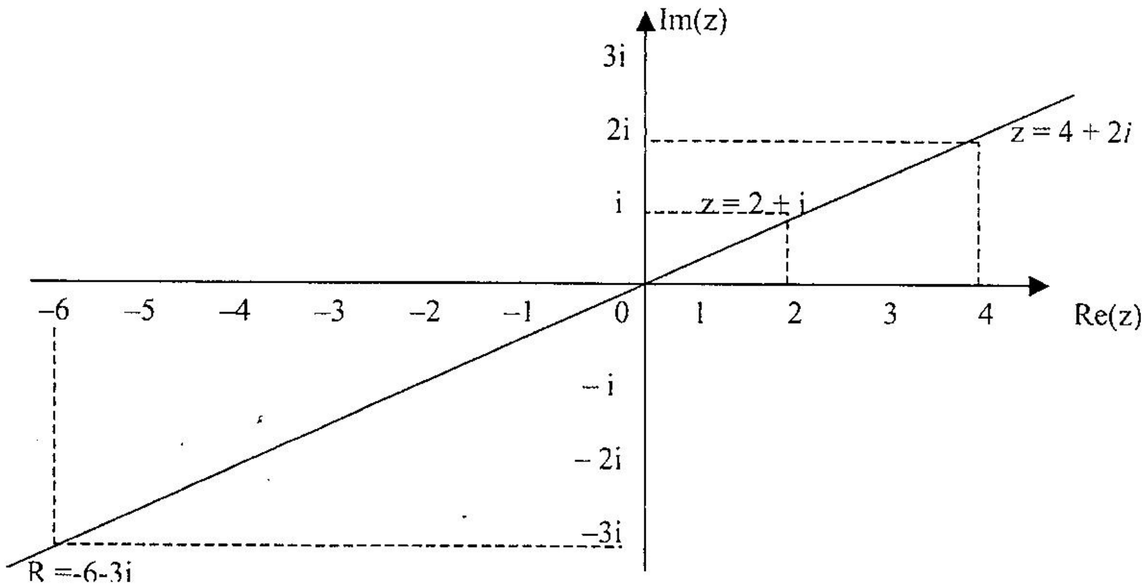
Let $z = 3 + 2i$, then $\bar{z} = 3 - 2i$. If we represent the points on an Argand diagram, we obtain



We notice from the above that the conjugate of a complex number is the reflection of that complex number in the line $y = 0$ or the x–axis.
In general for any $z \in C$, $\bar{z}$ is the reflection of z in the x–axis.

18.12.2 Geometrical representation of kz, where $k\in\Re$.

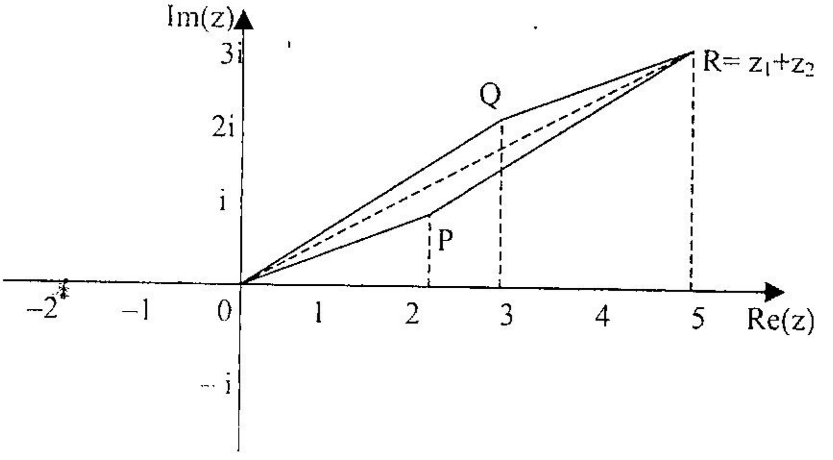
Let $P \equiv z = 2 + i$, if we represent the points $Q \equiv 2z$ and $R \equiv -3z$ on the Argand diagram, we obtain



We notice from the above that the points P and Q all lie on a straight line through the origin O. In general, for any complex number z, kz lies on the straight line passing through the origin and the point P representing Z.

18.12.3 Geometrical representation of $z_1 + z_2$ where $z_1, z_2 \in C$.

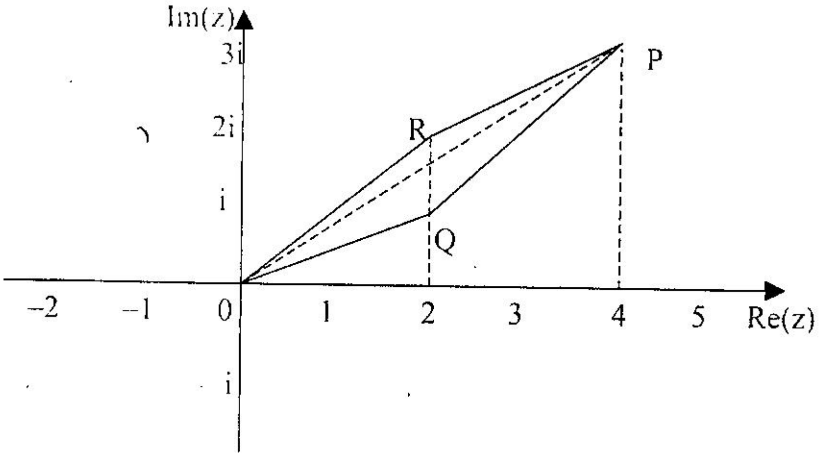
Let the points P and Q represent the complex numbers $z_1 = 2 + i$ and $z_2 = 3 + 2i$ respectively.
Then $z_1 + z_2 = 5 + 3i$.



