

Preface

“Pure Mathematics for A level Syllabus 9709 - Second Edition” is a course-book based on the new syllabus, written especially for the students preparing for Mathematics at the Higher School Certificate Level or G.C.E ‘A’ Level Examinations.

Features

The book includes notes, formulae and worked examples and follows a simple, logical and easy-to-understand approach.

Exercises

The exercises have been carefully constructed to reinforce the students learning of concepts and skills in Mathematics taught in the classroom. The questions are interesting, stimulating and challenging, and will help students develop skills and understanding necessary for them to excel in Mathematics.

Cambridge Questions

At the end of each chapter, Past Examination Questions have been included to familiarize the students with the type of questions being set by Cambridge.

Answers

To help the students, answers have been provided at the end of each question.

CONTENTS

1 Quadratics 1

- 1.1 Parabola 1
- 1.2 Completing the square 2
- 1.3 Stating the coordinates of the vertex of a quadratic function 4
- 1.4 Sketching the graph of a quadratic polynomial using completing the square 6
- 1.5 Discriminant of a quadratic polynomial 8
- 1.6 Solving quadratic equations 12
- 1.7 Solving linear and quadratic inequalities in one unknown 13
- 1.8 Rational fractions in inequalities 15
- 1.9 Simultaneous equations 17
- 1.10 Quadratic equations in some functions 18
- Past Examination Questions 19

2 Functions 20

- 2.1 Function, Domain, Co-domain and range 20
- 2.2 Composition of functions 24
- 2.3 One-one functions 26
- 2.4 Inverses of one-one functions 28
 - 2.4.1 Inverse of algebraic functions 28
 - 2.4.2 Inverse of exponential functions 29
 - 2.4.3 Inverse of logarithmic functions 30
 - 2.4.4 Inverse of quadratic functions 31
- 2.5 Relationship between a one-one function and its inverse 32
- Past Examination Questions 35

3 Coordinate geometry 37

Past Examination Questions 39

4 Circular measure 40

- 4.1 Length of arc and area of sector 42
- 4.2 Area of segment 42
- Past Examination Questions 46

5 Trigonometry 1 47

- 5.1 Graphs of sine, cosine and tangent 47
- 5.2 Exact values of trigonometric functions 51
- 5.3 Trigonometric identities 53
- 5.4 Solving trigonometric equations 55
- Past Examination Questions 58

6 Vectors I 59

- 6.1 Standard notations for vectors 59
- 6.2 Position vectors and displacement vectors 60
- 6.3 Magnitude of a vector 62
- 6.4 Unit vectors 64
- 6.5 Scalar product of two vectors 66
- 6.6 Angle between two directions 68
- 6.7 Perpendicular vectors 70
- Past Examination Questions 72

7

Series 73

- 7.1 Expansion of $(a+b)^n$, where n is a positive integer 73
- 7.2 Finding the n th term of an expansion 75
- 7.3 Finding the coefficient of a given term 76
- 7.4 Arithmetic progression 77
 - 7.4.1 Use of T_n to determine the n th term 77
 - 7.4.2 Use of T_n to determine the value of n 78
 - 7.4.3 Use of T_n to determine the value of a and d 79
 - 7.4.4 Use of T_n to determine the value of a and d 80
- 7.4 Sum of an AP 81
- 7.5 Geometric Progressions 84
 - 7.5.1 Use of T_n to determine the n th term 84
 - 7.5.2 Use of T_n to determine the value of n 85
 - 7.5.3 Use of T_n to determine the value of a and r 86
 - 7.5.4 Use of T_n to determine the value of a and r 87
- 7.6 Sum of a GP 88
- 7.7 Sum of infinity 90
- AP and GP - Past Examination Questions 92

8

Differentiation 1 94

- 8.1 Gradient of a curve 94
- 8.2 Derivative of ax^n 95
- 8.3 Derivative of $(ax+b)^n$ 98
- 8.4 Gradients 100
- 8.5 Tangents 102
- 8.6 Normals 104
- 8.7 Increasing and decreasing functions 106
- 8.8 Connected rates of change 108
- 8.9 Stationary points 110
- 8.10 Problems on maximum and minimum value 114

9**Integration 1 117**

- 9.1 Integral of ax^n 117
- 9.2 Integral of the type $(ax+b)^n$ 119
- 9.3 Problems involving the evaluation of a constant of integration 121
- 9.4 Evaluation definite integrals 123
- 9.5 Plane Areas 125
- 9.6 Volumes of revolution 128
- Area and Volume - Past Examination Questions 132

10 Algebra 133

- 10.1 Modulus functions 133
- 10.2 Modulus inequalities 135
- 10.3 Remainder and factor theorem 138
- 10.4 Partial fractions 141
 - 10.4.1 Distinct linear factors 141
 - 10.4.2 Distinct quadratic factors 143
 - 10.4.3 Repeated linear factors 144
- Partial fractions - Past Examination Questions 145
- 10.5 Expansion of $(1+x)^n$ 146
- Binomial Expansion - Past Examination Questions 148

11**Logarithmic and Exponential functions 149**

- 11.1 Laws of logarithms 149
- 11.2 Relationship between e^x and $\ln x$ and their graphs 152
- 11.3 Solving equations of the type $a^x = b$ and similar inequalities 154
- Linear law 156

12**Trigonometry 2 158**

- Compound angle formulae 158
- Expressing $a\cos\theta + b\sin\theta$ in the form $R\cos(\theta-\alpha)$ and similar forms 161
- Past Examination Questions 162

13**Differentiation 2 165**

- Derivative of exponential functions 165
- Derivative of logarithmic functions 167
- Derivative of trigonometric functions 168
- The product rule 170
- The Quotient rule 172
- Differentiation of implicit functions 174
- Differentiation of parametric equations 176
- Past Examination Questions 178

14	Integration II	181
14.1	Integration of exponential functions	181
14.2	Integration of the type $\frac{1}{ax+b}$	182
14.3	Integration of trigonometric functions	184
14.4	Use of double angle formula to integrate $\cos^2 x$ and $\sin^2 x$	186
14.5	Integration using partial fractions	188
14.6	Integration of the form $\frac{kf'(x)}{f(x)}$	190
14.7	Integration by parts	192
14.8	Integration using substitution	195
14.9	The trapezium rule	198
14.10	Use of sketch graphs to show underestimate or overestimate	202

15	Numerical solution of equations	204
15.1	Location of roots-searching for sign change	204
15.2	Location of roots – graphical considerations	206
15.3	Iterative methods	209
15.4	Given an iteration formula, how to find the original equation	212

16	Vectors 2	214
16.1	Vector equation of a line	214
16.2	Parallel, intersecting and skew lines	216
16.3	Cartesian equation of line	219
16.4	Angle between two lines	220
16.5	Finding the point of intersection of two lines	222
16.6	Finding the perpendicular or shortest distance from a point to a line	224
16.7	Equation of plane	226
16.8	Using cross-product to find a vector perpendicular to a plane	227
16.9	Using cross-product to find the equation of a plane	229
16.10	How to show that a point lies on a plane?	231
16.11	How to show that a line lies on a plane?	232
16.12	How to show that a line is parallel to a plane?	233
16.13	Finding the point of intersection of a line and a plane	234
16.14	Finding the perpendicular distance from a point to a plane	236
16.15	Angle between two planes	238
16.16	Angle between a line and a plane	240
16.17	How to find the equation of the line of intersection of two planes?	242

17	Differential Equations 244
17.1	Formulating a simple statement involving a rate of change as a differential equation 244
17.2	First order differential equation – variable separable 246
17.3	Particular solution 248
17.4	Progressive refinement of model 250
18	Complex Numbers 256
18.1	The complex number system 256
18.2	Addition of complex numbers 257
18.3	Subtraction of complex numbers 258
18.4	Multiplication of a complex number by a real number 259
18.5	Multiplication of two complex number 260
18.6	Conjugate of a complex number 261
18.7	Division of two complex numbers 263
18.8	Equal complex numbers 265
18.9	Solving polynomial equations having non-real roots 266
18.10	Square roots of a complex number 267
18.11	The Argand diagram 268
18.12	Geometrical properties 270
18.12.1	Geometrical interpretation of $\bar{z}$ 270
18.12.2	Geometrical interpretation of kz , where $k \in \mathbb{R}$ 270
18.12.3	Geometrical representation of $z_1 + z_2$ 271
18.12.4	Geometrical representation of $z_1 - z_2$ 271
18.13	Multiplication of a complex number by i 272
18.14	Modulus of a Complex Number 274
18.15	Argument of a Complex Number 276
18.16	Trigonometric form or polar form of a complex number 279
18.17	Loci 284
18.14	Past Examination Questions 288

List of formulae 290

CHAPTER 1

QUADRATICS

1.1 Parabola

All quadratic functions possess turning points which are either minimum or maximum, depending on the coefficient of x^2 .

The graph of a quadratic function (ax^2+bx+c) can be of two types depending on the sign of a .

When the coefficient of x^2 is positive, we obtain a curve as shown in **figure 1**.
For example, $3x^2 + 7x + 2$ will be a parabola with a minimum point.

However if the coefficient of x^2 is negative, we obtain a curve as shown in **figure 2**.
For example, $-x^2 + 5x + 6$ will be a parabola with a maximum point.

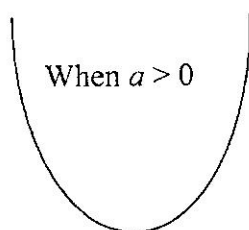


Figure 1

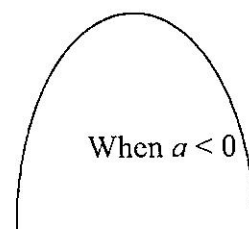


Figure 2

Exercise 1.1

Without any calculation, state whether the following quadratic polynomials will have a minimum or maximum point.

(a) $x^2 + 5x + 3$ [Ans: min] (e) $2(x-2)^2 + 5$ [Ans: min]

(b) $3x^2 + 7x + 1$ [Ans: min] (f) $1 - (x+2)^2$ [Ans: max]

(c) $4 - 5x - x^2$ [Ans: max] (g) $3(x-1)^2 + 6$ [Ans: min]

(d) $2 + 9x + x^2$ [Ans: min] (h) $1 + 4x - x^2$ [Ans: max]

1.2 Completing the square

To complete the square of a quadratic polynomial, we have to

- 1 Make sure that the coefficient of x^2 is one.
- 2 If the coefficient of x^2 is one, then add half the coefficient of x to the square
- 3 Subtract half the coefficient of x to the square
- 4 The first three terms form a perfect square and the remaining two give a constant.

1.2.1 Case 1 : coefficient of x^2 is 1

Example 1

Complete the square of the following quadratic polynomials.

(a) $x^2 + 4x + 10$

(b) $x^2 - 3x + 8$

Solution

(a) $x^2 + 4x + 10$

$= (x^2 + 4x + 4) + 10 - 4$ [we add half coefficient of x to the square and subtract]

Now, the first three terms form a perfect square.

Thus, we have

$$x^2 + 4x + 10 = (x + 2)^2 + 6$$

$$\begin{aligned} \text{(b) } x^2 - 3x + 8 &= (x^2 - 3x + \frac{9}{4}) + 8 - \frac{9}{4} \\ &= (x - \frac{3}{2})^2 + \frac{23}{4} \end{aligned}$$

Exercise 1.2.1

Complete the square of the following quadratic polynomials.

1 $x^2 + 6x + 15$ [Ans: $(x + 3)^2 + 6$]

2 $x^2 + 2x + 5$ [Ans: $(x + 1)^2 + 4$]

3 $x^2 - 4x + 10$ [[Ans: $(x - 2)^2 + 6$]

4 $x^2 + 8x - 1$ [Ans: $(x + 4)^2 - 17$]

5 $x^2 - 2\sqrt{3}x + 1$ [Ans: $(x - \sqrt{3})^2 - 2$]

6 $x^2 + 3x + 10$ [Ans: $(x + \frac{3}{2})^2 + \frac{31}{4}$]

7 $x^2 + 5x + 6$ [Ans: $(x + \frac{5}{2})^2 - \frac{1}{4}$]

8 $x^2 - x - 6$ [Ans: $(x + \frac{1}{2})^2 - \frac{25}{4}$]

1.2.2 Case 2 : coefficient of x^2 is not 1

Example 2

Complete the square of the following quadratic polynomials.

(a) $2x^2 + 4x + 5$ (b) $-x^2 + 5x + 7$

In these cases, we can see that the coefficients of x^2 are 2 and -1. So, we factorise and take 2 and -1 common. We have,

$$\begin{aligned}\text{(a)} \quad 2x^2 + 4x + 5 &= 2\left(x^2 + 2x + \frac{5}{2}\right) \\ &= 2\left(x^2 + 2x + 1 + \frac{5}{2} - 1\right) \\ &= 2\left[(x + 1)^2 + \frac{3}{2}\right] \\ &= 2(x + 1)^2 + 3\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad -x^2 + 5x - 6 &= -(x^2 - 5x + 6) \\ &= -(x^2 - 5x + \frac{25}{4} + 6 - \frac{25}{4}) \\ &= -\left[\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}\right] \\ &= \frac{1}{4} - \left(x - \frac{5}{2}\right)^2\end{aligned}$$

Exercise 1.2.2

Complete the square of the following quadratic polynomials.

1 $3x^2 - 7x + 10$ [Ans: $3\left(x - \frac{7}{6}\right)^2 + \frac{71}{12}$]

2 $2x^2 + 6x + 11$ [Ans: $2\left(x + \frac{3}{2}\right)^2 + \frac{13}{2}$]

3 $5x^2 - \sqrt{3}x + 2$ [Ans: $5\left(x - \frac{\sqrt{3}}{10}\right)^2 + \frac{37}{20}$]

4 $2x(x-2)$ [Ans: $2(x-1)^2 - 2$]

5 $3x^2 - 9x + 4$ [Ans: $3\left(x - \frac{3}{2}\right)^2 - \frac{11}{4}$]

6 $-x^2 - 3x + 2$ [Ans: $-\left(x + \frac{3}{2}\right)^2 + \frac{17}{4}$]

7 $4x(x+2)$ [Ans: $4(x+1)^2 - 4$]

1.3 Stating the coordinates of the vertex (minimum or maximum point) of a quadratic function

To state the coordinates of the vertex, that is the minimum or maximum point of a quadratic polynomial, use the following steps:

- 1 Complete the square
- 2 To find the x -coordinates of the turning point, equate the bracket to zero.
- 3 The y -coordinate is the constant term added to the perfect square.

Example 1

State the coordinates of the vertex (turning point) in each of the following quadratic polynomials:

(a) $(x - 2)^2 + 5$ (b) $2(x + 3)^2 - \frac{7}{4}$ (c) $9 - 3(x - 1)^2$

Solution

(a) $(x - 2)^2 + 5$

The x -coordinates is given by $x - 2 = 0 \Rightarrow x = 2$

And the y -coordinates is 5.

Hence, the minimum point is (2,5).

(b) $2(x + 3)^2 - \frac{7}{4}$

The x -coordinates is given by $x + 3 = 0 \Rightarrow x = -3$

And the y -coordinates is $-\frac{7}{4}$.

Hence, the minimum point is $(-3, -\frac{7}{4})$.

(c) $9 - 3(x - 1)^2$

The x -coordinates is given by $x - 1 = 0 \Rightarrow x = 1$

And the y -coordinates is 9.

In this case, it is a maximum point since there is a minus sign in front of the perfect square.

Hence, the maximum point is (1,9).

Example 2

Express $8 - 6x - x^2$ in the form $a - (x - b)^2$ and hence, or otherwise, find the maximum value of the function and the coordinates of the maximum point. (C)

Solution

$$8 - 6x - x^2 = a - (x - b)^2$$

$$8 - 6x - x^2 = a - (x^2 - 2bx + b^2)$$

$$8 - 6x - x^2 = a - x^2 + 2bx - b^2$$

Equating coefficients of like terms, we have

$$x^2] \quad -1 = -1$$

$$x] \quad -6 = 2b \Rightarrow b = -3$$

Constants] $8 = a - b^2$

$$8 = a - (-3)^2 \Rightarrow a = 17$$

$$\therefore 8 - 6x - x^2 = 17 - (x + 3)^2$$

The maximum value of the function is 17.

The coordinates of the maximum point is $(-3, 17)$.

Exercise 1.3

- 1 Complete the following table

Quadratic Polynomial	Nature of turning point	Coordinates of turning point	Max/min value of function
$(x-2)^2 + 1$	minimum	$(2, 1)$	1
$3 - 2(x+1)^2$	maximum	$(-1, 3)$	3
$4(x+3)^2 + 15$			
$6 - 5(x-2)^2$			
$(x+3)^2 - 9$			
$5 + (x-4)^2$			

- 2 Express $y = x^2 - 5x + 4$ in the form $a(x - b)^2 + c$ where a , b and c are constants. Hence state the minimum value of y .

$$[\text{Ans: } \left(x - \frac{5}{2}\right)^2 - \frac{9}{4}; \text{ minimum value} = -\frac{9}{4}]$$

- 3 Express $y = 2x^2 + 7x + 1$ in the form $a(x + b)^2 + c$ where a , b and c are constants. Hence state the minimum value of y .

$$[\text{Ans: } 2\left(x + \frac{7}{4}\right)^2 - \frac{41}{8}; \text{ minimum value} = -\frac{41}{8}]$$

- 4 Express $y = 2x^2 + 5x + 4$ in the form $a(x + b)^2 + c$ stating the numerical values of a , b and c . Hence, or otherwise, write down the coordinates of the minimum point on the graph of $y = 2x^2 + 5x + 4$.

$$[\text{Ans: } a = 2, b = 5/4, c = 7/8; (-5/4, 7/8)]$$

- 5 Show that $x^2 + x + 1$ is positive for all real x . (C)

- 6 Show that $x^2 - x + 1$ is positive for all real x . (C)

1.4 Sketching the graph of a quadratic polynomial using completing the square

To sketch the graph of a quadratic polynomial, make use of the following steps:

- 1 Complete the square
- 2 State the coordinates of the vertex of the graph
- 3 Using the Cartesian axes, draw a parabola

Example 1

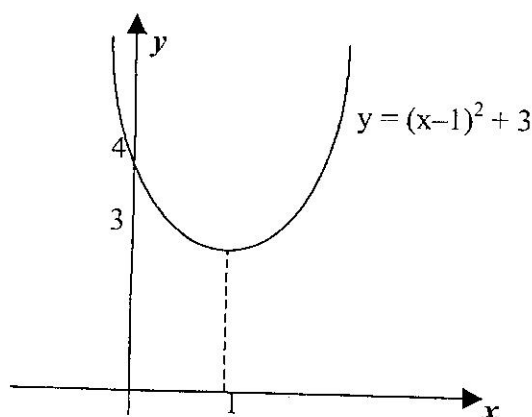
Sketch the graph of the quadratic polynomial $y = (x - 1)^2 + 3$

Solution

Clearly the graph has a minimum point and the coordinates of the vertex is (1,3)

When $x = 0$, $y = 4$

So the graph will have the following shape,



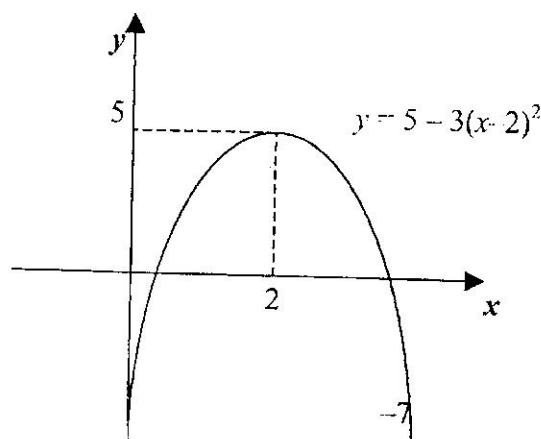
Example 2

Sketch the graph of the quadratic polynomial $y = 5 - 3(x - 2)^2$

Solution

Clearly the graph will have a maximum point and the coordinates of the vertex is (2,5).