Clearly OPQR forms a parallelogram. So, in general if $P \equiv z_1$ and $Q \equiv z_2$ are two complex numbers then $z_1 + z_2$ can be located by constructing a parallelogram using OP and OQ as the adjacent sides, where O is the origin. This method of locating the position of the point representing the sum of two complex numbers is called the parallelogram rule.

18.12.4 Geometrical representation of $z_1 - z_2$ where $z_1, z_2 \in C$.

Let the points P and Q represent the complex numbers $z_1 = 4 + 3i$ and $z_2 = 2 + i$ respectively.
Then $z_1 - z_2 = 2 + 2i$.

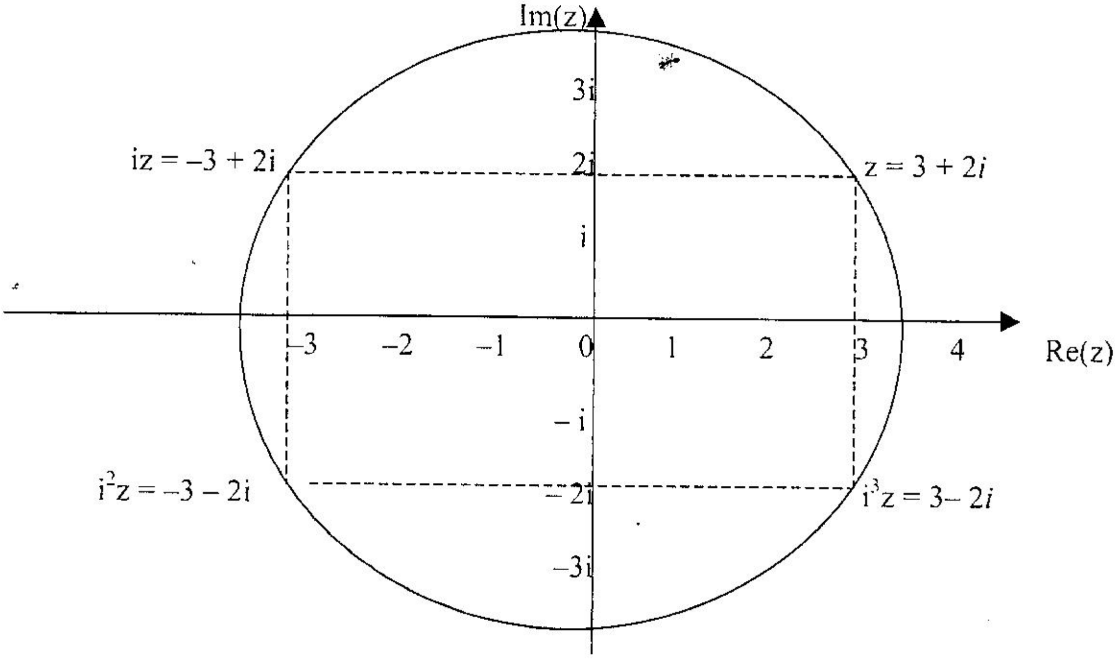


Clearly OPQR forms a parallelogram. So, in general if $P \equiv z_1$ and $Q \equiv z_2$ are two complex numbers then $z_1 - z_2$ can be located by constructing a parallelogram using OQ and OR as the adjacent sides, where O is the origin.

18.13 Multiplication of a complex number by i

Consider the complex number $z = 3 + 2i$. Now we are going to locate the points representing the complex numbers z, iz, i^2z, i^3z and i^4z on the Argand diagram.

We have
$$z = 3 + 2i$$
$$iz = i(3 + 2i) = -2 + 3i$$
$$i^2z = i(-2 + 3i) = -3 - 2i$$
$$i^3z = i(-3 - 2i) = 2 - 3i$$
$$i^4z = i(2 - 3i) = 3 + 2i$$



From the Argand diagram, we can see that the effect of multiplying a complex number z by i. The point representing iz is obtained by rotating z 90° anti–clockwise about the origin.