When $x = 0$, $y = -7$



Example 3

Sketch the graph of the quadratic polynomial $y = x^2 - 4x + 5$

Solution

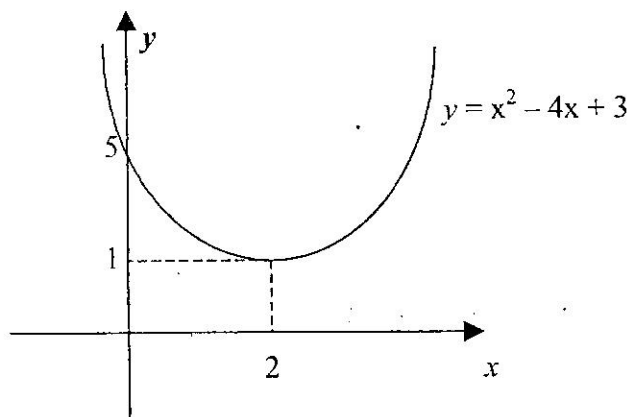
Completing the square, we have

$$\begin{aligned} \text{(a)} \quad x^2 - 4x + 5 &= (x^2 - 4x + 4) + 5 - 4 \\ &= (x - 2)^2 + 1 \end{aligned}$$

Clearly the graph has a minimum point and the coordinates of the vertex is (2, 1)

When $x = 0$, $y = 5$

So the graph will have the following shape

**Exercise 1.4**

Sketch the graph of the following quadratic polynomials.

(a) $y = (x - 1)^2 + 3$

(b) $y = (x - 2)^2 + 4$

(c) $y = 3 - (x - 4)^2$

(d) $y = 5 - (x + 1)^2$

(e) $y = 2(x + 1)^2 + 3$

(f) $y = 2(x - 3)^2 + 1$

(g) $y = 4 - (x + 3)^2$

(h) $y = 1 - (x + 3)^2$

(i) $y = x^2 - 6x + 10$

1.5 Discriminant of a quadratic polynomial

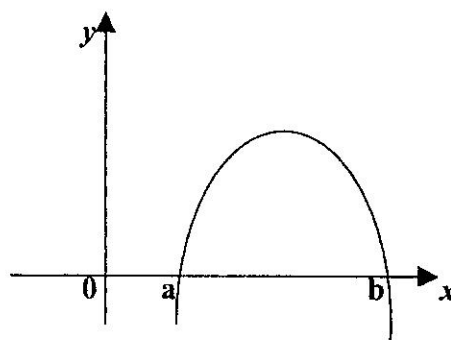
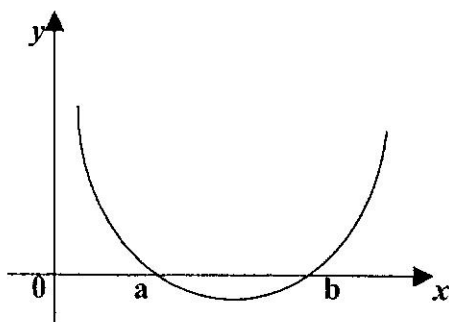
We know that the roots of a quadratic equation can be obtained by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ in the quadratic formula is called the **discriminant** of the equation as it determines the **type of roots** that the equation possesses.

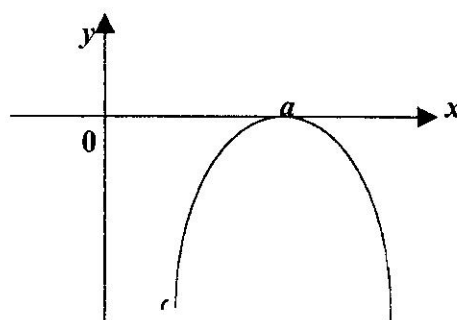
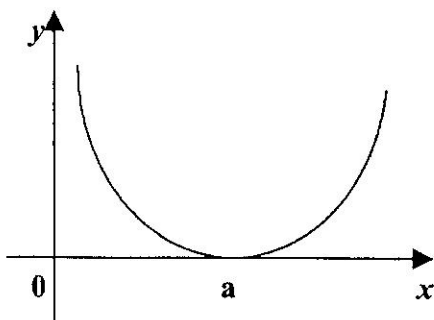
Case 1: When $b^2 - 4ac > 0$, there are two real values for $\pm\sqrt{b^2 - 4ac}$. Thus the equation has two **real and unequal roots or real and distinct roots**.

Graphically, the curve $y = ax^2 + bx + c$ cuts the x -axis at two different points.



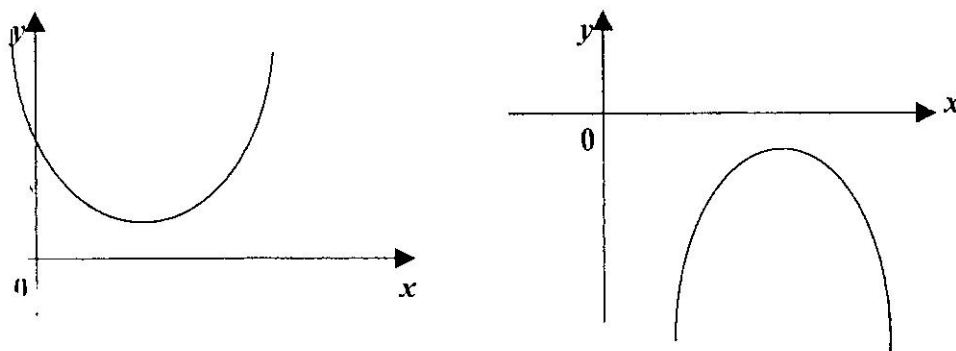
Case 2: When $b^2 - 4ac = 0$, then $x = -\frac{b}{2a}$. Thus both roots of the equation are real and

equal. Graphically, the curve touches the x -axis at $x = -\frac{b}{2a}$, that is, the x -axis is a tangent to the curve. These roots are called **repeated or coincident roots**.



Case 3: When $b^2 - 4ac < 0$, then we have **no real roots**. The square root of a negative number is known as a complex number and we say that the equation has **complex or imaginary** roots.

Graphically, the curve does not cut or touch the x-axis. It lies entirely above or below the axis.



From the above we conclude that the quadratic equation $ax^2 + bx + c = 0$,

	Nature of roots	No. of real roots
$b^2 - 4ac > 0$	Real and distinct	2
$b^2 - 4ac = 0$	Real and Equal	1
$b^2 - 4ac < 0$	Complex, imaginary or not real	0
$b^2 - 4ac \geq 0$	Real	1 or 2

Example 1

Determine the number of real roots of the following quadratic equations:

(a) $x^2 - 4x + 3 = 0$ (b) $x^2 + 4x + 4 = 0$ (c) $4x^2 + 5\sqrt{3}x + 5 = 0$ (C)

Solution

(a) $x^2 - 4x + 3 = 0$

Here $a = 1$, $b = -4$ and $c = 3$

Now $b^2 - 4ac = (-4)^2 - 4(1)(3) = 4$

Since $b^2 - 4ac > 0$, there are **two** roots.

(b) $x^2 + 4x + 4 = 0$

Here $a = 1$, $b = 4$ and $c = 4$

Now $b^2 - 4ac = (4)^2 - 4(1)(4) = 0$

Since $b^2 - 4ac = 0$, there is only **one** root.

$$(c) \ 4x^2 - 5\sqrt{3}x + 5 = 0$$

$$\text{Here } a = 4 \quad b = -5\sqrt{3} \text{ and } c = 5$$

$$\text{Now } b^2 - 4ac = (-5\sqrt{3})^2 - 4(4)(5) = 75 - 80 = -5$$

Since $b^2 - 4ac < 0$, the roots are not real or imaginary.

The number of real roots = 0.

Example 2

Find the set of values of a for which $ax^2 + 3x + 4$ is positive for all real values of x . (C)

Solution

Since $ax^2 + 3x + 4$ is positive for all real values of x , the graph of the quadratic polynomial will lie entirely above the x -axis. So, there will be no real roots.

$$\text{Hence } b^2 - 4ac < 0.$$

$$\text{Here } a = a \quad b = 3 \text{ and } c = 4$$

$$3^2 - 4(a)(4) < 0$$

$$9 - 16a < 0$$

$$-16a > -9$$

$$a > \frac{9}{16}$$

Example 3

Prove that, for all real values of a ($a > 0$), the equation $x^2 + 2ax + 2a^2 + a + 1 = 0$ has no real roots for x . (C)

Solution

$$\text{Here } a = 1, \ b = 2a \text{ and } c = 2a^2 + a + 1$$

$$\text{Now } b^2 - 4ac = (2a)^2 - 4(1)(2a^2 + a + 1)$$

$$= 4a^2 - 8a^2 - 4a - 4$$

$$= -4a^2 - 4a - 4$$

Since each and every term is negative, $b^2 - 4ac < 0$.

Hence the equation has no real roots.