Example

Simplify $\dfrac{(1+i)^5}{(1-i)^4}$

Solution

From the Argand diagram,

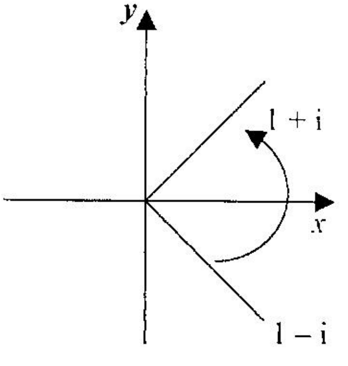
$1 + i = i(1 - i)$

$(1 + i)^5 = i^5(1 - i)^5 = i(1 - i)^5$

Hence $\dfrac{(1+i)^5}{(1-i)^4}$

$= \dfrac{i(1-i)^5}{(1-i)^4}$

$= i(1 - i) = 1 + i.$



Exercise 18.13

1 Simplify $\dfrac{(1+i)^4}{(1-i)^3}$ [Ans : $1-i$]

2 Simplify $\dfrac{(3-4i)^5}{(4+3i)^3}$ [Ans : $24-7i$]

3 Simplify $\dfrac{(3+2i)^3}{(-2+3i)^4}$ [Ans : $\dfrac{3-2i}{13}$]

4 Simplify $\dfrac{(7+8i)^{10}}{(-8+7i)^{11}}$ [Ans : $\dfrac{8+7i}{113}$]

5 Simplify $\dfrac{(1+i)^4}{(2-2i)^3}$ [Ans : $\dfrac{1-i}{8}$]

6 Simplify $\dfrac{(2+3i)^7}{(3-2i)^6}$ [Ans : $-2-3i$]

7 Simplify $\dfrac{(3+4i)^3}{(4-3i)^2}$ [Ans : $-3-4i$]

8 Simplify $\dfrac{(5+4i)^9}{(-4+5i)^{10}}$ [Ans : $\dfrac{-5+4i}{41}$]

9 Simplify $\dfrac{(2+5i)^6}{(5-2i)^5}$ [Ans : $-5+2i$]

10 Simplify $\dfrac{(7+3i)^5}{(-3+7i)^6}$ [Ans : $\dfrac{-7+3i}{58}$]

18.14 Modulus of a complex number

If z = x + yi, then the modulus of z, written as $|z|$ is given by $|z| = |x+yi| = \sqrt{x^2+y^2}$

Example 1
Find the modulus of the following complex numbers
(a) 3 + 4i (b) −12 + 5i
Solution

(a) $|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

(b) $|-12+5i| = \sqrt{(-12)^2+5^2} = \sqrt{169} = 13$

Example 2
Find the modulus of the following complex numbers
(a) (3 + 4i) (1 + i) **(b)** (12 + 5i) (2 + i)
Solution

(a) $|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore\ |(3+4i)(1+i)| = 5\sqrt{2}$

As we can see, there is no need to multiply the two complex numbers. We can find the modulus of each complex number separately and multiply.
Thus $|z_1 z_2| = |z_1|\times|z_2|$

(b) $|12+5i| = \sqrt{12^2+5^2} = \sqrt{169} = 13$

$|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$

$\therefore\ |(12+5i)(2+i)| = 13\sqrt{5}$

Example 3
Find the modulus of the complex number $z = \dfrac{6+8i}{3-4i}$.

Solution

$|6+8i| = \sqrt{6^2+8^2} = \sqrt{100} = 10$

$|3-4i| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$

$\therefore\ \left|\dfrac{6+8i}{3-4i}\right| = \dfrac{10}{5} = 2$

As we can see, there is no need to express z in the form x + yi, unless specified in a particular question. We can find the modulus of each complex number separately and divide.

Thus $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

Exercise 18.14

Find the modulus of the following complex numbers

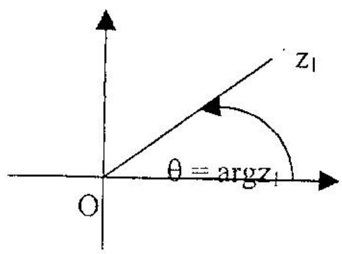
1	$8 + 15i$	[Ans : 17]
2	$24 + 7i$	[Ans : 25]
3	$2 + 5i$	[Ans : $\sqrt{29}$]
4	$1 - 3i$	[Ans : $\sqrt{10}$]
5	$2 + \sqrt{3}i$	[Ans : $\sqrt{7}$]
6	$1 + \sqrt{5}i$	[Ans : $\sqrt{6}$]
7	$(3 + 4i)(6 - 8i)$	[Ans : 50]
8	$(12 - 5i)(3 - 4i)$	[Ans : 65]
9	$(1 + i)(3 - 5i)$	[Ans : $2\sqrt{17}$]
10	$(10 + 3i)(1 + 2i)$	[Ans : $\sqrt{545}$]
11	$\frac{12+5i}{1-i}$	[Ans : $\frac{13}{2}\sqrt{2}$]
12	$\frac{2+i}{3-i}$	[Ans : $\frac{\sqrt{2}}{2}$]
13	$\frac{3+4i}{5-2i}$	[Ans : $\frac{5}{29}\sqrt{29}$]
14	$\frac{4+i}{2i}$	[Ans : $\frac{\sqrt{17}}{2}$]
15	$\frac{6-8i}{5i}$	[Ans :2]

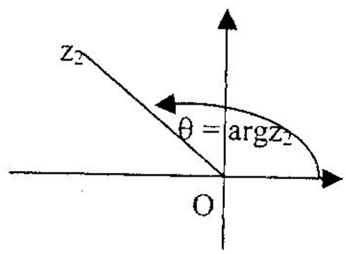
18.15 Argument of a complex number

If $z = x + yi$, then the argument of z, written as $\arg z$ is given by $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

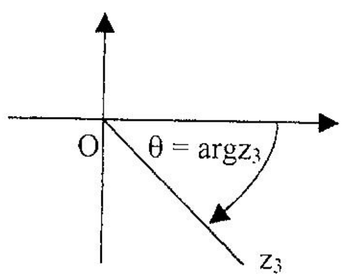
A unique value of θ satisfying the inequalities $-\pi < \theta < \pi$ is called the principal value of $\arg z$.

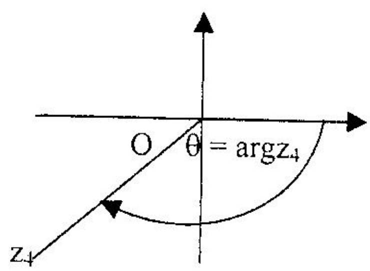
The following Argand diagrams will help the reader to understand much better the concept of argument.





If a complex number lies in the first or second quadrants, the argument will be positive.





If a complex number lies in the third or fourth quadrants, the argument will be negative. As we can see from the above diagrams, the argument of any complex number Z is the angle that OZ makes with the positive real axis.

Example 1

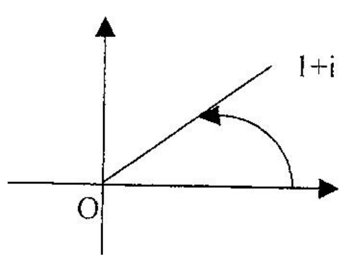
Find the argument of the following complex numbers

(a) $1 + i$ (b) $-1 + \sqrt{3}\,i$

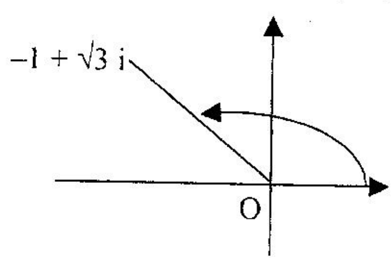
Solution

It is important to locate the complex number in an Argand diagram as to have an idea whether the argument is positive or negative.

(a) $\arg(1 + i) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$



(b) $\arg(-1 + \sqrt{3}\,i) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3}$



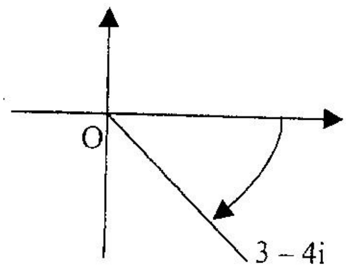
Example 2

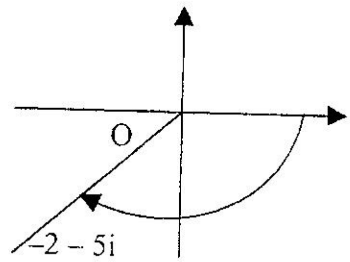
Find the argument of the following complex numbers

(a) $3 - 4i$ **(b)** $-2 - 5i$

Solution

(a) $\arg(3 - 4i) = \tan^{-1}\left(\frac{-4}{3}\right) = -0.93$ rad **(b)** $\arg(-2 - 5i) = \tan^{-1}\left(\frac{-5}{-2}\right) = -1.95$ rad





Example 2

Find the argument of the following complex numbers

(a) $(1 + i)(\sqrt{3} + i)$ **(b)** $(5 + 2i)(1 - i)$

Solution

(a) $\arg(1 + i) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$ rad

$\arg(\sqrt{3} + i) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ rad

$\therefore \arg\left|(1+i)(\sqrt{3}+i)\right| = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

As we can see, there is no need to multiply the two complex numbers. We can find the argument of each complex number separately and add (recall properties of log).

Thus $\mathbf{\arg z_1 z_2 = \arg z_1 + \arg z_2}$

(a) $\arg(5 + 2i) = \tan^{-1}\left(\frac{2}{5}\right) = 0.38$ rad

$\arg((1 - i) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$ rad

$\therefore \arg\left|(1+i)(\sqrt{3}+i)\right| = 0.38 - (-\frac{\pi}{4}) = 1.17$ rad

As we can see, there is no need to divide the two complex numbers. We can find the argument of each complex number separately and subtract (recall properties of log).

Thus $\mathbf{\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2}$

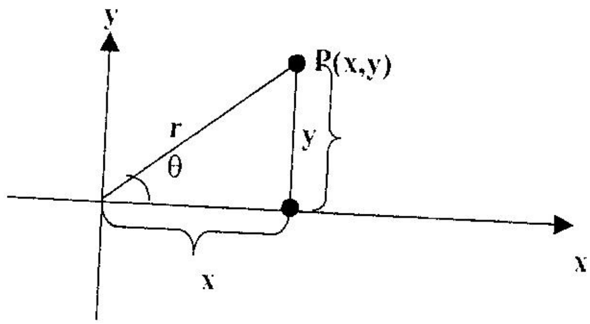
Exercise 18.15

Find the argument of the following complex numbers

1	$2 + 2i$	[Ans : $\frac{\pi}{4}$]
2	$2 + 7i$	[Ans : 1.29 rad]
3	$\sqrt{2} + 3i$	[Ans : 1.13 rad]
4	$1 - 3i$	[Ans : –1.24 rad]
5	$2 - \sqrt{5}i$	[Ans : –0.84 rad]
6	$-1 + \sqrt{3}i$	[Ans : $\frac{2\pi}{3}$]
7	$-3 + 4i$	[Ans : 2.21 rad]
8	$-1 - 5i$	[Ans : – 1.77 rad]
9	$(1 + i)(1 + \sqrt{3}i)$	[Ans : $\frac{7\pi}{12}$]
10	$(2 + i)(1 - i)$	[Ans : 1.25 rad]
11	$\frac{1+i}{3-i}$	[Ans : 1.11 rad]
12	$\frac{2+3i}{3+2i}$	[Ans : 0.39 rad]
13	$\frac{1-2i}{2i}$	[Ans : – 2.68 rad]
14	$\frac{i}{1+2i}$	[Ans : 0.46 rad]