Exercise 1.5

Use the discriminant of a quadratic polynomial to determine the number of real roots of the following equations:

- 1 $x^2 - 4x + 3 = 0$ [Ans : 2]
- 2 $3x^2 + 7x + 2 = 0$ [Ans : 2]
- 3 $x^2 + 4x + 4 = 0$ [Ans : 1]
- 4 $x^2 + x + 5 = 0$ [Ans : 0]
- 5 $x^2 - 6x + 9 = 0$ [Ans : 1]
- 6 $x^2 - 4 = 0$ [Ans : 2]
- 7 $4x^2 - 5(\sqrt{3})x + 5 = 0$ (C) [Ans : 0]
- 8 $x^2 + 9 = 0$ [Ans : 0]
- 9 $2x^2 - 7x + 3 = 0$ [Ans : 2]
- 10 $x^2 + 2x + 1 = 0$ [Ans : 1]
- 11 Find the set of values of a for which $ax^2 + 4x + 5 = 0$ is positive for all real values of x . [Ans : $a > 4/5$]
- 12 Find the set of values of k for which $kx^2 - 3x + 2 = 0$ is positive for all real values of x . [Ans : $k > 9/8$]
- 13 Find the set of values of k for which $k + 2x - 3x^2$ is negative for all real values of x . [Ans : $k < -1/3$]
- 14 Prove that, if $x^2 > k(x + 1)$ for all real values of x , then $-4 < k < 0$. (C)
- 15 Given that $x^2 + ax + b = (x+p)^2 + q$, find the constants p and q in terms of a and b . Deduce that the equation $x^2 + ax + b = 0$ has no real solutions if $b > \frac{1}{4}a^2$. (C)
[Ans : $p = a/2$, $q = b - \frac{1}{4}a^2$]
- 16 Find the set of values of the real number a for which the equation $x^2 - 2ax + a = 0$ has real solutions for x . [Ans : $0 < a < 1$]

1.6 Solving quadratic equations

To solve a quadratic equation, we may use the method of factorization, completing the square or make use of the quadratic formula.

Example 1

Solve the equation $12x^2 + x - 6 = 0$

Solution

$$12x^2 + x - 6 = (4x + 3)(3x - 2)$$

Either $x = -\frac{3}{4}$ or $x = \frac{2}{3}$ (Ans)

Example 2

Solve the equation $x^2 + 2x - 5 = 0$

Solution

Since the equation cannot be factorised, we make use of the quadratic formula

Here $a = 1$, $b = 2$ and $c = -5$

Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we have

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times -5}}{2(1)}$$

either $x = 1.45$ or $x = -3.45$

Example 2 can also be solved by the method of completing the square

$$x^2 + 2x - 5 = (x+1)^2 - 6 = 0$$

$$(x+1)^2 = 6$$

$$x+1 = \pm\sqrt{6}$$

either $x = 1.45$ or $x = -3.45$

Exercise 1.6

Solve the following quadratic equations using any method of your choice.

1 $x^2 + 5x + 1 = 0$ [Ans : -0.209, -4.79]

2 $2x^2 + 7x + 3 = 0$ [Ans : -3, - $\frac{1}{2}$]

3 $15 - 2a - a^2 = 0$ [Ans : 3, -5]

4 $2x^2 + 7x + 1 = 0$ [Ans : -0.149, -3.35]

5 $\sqrt{5}x^2 + 9x + 2 = 0$ [Ans : -0.24, -3.79]

6 $\sqrt{2}x^2 + 5x + 3 = 0$ [Ans : -0.77, -2.77]

1.7 Solving linear and quadratic inequalities in one unknown

1.7.1 Linear inequalities

Example 1

Find the set of values of x which satisfy inequality $x - 4 < 3 - x$

Solution

$$x - 4 < 3 - x$$

$$2x < 7$$

$$x < \frac{7}{2} \text{ (Answer)}$$

Example 2

Solve the inequalities $3(3 - 2x) < 3 + x$

Solution

$$3(3 - 2x) < 3 + x$$

$$9 - 6x < 3 + x$$

$$-6x - x < 3 - 9$$

$$7x < -6$$

$$7x > 6 \text{ (sign change)}$$

$$\text{Therefore } x > \frac{6}{7}$$

1.7.2 Quadratic inequalities

To solve a quadratic inequalities, the following steps will be useful.

Step 1 Rearrange the inequalities in such a way that one side is always zero.

Step 2 Factorise the quadratic expression.

Step 3 Find the critical values of the expression by equating each bracket to zero.

Step 4 By drawing a table, select the right answer.

Example 1

Solve the inequality $x(x + 1) < 12$ (C)

Solution

The given inequality is $x(x + 1) < 12$

It can be written as $x^2 + x - 12 < 0$

$$(x + 4)(x - 3) < 0$$

Critical values of x are -4 and 3 .

	$x < -4$	$-4 < x < 3$	$x > 3$
$x + 4$	-	+	+
$x - 3$	-	-	+
$(x + 4)(x - 3)$	+	-	+

Therefore the solution set is $\{ x \in \mathbb{R} : -4 < x < 3 \}$

Exercise 1.7

Solve the following inequalities

- 1 $2x^2 - 7x < -3$ [Ans : $\frac{1}{2} < x < 3$]
- 2 $(x - 2)(x - 5) > 0$ [Ans : $x < 2$ or $x > 5$]
- 3 $(x - 2)(x - 3)(x - 5) > 0$ [Ans : $2 < x < 3$ or $x > 5$]
- 4 $3x^2 - 7x < -2$ [Ans : $\frac{1}{3} < x < 2$]
- 5 $\frac{(x+1)(x-2)}{(x-1)} < 0$ [$x < -1$ or $1 < x < 2$]
- 6 $\frac{(2x-1)(3x-5)}{(x-2)} > 0$ [$\frac{1}{2} < x < \frac{5}{3}$ or $x > 2$]
- 7 $\frac{(x+1)(x-2)}{x} < 0$ [$x < -1$ or $0 < x < 2$]
- 8 $(1 - x)^2 \geq 17 - 2x$ [Ans : $x \leq -4$ or $x \geq 4$]
- 9 $(x - 1)(5x + 4) > 2(x - 1)$ [Ans : $x < -\frac{2}{5}$ or $x > 1$]
- 10 $(1 - x)(4 - x) > x + 11$ [Ans : $x < -1$ or $x > 7$]
- 11 $5x^2 < 3x + 2$ [Ans : $-\frac{2}{5} < x < 1$]
- 12 $(x+2)^2 < x(4-x) + 40$ [Ans : $-\sqrt{18} < x < \sqrt{18}$]
- 13 $(2x - 1)(x + 1) < 0$ [Ans : $-1 < x < \frac{1}{2}$]
- 14 $x^4 - x^3 \geq 0$ (C) [Ans : $x < -1$ or $0 \leq x < 1$]
- 15 $x^3 < 6x - x^2$ (C) [Ans : $x < -3$ or $0 < x < 2$]
- 16 $x(x+1) < 12$ (C) [Ans : $-4 < x < 3$]
- 17 $x^2 - 9 \geq (x + 3)(x^2 - 3x + 1)$ (C) [Ans : $x \leq 2$]

1.8 Rational fractions in inequalities

Consider the inequality $\frac{x+2}{x-5} < 3$

The initial problem here is that we do not know whether $x - 5$ is negative or positive. This prevents the apparently obvious step of multiplying both sides by $x - 5$. Thus, in this case the following steps would be useful in solving an inequality.

Step 1 Bring all the functions on one side. One side must always be 0.

[**Never cross multiply**]

Step 2 Take L. C. M, simplify and factorise.

Step 3 Find critical values and construct a table as usual.

Example 2

Solve the inequality $3x + 5 > \frac{2}{x}$ (C)

Solution

$$3x + 5 > \frac{2}{x}$$

$$3x + 5 - \frac{2}{x} > 0$$

$$\frac{(3x+5)x-2}{x} > 0$$

$$\frac{3x^2+5x-2}{x} > 0$$

$$\frac{(3x-1)(x+2)}{x} > 0$$

Critical values of x are -2 , 0 and $\frac{1}{3}$.

	$x < -2$	$-2 < x < 0$	$0 < x < \frac{1}{3}$	$x > \frac{1}{3}$
$3x-1$	-	-	-	+
$x+2$	-	+	+	+
x	-	-	+	+
$\frac{(3x-1)(x+2)}{x}$	-	+	-	+

Therefore the solution set is $\{ x \in \mathbb{R} : -2 < x < 0 \text{ or } x > \frac{1}{3} \}$

INEQUALITIES – PAST EXAMINATION QUESTIONS

Solve the inequalities.

1 $\frac{x+1}{x-1} < 4$ [Ans : $x < 1$ or $x > 5/3$]

2 $\frac{2x}{x+1} > x$ [Ans : $x < -1$ or $0 < x < 1$]

3 $\frac{x-5}{1-x} > 1$ [Ans : $1 < x < 3$]

4 $\frac{10}{x} < 19 - 6x$ [Ans : $x < 0$ or $2/3 < x < 5/2$]

5 $\frac{4-x}{x-2} > 3$ [Ans : $2 < x < 5/2$]

6 $x + 1 < \frac{2}{x}$ [Ans : $x < -2$ or $0 < x < 1$]

7 $\frac{12}{x-3} < x + 1$ [Ans : $-3 < x < 3$ or $x > 5$]

8 $\frac{x-1}{x+1} > \frac{x}{6}$ [Ans : $x < -1$ or $2 < x < 3$]

9 $\frac{3}{2} < \frac{2x-3}{x-5}$ [Ans : $x < -9$ or $x > 5$]

10 $\frac{1}{x-1} < \frac{1}{x+1}$ [Ans : $-1 < x < 1$]

11 $\frac{2}{x+1} > x$ [Ans : $x < -2$ or $-1 < x < 1$]

12 $\frac{15}{x-2} > 3x - 2$ [Ans : $x < -1$ or $2 < x < 11/3$]

13 $\frac{3}{1-x} < 5 - 4x$ [Ans : $x < 1/4$ or $1 < x < 2$]

1.9 Simultaneous equations

In this section we are going to solve a pair of simultaneous equations in which one is linear and the other is quadratic.