18.16 Trigonometric form or polar form of a complex number

Let the point P(x,y) represent the complex number $z = x + yi$. Also let $OP = r$ and θ be the angle which OP makes with the x-axis, as shown below.



From the diagram, $r = \sqrt{x^2 + y^2}$

Again $x = r\cos\theta$

$y = r\sin\theta$

We can write $z = x + yi$ as

$z = r\cos\theta + r\sin\theta i = r(\cos\theta + i\sin\theta)$

And $\bar{z} = r\cos\theta - r\sin\theta i = r[\cos(-\theta) + i\sin(-\theta)]$

Example 1

Express the following complex numbers in trigonometric form

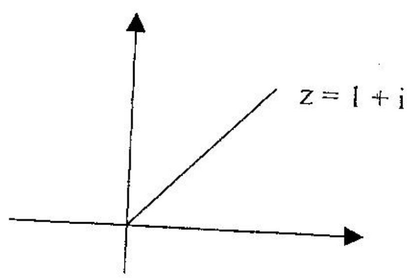
(a) $1 + i$ (b) $-1 + \sqrt{3}\,i$

Solution

(a) $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

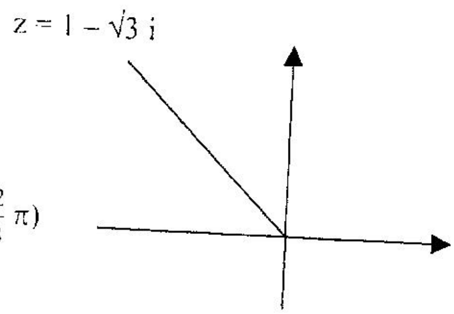
$\therefore 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



(b) $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \frac{2}{3}\pi$

$\therefore -1 + \sqrt{3} = 2\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$



Example 2

Express the following complex numbers in trigonometric form

(a) $\sqrt{3} - i$ (b) $-1 - i$

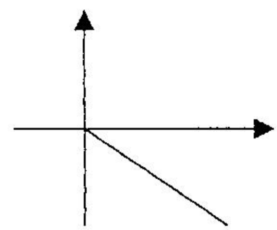
Solution

(a) $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = -\frac{\pi}{4}$

$\therefore\ \sqrt{3} - 1 = \sqrt{2}\,[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})]$

$= \sqrt{2}\,(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$

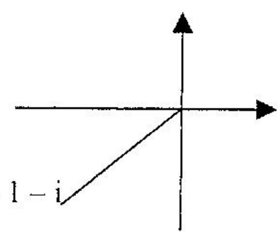


$z = \sqrt{3} - i$

(b) $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \frac{5}{4}\pi$

$\therefore\quad -1 - i = 2\,(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi)$



$z = -1 - i$

Example 3

Find the modulus of the complex number $\frac{7+3i}{5-2i}$, and show that its argument is $\frac{\pi}{4}$. Hence express $z = \frac{7+3i}{5-2i}$ in polar form.

Solution

Let $z = \frac{7+3i}{5-2i}$

Then $z = \frac{7+3i}{5-2i} \cdot \frac{5+2i}{5+2i}$

$= \frac{35+14i+15i-6}{25+4}$

$= \frac{29+29i}{29}$

$= 1 + i$

Hence $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\text{Arg}(z) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$ **(shown)**

$\therefore\ \frac{7+3i}{5-2i} = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

Exercise 18.16

Express the following complex numbers in trigonometric form

1 $1 + \sqrt{3}\,i$

[Ans : $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$]

2 $3 + 4i$

[Ans : $5(\cos 53.1° + i\sin 53.1°)$]

3 $3 - 3i$

[Ans : $\sqrt{18}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$]

4 $-5 + 5i$

[Ans : $\sqrt{50}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$]

5 $1 + 2i$

[Ans : $\sqrt{5}(\cos 63.4° + i\sin 63.4°)$]

6 $-6 - 6i$

[Ans : $\sqrt{72}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$]

7 $3 - 4i$

[Ans : $5(\cos 53.1° - i\sin 53.1°)$]

8 $10i$ **(C)**

[Ans : $10\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$]

9 $-3 + 4i$ **(C)**

[Ans : $5(\cos 126.9° - i\sin 126.9°)$]

10 $\frac{10i}{-3+4i}$ **(C)**

[Ans : $2(\cos 36.9° - i\sin 36.9°)$]

18.17 Modulus and argument of z^n

If $z = x + iy$, then $|z^n| = |z|^n$ and $\arg(z^n) = n \arg z$

Example 1
If $z = 3 + 4i$, find the modulus of z^3.
Solution
$|z| = \sqrt{3^2 + 4^2} = 5$

Hence $|z^3| = 5^3 = 125$

Example 2
If $z = 1 + i$, find the modulus of z^5.
Solution
$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Hence $|z^5| = (\sqrt{2})^5 = 4\sqrt{2}$

Example 3
If $z = 2 + 2i$, find the argument of z^3.
Solution
$\text{Arg } z = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\therefore \arg z^3 = 3 \arg z = \frac{3\pi}{4}$

Example 4
If $z = 1 + \sqrt{3}i$, find the modulus of z^2.
Solution
$\text{Arg } z = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$\therefore \arg z^2 = 2 \arg z = \frac{2\pi}{3}$

Example 5
Given that $z = 1 + i\sqrt{3}$, find $|z|$ and arg z. Hence, or otherwise, show that $z^5 - 16z^* = 0$.
(C)
Solution
$|z| = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\arg z = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Now $z^5 = 2^5 (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^5 = 32 (\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi)$

$$= 32\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= 16 - 16\sqrt{3}\,i$$

Also $z^* = 2 (\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) \quad = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}\ i$

Hence $z^5 - 16z^* = 16 - 16\sqrt{3}\,i - 16(1 - \sqrt{3}\,i) = 0$ **(shown)**

Exercise 18.17

1 If $z = 6 + 8i$, find the modulus of z^3. [Ans : 1000]

2 If $z = 12 - 5i$, find the modulus of z^2. [Ans : 169]

3 If $z = 1 + \sqrt{3}\,i$, find the modulus of z^5. [Ans : 32]

4 If $z = \sqrt{3} - i$, find the modulus of z^6. [Ans : 64]

5 If $z = 8i$, find the argument of z^2. [Ans : π]

6 If $z = 2 + i$, find the argument of z^5.

[Ans : 132.8° or 2.32 rad]

7 If $z = 3 + 2\,i$, find the argument of z^4.

[Ans : 134.8° or 2.35 rad]

8 Given that $z = \sqrt{3} + i$, find $|z|$ and arg z. Hence, or otherwise, show that $z^6 = -32$.

9 Given that $z = 1 + i$, find $|z|$ and arg z. Hence, or otherwise, show that $z^2 = 2i$.

18.18 Loci

Case 1
$|z-z_1| = a$, is a circle centre z_1 and radius a.

Example 1
Sketch on the Argand diagram the locus of

(a) $|z| = 3$

(b) $|z-1| = 1$

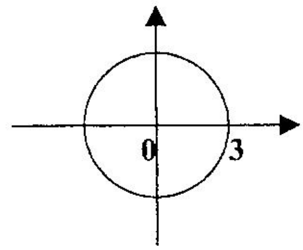
(c) $|z-1-i|^2 = 2$ (C)

Solution

(a) $|z| = 3$

$|z-(0+0i)| = 3$

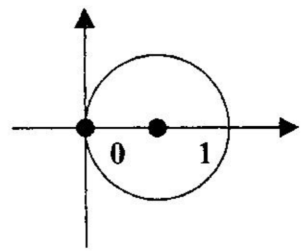
It is circle centre (0,0) and having radius 3.



(b) $|z-1| = 1$

$|z-(1+0i)| = 1$

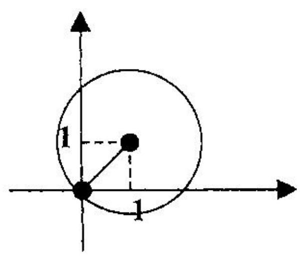
It is circle centre (1,0) and having radius 1.



(c) $|z-1-i|^2 = 2$

$|z-(1+i)| = \sqrt{2}$

It is circle centre (1,1) and having radius $\sqrt{2}$.



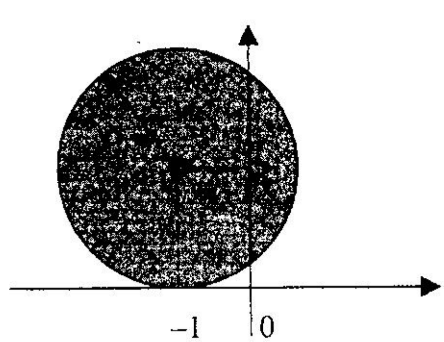
Example 2
Indicate and describe the regions defined by the inequality $|z+1-2i| \le 2$.

Solution

$|z+1-2i| \le 2$

$|z-(-1+2i)| \le 2$

It describes all points inside the circle centre (−1,2) and having radius 2.



Case 2

$|z - z_1| = |z - z_2|$ represents the perpendicular bisector of the line joining z_1 to z_2.