Example 1

Solve the simultaneous equations

$$y = x - 2, y^2 = x \quad (\text{C})$$

Solution

$$y = x - 2 \quad \dots (1)$$

$$y^2 = x \quad \dots (2)$$

Substituting (1) in (2), we have

$$(x - 2)^2 = x$$

$$x^2 - 4x + 4 = x$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$\text{either } x = 1 \text{ or } x = 4$$

$$\text{From (1) } y = -1 \text{ or } y = 2$$

Thus we have as solutions (1, -1) and (4, 2).

Example 2

- (i) Solve the simultaneous equations $y = x^2 - 3x + 2$, $y = 3x - 7$
(ii) Interpret your solution to part (i) geometrically. (C)

Solution

$$(i) \quad y = x^2 - 3x + 2 \quad \dots (1)$$

$$y = 3x - 7 \quad \dots (2)$$

Substituting (2) in (1), we have

$$3x - 7 = x^2 - 3x + 2$$

$$x^2 - 6x + 9 = 0$$

$$(x - 3)(x - 3) = 0$$

$$x = 3 \text{ or } x = 3$$

$$\text{From (2) } y = 2$$

Thus we have as the only solution (3, 2).

- (ii) The line $y = 3x - 7$ is a tangent to the curve $y = x^2 - 3x + 2$.

Exercise 1.9

Solve the following simultaneous equations:

- | | | |
|---|---|--------------------------------|
| 1 | $x + y = 12, y^2 = 8x + 9$ | [Ans : (5, 7) ; (27, -15)] |
| 2 | $3x + y = 1, x^2 + y^2 = 5$ | [Ans : (1, -2) ; (-2/5, 11/5)] |
| 3 | $x + y = 5, xy = x + 3$ | [Ans : (1, 4) ; (3, 2)] |
| 4 | $x + 2y = 2, x^2 + xy = 20$ | [Ans : (4, 1) ; (-10/3, -8/3)] |
| 5 | $3x - y = 7, x^2 + xy - y^2 = 1$ | [Ans : (5, 8) ; (2, -1)] |
| 6 | $2^x - 3^y, x + y = 1 \quad (\text{C})$ | [Ans : (0.63, 0.37)] |

1.10 Quadratic Equations in some functions

Sometimes we may come across equations of the form $x^4 - 5x^2 + 4 = 0$. This equation can be rewritten as $(x^2)^2 - 5(x^2) + 4 = 0$, which is a quadratic equation in x^2 .

Similarly, $y - 2\sqrt{y} - 3 = 0$ is a quadratic in $\sqrt{y}$. To solve such equations, we have to use the middle term as substitution.

Example 1

Solve the equation $x^4 - 29x^2 + 100 = 0$

Solution

$$x^4 - 29x^2 + 100 = 0$$

Let $x^2 = y$,

We have $y^2 - 29y + 100 = 0$

$$(y - 25)(y - 4) = 0$$

Either $y = 25$ or $y = 4$

$$\therefore \begin{array}{ll} x^2 = 25 & \text{or } x^2 = 4 \\ x = \pm 5 & \text{or } x = \pm 2 \end{array}$$

Example 2

Solve the equation $x - 5\sqrt{x} + 6 = 0$

Solution

$$x - 5\sqrt{x} + 6 = 0$$

Let $\sqrt{x} = y$,

We have $y^2 - 5y + 6 = 0$

$$(y - 2)(y - 3) = 0$$

Either $y = 2$ or $y = 3$

$$\therefore \begin{array}{ll} \sqrt{x} = 2 & \text{or } \sqrt{x} = 3 \\ x = 4 & \text{or } x = 9 \end{array}$$

Example 3

Solve the equation $r^6 - 9r^3 + 8 = 0$

Solution

$$r^6 - 9r^3 + 8 = 0$$

Let $r^3 = x$,

We have $x^2 - 9x + 8 = 0$

$$(x - 1)(x - 8) = 0$$

Either $x = 1$ or $x = 8$

$$\therefore \begin{array}{ll} r^3 = 1 & \text{or } r^3 = 8 \\ r = 1 & \text{or } r = 2 \end{array}$$

QUADRATICS – PAST EXAMINATION QUESTIONS

- 1 The function f is defined by $f: x \rightarrow x^2 + kx + 1$, ($x \in \mathbb{R}$), where k is a real constant. Find the set of values of k such that the range of f is a subset of $\mathbb{R}^+$.
[Ans : $-2 < k < 2$]
- 2 Express $9x^2 - 36x + 52$ in the form $(Ax - B)^2 + C$, where A , B and C are integers. Hence, or otherwise, find the set of values taken by $9x^2 - 36x + 52$ for $x \in \mathbb{R}$.
[Ans : $(3x-6)^2 + 16$, $y > 16$]
- 3 (i) Express $9x^2 + 12x + 7$ in the form $(ax + b)^2 + c$, where a , b and c are constants whose values are to be found.
(ii) Find the set of values of x for which $9x^2 + 12x + 7 > 19$.
(iii) Find the set of values taken by $\frac{1}{9x^2 + 12x + 7}$ for real values of x .
[Ans : (i) $a=3$, $b=2$, $c=3$; (ii) $x < -2$ or $x > \frac{2}{3}$ (iii) $0 < y \leq \frac{1}{3}$]
- 4 Sketch the graph of the curve with equation $y = x(1 - x)$. Determine the greatest and least values of y when $-1 \leq x \leq 1$.
[Ans : $\frac{1}{4}$, -2]
- 5 It is given that $f(x) = 4x^2 + 6x - 6$. Express $f(x)$ in the form $A(x+B)^2 + C$, where A , B and C are constants. Hence or otherwise,
(a) find the exact solutions of $f(x) = 0$,
(b) find the exact solutions of $f(x - \frac{3}{4}) = 0$,
(c) find the exact coordinates of the minimum point on the graph of $y = f(x)$.

$$[\text{Ans : } 4, \frac{3}{4}, -\frac{33}{4} \quad (\text{a}) \quad \frac{-3 \pm \sqrt{33}}{4} \quad (\text{b}) \quad \pm \frac{\sqrt{33}}{4} \quad (\text{c}) \quad \left(-\frac{3}{4}, -\frac{33}{4}\right)]$$

- 6 Sketch the graph of $y = x^2 + 2x - 11$ and hence obtain the graph of $\frac{1}{x^2 + 2x - 11}$. State the range of the function in each case.
[Ans : $y \leq -10$; $-\frac{1}{10} \leq y \leq 0$]
- 7 Show that, if λ is positive but not greater than 3, the roots of the equation $(\lambda - 2)x^2 - (8 - 2\lambda)x - (8 - 3\lambda) = 0$ are real. Find the range of λ for which one root is real and positive, the other root is real and negative.

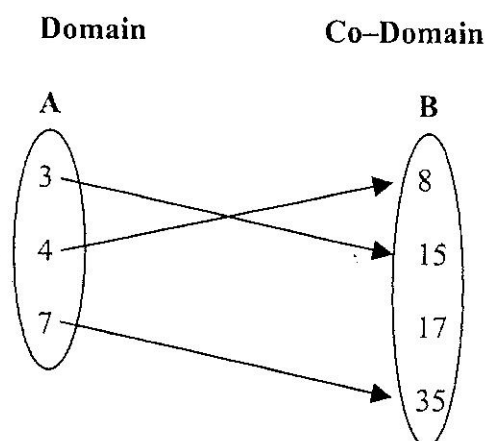
$$[\text{Ans : } 2 < \lambda < \frac{8}{3}]$$

CHAPTER 2

FUNCTIONS

2.1 Function, Domain and range

A function is a rule which maps a single number to another single number.



The elements of the Set A are connected to the elements of the Set B by the relation “is a factor of.” For example, 3 is a factor of 15, 4 is a factor of 8 and 7 is a factor of 35. So, each element in the Set A (Domain) has a unique image in the Set B (Co-domain). This type of relation is called a **function**.

If we look at the function $f : x \rightarrow \ln x$, the choice of x is limited as only positive numbers give a real result. So, we write $f : x \rightarrow \ln x, x > 0$.

Similarly for the function $g : x \rightarrow \sqrt{x}$, the choice of x is also limited as only positive numbers give a real result. So, we write $g : x \rightarrow \sqrt{x}, x \geq 0$.

The sets of values $x > 0$ and $x \geq 0$ are called the **domain** of the functions f and g respectively.

The **range** is the set of output numbers. In fact the range is a **subset** of the co domain. In the above relation the range is $\{8, 15, 35\}$.

We are now going to take a few examples to illustrate in a much better way the difference between range and domain.

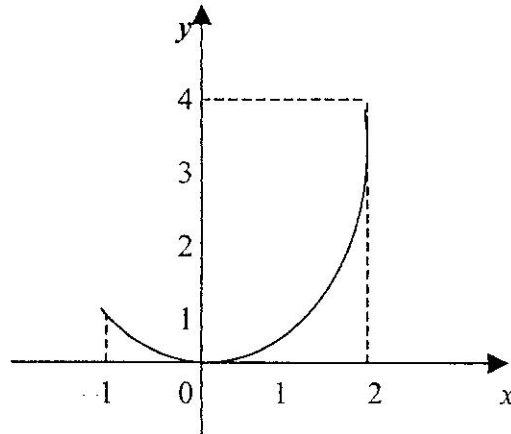
Example 1

Sketch $f: x \rightarrow x^2, -1 \leq x \leq 2$ and state the range of f .