Example 1

Sketch on the Argand diagram the locus of

(a) $|z-1| = |z-i|$ (C)

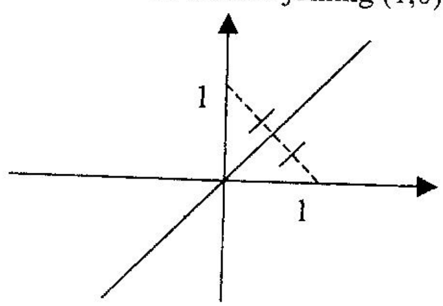
(b) $|z| = |z - 1 + \sqrt{3i}|$

Solution

(a) $|z-1| = |z-i|$

$|z-(1+0i)| = |z-(0+i)|$

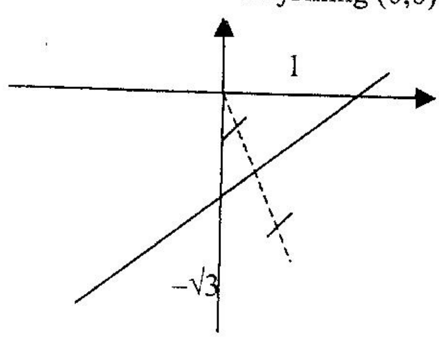
It is perpendicular bisector of the line joining (1,0) to (0,1).



(b) $|z| = |z-1+\sqrt{3i}|$

$|z-(0+0i)| = |z-(1-\sqrt{3i})|$

It is perpendicular bisector of the line joining (0,0) to $(1, -\sqrt{3})$.



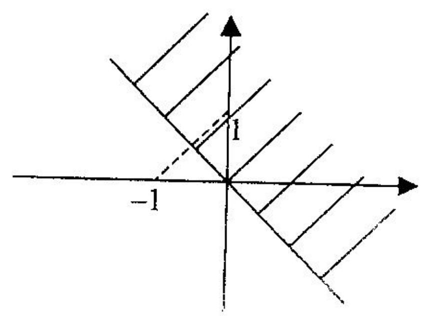
Example 2

Indicate and describe the regions defined by the inequality $|z+1| \le |z-i|$.

Solution

$|z-(-1+0i)| \le |z-(0+i)|$

It is a half plane to the right of the perpendicular bisector.



Case 3

$\arg(z - z_1) = \alpha$ represents a half line having one end at z_1 and making an angle of α with the x–axis.

Example 1

Sketch and describe the locus of

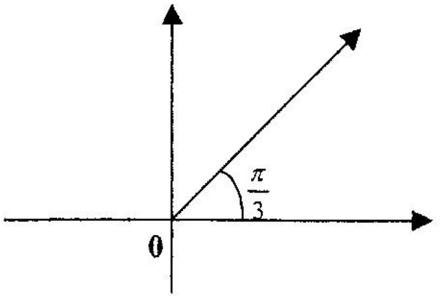
(a) $\arg z = \frac{\pi}{3}$

(b) $\arg(z-2) = 120°$

Solution

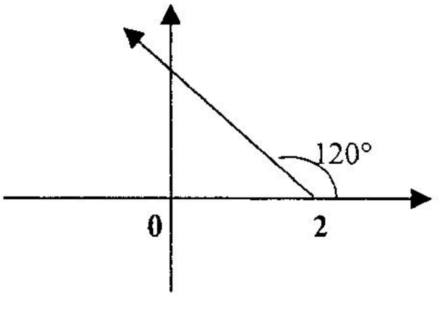
(a) $\arg[z - (0 + 0i)] = \frac{\pi}{3}$

It represents a half line having one end at (0,0) and making an angle of $\frac{\pi}{3}$ with the x–axis.



(b) $\arg[z - (2 + 0i)] = 120°$

It represents a half line having one end at (2,0) and making an angle of 120° with the x–axis.



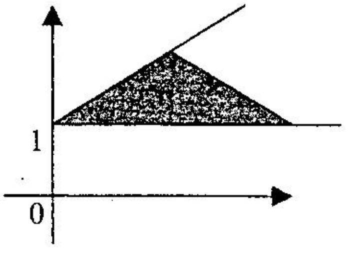
Example 2

Indicate and describe the regions defined by the inequality $0 \le \arg(z - i) \le \frac{\pi}{6}$.

Solution

It is the region bounded by the half lines

$\arg(z - i) = 0$ and $\arg(z - i) = \frac{\pi}{6}$.



Exercise 18.18

Sketch (describe as well) on the Argand diagram the locus of z if :

1 $|z| = 2$

2 $|z| = 1.5$

3 $|z-3| = 1$

4 $|z-2i| = 2$

5 $|z-1-i| = 2$

6 $|z+2-i| = 1$

7 $|z+3+2i| = 2$

8 $|z+1+i| = \sqrt{2}$

Sketch (describe as well) on the Argand diagram the locus of z if :

9 $|z-1| = |z-3|$

10 $|z| = |z-2i|$

11 $|z+2| = |z-4i|$

12 $|z-i| = |z+3|$

13 $\arg z = \dfrac{\pi}{6}$

14 $\arg z = -\dfrac{\pi}{4}$

15 $\arg(z+1) = \dfrac{2}{5}\pi$

16 $\arg(z-1-2i) = \dfrac{1}{3}\pi$

17 $\arg(z+3-i) = \dfrac{\pi}{2}$

18 $\arg(z-1+2i) = \pi$

Indicate and describe the regions defined by the inequalities in the Argand diagram.

19 $|z-3| \leq 1$

20 $|z-2+i| \leq 2$

21 $|z-3| \geq |z-1|$

22 $|z-3i| \leq |z-1|$

23 $0 \leq \arg(z-1) \leq \dfrac{\pi}{3}$.

24 $\dfrac{\pi}{3} \leq \arg(z-2) \leq \dfrac{\pi}{2}$.