Solution

For the function f , the table values of x and $f(x)$ is constructed as follows:

x	-1	0	1	2
$f(x)$	1	0	1	4



From the graph we can see that the range of f is $0 \leq f \leq 4$.

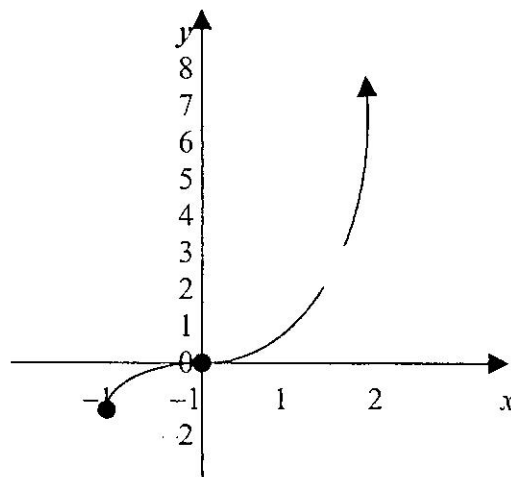
Example 2

Sketch $f: x \rightarrow x^3, x \geq -1$ and state the range of f .

Solution

For the function f , the table values of x and $f(x)$ is constructed as follows:

x	-1	0	1	2	3
$f(x)$	-1	0	1	8	27



The arrow indicates that the graph will increase up to infinity. So, the range of f is given by $f \geq -1$.

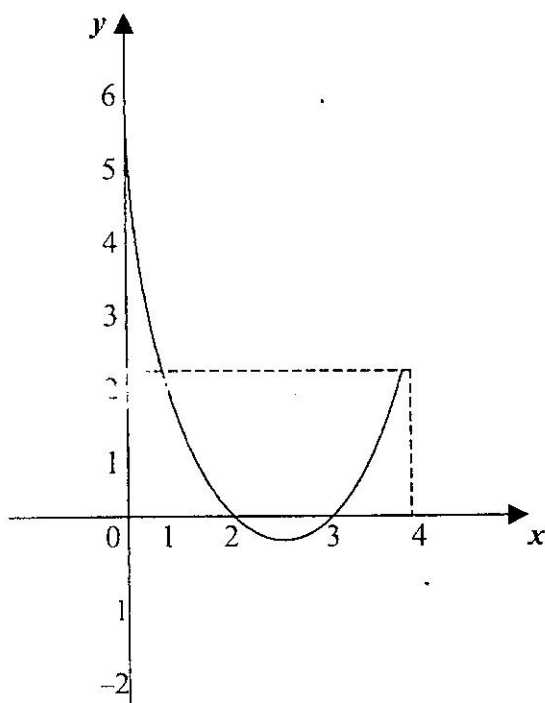
Example 3

Sketch $f: x \mapsto x^2 - 5x + 6$, $0 \leq x \leq 4$ and state the range of f .

Solution

For the function f , the table values of x and $f(x)$ is constructed as follows:

x	0	1	2	3	4
$f(x)$	6	2	0	0	2



To be able to state the range of the function we have to find the vertex of the graph. Completing the square, we have

$$\begin{aligned} x^2 - 5x + 6 &= x^2 - 5x + \frac{25}{4} + 6 - \frac{25}{4} \\ &= \left(x - \frac{5}{2}\right)^2 - \frac{1}{4} \end{aligned}$$

The coordinates of the minimum point is $\left(\frac{5}{2}, -\frac{1}{4}\right)$.

Hence the range of the f is $-\frac{1}{4} \leq f \leq 6$.

Exercise 2.1

Sketch the graph of the following functions and hence state their range.

- 1 $f: x \rightarrow x^2, x \in \mathbb{R}, -1 \leq x \leq 3.$
[Ans : $0 \leq f \leq 9$]
- 2 $g: x \rightarrow x^3, x \in \mathbb{R}, -1 \leq x \leq 2.$
[Ans : $-1 \leq g \leq 8$]
- 3 $f: x \rightarrow \frac{1}{2}x^2, x \in \mathbb{R}, x \geq 0.$
[Ans : $f \geq 0$]
- 4 $g: x \rightarrow \frac{1}{3}x^3, x \in \mathbb{R}, x \geq 2.$
[Ans : $g \geq 8/3$]
- 5 $f: x \rightarrow (x-1)^2, x \in \mathbb{R}, x \geq 1.$
[Ans : $f \geq 0$]
- 6 $f: x \rightarrow x^2 - 5x + 6, x \in \mathbb{R}, x \leq 3.$
[Ans : $f \geq -\frac{1}{4}$]
- 7 $f: x \rightarrow |x-2|, x \in \mathbb{R}, -3 \leq x \leq 7.$
[Ans : $0 \leq f \leq 5$]
- 8 $f: x \rightarrow |2x-3|, x \in \mathbb{R}, -2 \leq x \leq 6.$
[Ans : $0 \leq f \leq 9$]
- 9 $f: x \rightarrow x^2 - 2x, x \in \mathbb{R}, x \leq 1.$
[Ans : $f \geq -1$]
- 10 $f: x \rightarrow x^2 + 6x, x \in \mathbb{R}, x \geq -3.$
[Ans : $f \geq -9$]
- 11 $f: x \rightarrow x^2 + 1, x \in \mathbb{R}, x \geq 0.$
[Ans : $f \geq 1$]
- 12 $f: x \rightarrow x^3 + 2, x \in \mathbb{R}, x \geq 0.$
[Ans : $f \geq 2$]

2.2 Composition of functions

When two functions are combined together, we talk of composite functions.

Example 1

Given $f: x \rightarrow x^2$, $x \in \mathbb{R}$ and $g: x \rightarrow e^{\sqrt{x}}$, $x > 0$, find

(a) $fg(x)$ (b) $gf(x)$.

Solution

(a) $fg(x) = f(e^{\sqrt{x}}) = (e^{\sqrt{x}})^2 = e^{2\sqrt{x}}$, $x > 0$.

(b) $gf(x) = g(x^2) = e^{\sqrt{x^2}} = e^x$, $x \in \mathbb{R}$.

Example 2

Given $f: x \rightarrow 2x + 3$, $x \in \mathbb{R}$ and $g: x \rightarrow \frac{5}{x}$, $x \neq 0$, find

(a) $f^2(x)$

(b) $fgf(x)$.

Solution

(a) $fg(x) = ff(x)$
 $= f(2x + 3)$
 $= 2(2x + 3) + 3$
 $= 4x + 9$, $x \in \mathbb{R}$

(b) $gf(x) = g(2x + 3)$
 $= \frac{5}{2x+3}$, $x \neq -\frac{3}{2}$

(c) $fgf(x) = f\left(\frac{5}{2x+3}\right)$
 $= 2\left(\frac{5}{2x+3}\right) + 3$
 $= \frac{10}{2x+3} + 3$
 $= \frac{10 + 3(2x+3)}{2x+3}$
 $= \frac{6x+19}{2x+3}$, $x \neq -\frac{3}{2}$

Exercise 2.2

In questions 1 – 10, find $fg(x)$ and $gf(x)$ for functions f and g defined on $\mathbb{R}$, $x \geq 0$.

1 $f: x \rightarrow x^2$, $g: x \rightarrow 3x + 1$
[Ans : $fg = (3x + 1)^2$; $gf = 3x^2 + 1$]

2 $f: x \rightarrow x^2 - 2$, $g: x \rightarrow 2x + 1$
[Ans : $fg = 4x^2 + 4x - 1$; $gf = 2x^2 - 3$]

3 $f: x \rightarrow x^2$, $g: x \rightarrow e^{2x}$
[Ans : $fg = e^{4x}$; $gf = e^{2x^2}$]

4 $f: x \rightarrow \ln(x + 2)$, $g: x \rightarrow 3x + 4$
[Ans : $fg = \ln(3x + 6)$; $gf = 3 \ln(x + 2) + 4$]

5 $f: x \rightarrow \ln(x - 1)$, $g: x \rightarrow 2x + 5$
[Ans : $fg = \ln(2x + 4)$; $gf = 2 \ln(x - 1) + 5$]

6 $f: x \rightarrow x^3$, $g: x \rightarrow e^{x+2}$
[Ans : $fg = e^{3x+6}$; $gf = e^{x^3+2}$]

7 $f: x \rightarrow \frac{x}{x+1}$, $g: x \rightarrow 2x + 1$
[Ans : $fg = \frac{2x+1}{2x+2}$; $gf = \frac{3x+1}{x+1}$]

8 $f: x \rightarrow \frac{3x}{x+2}$, $g: x \rightarrow 2x + 5$
[Ans : $fg = \frac{6x+15}{2x+7}$; $gf = \frac{11x+10}{x+2}$]

9 $f: x \rightarrow \frac{3}{x+4}$, $g: x \rightarrow 2x + 4$
[Ans : $fg = \frac{3}{2x+8}$; $gf = \frac{4x+22}{x+4}$]

10 $f: x \rightarrow \frac{x+7}{x+1}$, $g: x \rightarrow 3x + 1$
[Ans : $fg = \frac{3x+8}{3x+2}$; $gf = \frac{4x+22}{x+1}$]

2.3 One-One function

2.3.1 The horizontal line test

A function f is said to be one-one if a horizontal line cuts the graph of f only once. That is, for each value of x there is only one corresponding value of y .