Complex Numbers – Past Examination Questions

1 Given that $w = \frac{3-2i}{4-i}$, express w in the form x + iy, and find $|w|$ and arg w.

[Ans : $\frac{14}{17} - \frac{5}{17}i$; $|w| = \sqrt{\frac{13}{17}}$; arg w = −19.7°]

2 Find the modulus of the complex number $\frac{2i}{3-4i}$, and show that the argument, in radians, is 2.5, correct to one decimal place. [Ans : $\frac{2}{5}$]

3 Find the modulus and argument of the complex numbers
(a) $(1 - i)(4 + 3i)$
(b) $\frac{1+7i}{1+i}$

[Ans : (a) $5\sqrt{2}$, 8.13°; (b) 5, 36.9°]

4 If $a = 3 - i$ and $b = 1 + 2i$, find the moduli of :
(i) $2a + 3b$
(ii) $\frac{a}{2b}$ [Ans: $\sqrt{97}$, $\frac{\sqrt{2}}{2}$]

5 Find the modulus and argument of the complex numbers
(a) $(2 + i)(3 - 4i)$
(b) $\frac{-2+4i}{3+i}$ [Ans : (a) 11.2, −26.6°; (b) $\sqrt{2}$, 98.1°]

6 Find the modulus and argument of the complex numbers $z = \frac{13-5i}{4-9i}$.

[Ans : $\sqrt{2}$, $\frac{\pi}{4}$]

7 If $a = 3 + 4i$ and $b = 12 - 5i$, find the modulus and argument of :
(i) a (ii) $\frac{b}{a}$

[Ans : (i) 5, 53.1°; (ii) $\frac{13}{5}$, −75.7°]

8 (a) Two complex numbers w are z such that
$w + z = 6 + 2i$,
$w - 3z = \frac{20}{2-i}$
Find w and z, giving each answer in the form x + yi.

[Ans : $w = \frac{13}{2} + \frac{5}{2}i$; $z = -\frac{1}{2} - \frac{1}{2}i$]

(b) On a Argand diagram, mark and label clearly the points P and Q representing the complex numbers p and q respectively, where

$p = \cos(\frac{1}{4}\pi) + i\sin(\frac{1}{4}\pi)$

$q = 2\cos(\frac{1}{4}\pi) + i\sin(\frac{1}{4}\pi)$.

Find the moduli of the complex numbers a, b, c, d, e, where

$$a = p^4$$
$$b = q^2$$
$$c = -ip$$
$$d = \frac{1}{q}$$
$$e = p + p^*$$

On your Argand diagram mark and label the points A, B, C, D, E representing these complex numbers.

9(a) Find the argument (in the interval $-\pi<\theta<\pi$) of $\left(\frac{3}{5}+\frac{4}{5}i\right)^{20}$, giving two places of decimal in your answer.

Find the algebraic form, $\left(\frac{3}{5}+\frac{4}{5}i\right)^{20} - \left(\frac{3}{5}-\frac{4}{5}i\right)^{20}$, giving two places of decimal in your answer.

[Ans : –0.30 rad ; –0.59 i]

(b) Given that $(x + iy)^2 = -5 + 12i$, where x and y are real, find the set of possible values of x + iy.

By completing the square, or otherwise, solve

$$z^2 + 4z = -9 + 12i.$$

[Ans: ± (2 + 3i) ; z = 3i ; z = – 4 – 3i]

10 The complex number z is given by $z = \cos\theta + i\sin\theta$, where $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

Show that $|1+z| = 2\cos(\frac{1}{2}\theta)$ and $\arg(1+z) = \frac{1}{2}\theta$.

Hence, or otherwise, show that $\frac{1}{1+z} = \frac{1}{2}[1 - i\tan(\frac{1}{2}\theta)]$.

Describe the locus of z and the locus of the point representing $\frac{1}{1+z}$ in an Argand diagram as θ varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

[Ans : locus of z is a semicircle radius 1

locus of $\frac{1}{1+z}$ must lie on the line x = [illegible]

LIST OF FORMULAE

PURE MATHEMATICS

Algebra

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d, \qquad S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$$

For an geometric series:

$$u_n = ar^{n-1}, \qquad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1), \qquad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$$

Binomial expansion:

$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$, where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 \ldots$, where n is rational and $|x| < 1$

Trigonometry

Arc length of circle $= r\theta$ (θ in radius)

Area of sector of circle $= \frac{1}{2}r^2\theta$ (θ in radius)

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1 + \tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \cos A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2\sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Principal values:

$$-\frac{1}{2}\pi \leq \sin^{-1} x \leq \frac{1}{2}\pi$$

$$0 \leq \cos^{-1} x \leq \pi$$

$$-\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi$$

Differentiation

f(x)	f'(x)
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
uv	$u\frac{dv}{dx}+v\frac{du}{dx}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

If $x = f(t)$ and $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Integration

f(x)	$\int$ f(x) dx
x^n	$\frac{x^{n+1}}{n+1}+c \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec x$	$\tan x + c$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\lvert f(x)\rvert + c$$

Vectors

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert a\rvert\lvert b\rvert\cos\theta$$

Numerical integration

Trapezium rule:

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h\{y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n\}, \text{ where } h = \frac{b-a}{n}.$$