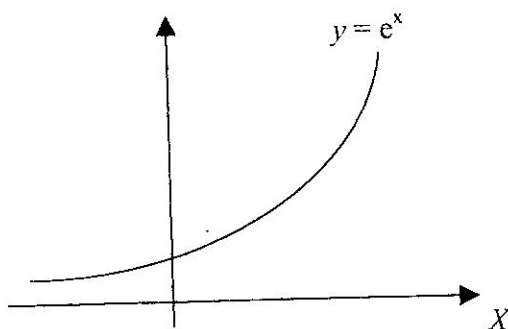


Figure 1

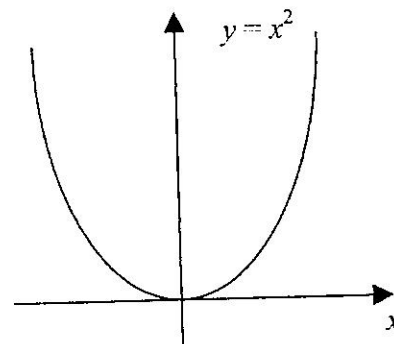


Figure 2

Figure 1 shows the graph of $y = e^x$. Clearly it is a one-one function as a horizontal line cuts the graph of the function only once.

Figure 2 shows the graph of $y = x^2$. Clearly it is not a one-one function as a horizontal line cuts the graph of the function twice.

Yet there exists another way of showing how a function is a one-one function.

Statement : A function f is said to be one-one if there exist two values of x , say x_1 and x_2 , where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
In simple terms if we assume $f(x_1) = f(x_2)$, we have to show that $x_1 = x_2$.

Example

Show that $f(x) \rightarrow \frac{x}{x+1}$, $x \neq -1$, is a one-one function.

Solution

Consider

$$f(x_1) = f(x_2)$$

$$\frac{x_1}{x_1 + 1} = \frac{x_2}{x_2 + 1}$$

$$x_1(x_2 + 1) = x_2(x_1 + 1)$$

$$x_1x_2 + x_1 = x_2x_1 + x_2$$

$$\text{Hence } x_1 = x_2$$

Since $x_1 = x_2$, $f(x)$ is a one-one function.

Exercise 2.3

Show graphically, or otherwise, that the following are one-one functions.

1 $f(x) \rightarrow x^2, x \geq 0.$

2 $f(x) \rightarrow e^x + 2, x \in \mathbb{R}.$

3 $f(x) \rightarrow x^2, x > 0.$

4 $f(x) \rightarrow x^3, x \in \mathbb{R}.$

5 $f(x) \rightarrow \frac{1}{x}, x \in \mathbb{R}, x \neq 0.$

6 $f(x) \rightarrow \frac{4}{x^2}, x \geq 0.$

7 $f(x) \rightarrow \frac{x}{x+1}, x \neq -1.$

8 $f(x) \rightarrow \frac{2x}{x-2}, x \neq 2.$

9 $f(x) \rightarrow \frac{x}{x-5}, x \neq 5.$

10 $f(x) \rightarrow \frac{x}{x+3}, x \neq -3.$

11 $f(x) \rightarrow (x-1)^2, x \geq 1.$

12 $f(x) \rightarrow \ln x, x > 0.$

13 $f(x) \rightarrow \sqrt{x}, x \geq 0.$

14 $f(x) \rightarrow \ln(x-3), x > 3.$

15 $f(x) \rightarrow e^{x+3}, x \in \mathbb{R}.$

2.4 Inverses of one-one functions

When does the inverse of a function exist?

This is a very important question as it is not possible or feasible to find the inverse of any function. A function has an inverse or the inverse of a function exist if and only if it is a one-one function. If a given function is not one-one, the inverse does not exist.

To find the inverse of a function, make use of the following steps:

Step 1 Equate the given function to y (say)

Step 2 Make x the subject of formula

Step 3 Replace y by x .

2.4.1 Inverse of algebraic functions

Example

Given that $f(x) = \frac{2x}{x+3}$, $x \neq -3$, find $f^{-1}(x)$.

Solution

$$\begin{aligned}\text{Let } \frac{2x}{x+3} &= y \\ 2x &= y(x+3) \\ 2x &= xy + 3y \\ 2x - xy &= 3y \\ x(2-y) &= 3y \\ x &= \frac{3y}{2-y}\end{aligned}$$

$$\text{Hence } f^{-1}(x) = \frac{3x}{2-x}, x \neq 2.$$

Exercise 2.4.1

In each of the following questions find $f^{-1}(x)$, stating the domain in each case.

- | | | |
|---|--|--|
| 1 | $\frac{3x+1}{2x}, x \in R, x \neq 0.$ | $[\text{Ans : } f^{-1}(x) = \frac{1}{2x-3}, x \neq \frac{3}{2}]$ |
| 2 | $\frac{4x+1}{2x+5}, x \in R, x \neq -\frac{5}{2}.$ | $[\text{Ans : } f^{-1}(x) = \frac{5x-1}{4-2x}, x \neq 2]$ |
| 3 | $\frac{6x}{x+2}, x \in R, x \neq -2.$ | $[\text{Ans : } f^{-1}(x) = \frac{2x}{6-x}, x \neq 6]$ |
| 4 | $\frac{3x}{x+5}, x \in R, x \neq -5.$ | $[\text{Ans : } f^{-1}(x) = \frac{5x}{3-x}, x \neq 3]$ |
| 5 | $\frac{x+1}{x}, x \in R, x \neq 0.$ | $[\text{Ans : } f^{-1}(x) = \frac{1}{x-1}, x \neq 1]$ |
| 6 | $\frac{x}{2x-1}, x \in R, x \neq \frac{1}{2}.$ | $[\text{Ans : } f^{-1}(x) = \frac{x}{2x-1}, x \neq \frac{1}{2}]$ |

2.4.2 Inverse of exponential functions

To find the inverse of exponential functions we have to apply the same steps given in the previous page.

Example 1

Given that $f(x) = e^{3x}$, $x \in \mathbb{R}$, find $f^{-1}(x)$.

Solution

$$\begin{aligned}\text{Let } e^{3x} &= y \\ 3x &= \ln y \\ x &= \frac{\ln y}{3}\end{aligned}$$

$$\text{Hence } f^{-1}(x) = \frac{\ln x}{3}, x > 0.$$

Example 2

Given that $f(x) = 2e^{x-1}$, $x \in \mathbb{R}$, find $f^{-1}(x)$.

Solution

$$\begin{aligned}\text{Let } 2e^{x-1} &= y \\ e^{x-1} &= \frac{y}{2} \\ x-1 &= \ln\left(\frac{y}{2}\right) \\ x &= \ln\left(\frac{y}{2}\right) + 1\end{aligned}$$

$$\text{Hence } f^{-1}(x) = \ln\left(\frac{x}{2}\right) + 1, x > 0.$$

Exercise 2.4.2

In each of the following questions find $f^{-1}(x)$, stating the domain in each case.

- | | | |
|---|--|--|
| 1 | $f(x) = e^{2x}$, $x \in \mathbb{R}$ | [Ans : $\frac{\ln x}{2}$, $x > 0$] |
| 2 | $f(x) = 3e^x$, $x \in \mathbb{R}$ | [Ans : $\frac{1}{3} \ln\left(\frac{x}{3}\right)$, $x > 0$] |
| 3 | $f(x) = e^{x+2}$, $x \in \mathbb{R}$ | [Ans : $\ln x - 2$, $x > 0$] |
| 4 | $f(x) = 2e^{x-3}$, $x \in \mathbb{R}$ | [Ans : $\ln\left(\frac{x}{2}\right) + 3$, $x > 0$.] |
| 5 | $f(x) = \frac{1}{5}e^{x+1}$, $x \in \mathbb{R}$ | [Ans : $\ln(5x) + 1$, $x > 0$.] |
| 6 | $f(x) = 3e^{x+4} - 2$, $x \in \mathbb{R}$ | [Ans : $\ln\left(\frac{x+2}{3}\right) + 4$, $x > -2$.] |

2.4.3 Inverse of logarithmic functions

To find the inverse of logarithmic functions we have to apply the same steps given in the previous pages.

Example 1

Given that $f(x) = \ln(x - 2)$, $x > 2$, find $f^{-1}(x)$.

Solution

$$\text{Let } \ln(x - 2) = y$$

$$x - 2 = e^y$$

$$x = e^y + 2$$

$$\text{Hence } f^{-1}(x) = e^x + 2, x \in \mathbb{R}.$$

Example 2

Given that $f(x) = 2 \ln x - 1$, $x > 0$, find $f^{-1}(x)$.

Solution

$$\text{Let } 2 \ln x - 1 = y$$

$$2 \ln x = y + 1$$

$$\ln x = \frac{y+1}{2}$$

$$x = e^{\frac{y+1}{2}}$$

$$\text{Hence } f^{-1}(x) = e^{\frac{x+1}{2}}, x \in \mathbb{R}.$$

Exercise 2.4.3

In each of the following questions find $f^{-1}(x)$, stating the domain in each case.

$$1 \quad f(x) = 3 \ln x, x > 0 \quad [\text{Ans : } e^{\frac{x}{3}}, x \in \mathbb{R}]$$

$$2 \quad f(x) = \ln(x - 4), x > 4 \quad [\text{Ans : } e^x + 4, x \in \mathbb{R}]$$

$$3 \quad f(x) = 5 \ln(x - 2), x > 2 \quad [\text{Ans : } e^{\frac{x}{5}} + 2, x \in \mathbb{R}]$$

$$4 \quad f(x) = \ln(x + 1) + 3, x > -1 \quad [\text{Ans : } e^{x-3} + 1, x \in \mathbb{R}]$$

$$5 \quad f(x) = \ln(x + 1) - 3, x > -1 \quad [\text{Ans : } e^{x+3} - 1, x \in \mathbb{R}]$$

$$6 \quad f(x) = \ln(2x - 3), x > \frac{3}{2} \quad [\text{Ans : } e^{\frac{x+3}{2}}, x \in \mathbb{R}]$$

2.4.4 Inverse of quadratic functions

To find the inverse of a quadratic function, we have to apply the same steps mentioned in the previous pages. In addition, we have to complete the square so as to make x the subject of formula.

Example 1

Given that $f(x) = x^2 - 2x$, $x \geq 1$, find $f^{-1}(x)$.

Solution

$$\begin{aligned}\text{Let } x^2 - 2x &= y \\ (x - 1)^2 - 1 &= y \\ (x - 1)^2 &= y + 1 \\ x - 1 &= \sqrt{y + 1} \\ x &= \sqrt{y + 1} + 1,\end{aligned}$$

$$\text{Hence } f^{-1}(x) = \sqrt{x + 1} + 1, x \geq -1.$$

Example 2

Given that $f(x) = x^2 + 6x$, $x \geq -3$, find $f^{-1}(x)$.

Solution

$$\begin{aligned}\text{Let } x^2 + 6x &= y \\ (x + 3)^2 - 9 &= y \\ (x + 3)^2 &= y + 9 \\ x + 3 &= \sqrt{y + 9} \\ x &= \sqrt{y + 9} - 3,\end{aligned}$$

$$\text{Hence } f^{-1}(x) = \sqrt{x + 9} - 3, x \geq -9.$$

Exercise 2.4.4

In each of the following questions find $f^{-1}(x)$, stating the domain in each case.

- 1 $f(x) = x^2 - 4x$, $x \geq 2$ [Ans : $2 + \sqrt{x + 4}$, $x \geq -4$]
- 2 $f(x) = x^2 + 8x$, $x \geq -4$ [Ans : $-4 + \sqrt{x + 16}$, $x \geq -16$]
- 3 $f(x) = x^2 - 5x + 6$, $x \geq 3$ [Ans : $\sqrt{x + \frac{1}{4}} + \frac{5}{2}$, $x \geq -\frac{1}{4}$]
- 4 $f(x) = 2x^2 - 7x + 6$, $x \leq \frac{1}{2}$ [Ans : $\sqrt{\frac{1}{2}\left(x + \frac{1}{8}\right)} + \frac{7}{4}$, $x \geq -\frac{1}{8}$]
- 5 $f(x) = x^2 - 2x + 1$, $x \leq 1$ [Ans : $\sqrt{x} + 1$, $x \geq 0$]

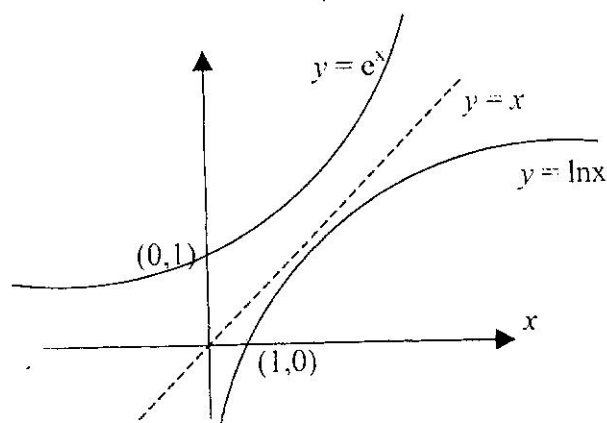
2.5 Relationship between a one-one function and its inverse

By now the reader must be familiar with one-one functions. We shall attempt to show graphically the relationship between a one-one function and its inverse.

Consider for example, a function $f: x \rightarrow e^x$, $x \in \mathbb{R}$.

Now $f^{-1}(x) = \ln x$, $x > 0$.

If we sketch the graph of f and f^{-1} on a single diagram, we obtain



We can see from the diagram that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$ (shown dotted in the diagram).

So, in general the graph of f^{-1} can be obtained from the graph of f by a reflection in the line $y = x$. Under such a transformation all points (x, y) are mapped onto (y, x) . In other words we have to interchange the coordinates of f to obtain the coordinates of f^{-1} .

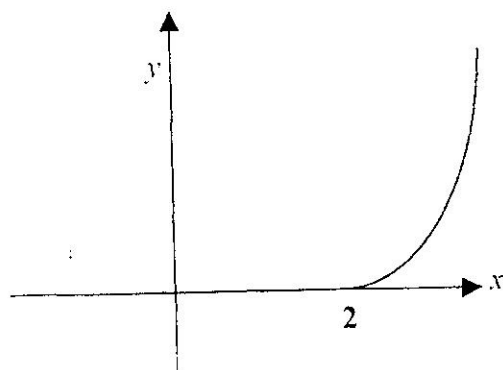
For example, $(1, 0)$ is mapped onto $(0, 1)$

$$(2, 3) \rightarrow (3, 2)$$

$$(3, 3) \rightarrow (3, 3)$$

Example 1

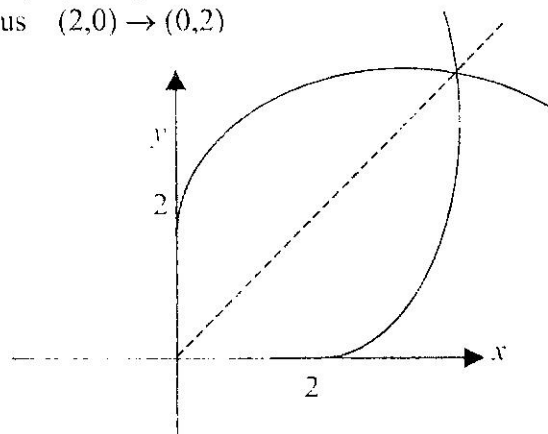
The diagram shows the graph of $y = g(x)$. Sketch the graph of $g^{-1}(x)$.



Solution

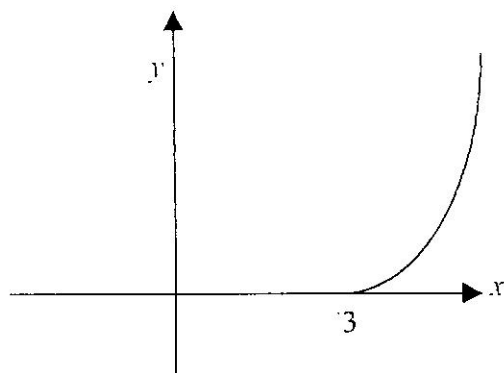
The graph of g^{-1} can be obtained by a reflection in the line $y = x$.

Thus $(2,0) \rightarrow (0,2)$

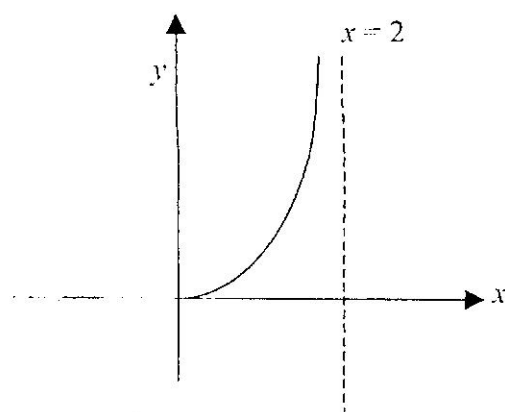
**Exercise 2.5**

Sketch the graph of the inverse functions in each of the following cases.

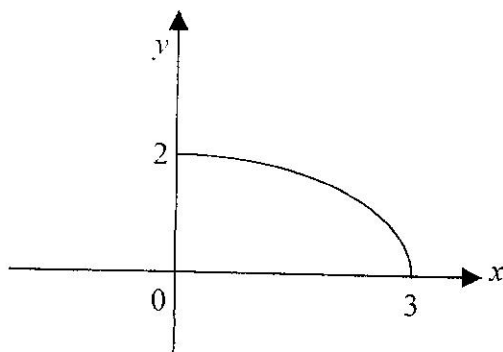
- 1 The diagram shows the graph of $y = f(x)$. Sketch the graph of $f^{-1}(x)$.



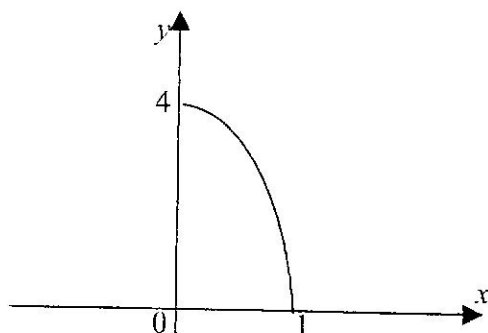
- 2 The diagram shows the graph of $y = f(x)$. Sketch the graph of $f^{-1}(x)$.



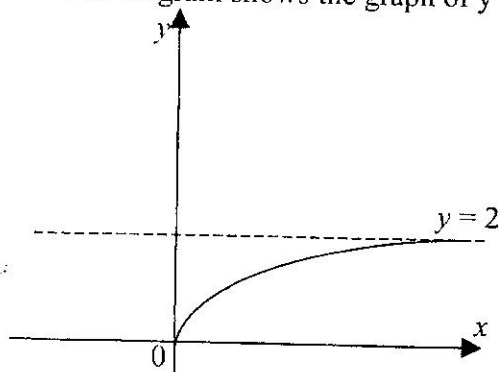
- 3 The diagram shows the graph of $y = g(x)$. Sketch the graph of $g^{-1}(x)$.



- 4 The diagram shows the graph of $y = f(x)$. Sketch the graph of $f^{-1}(x)$.



- 5 The diagram shows the graph of $y = f(x)$. Sketch the graph of $f^{-1}(x)$.



For questions 6 – 10, sketch the graph of the given functions and hence sketch the graph of inverse function.

- 6 $f(x) \rightarrow (x - 2)^2, x \geq 2.$
 7 $f(x) \rightarrow x^3, x \in \mathbb{R}, x \geq 0.$
 8 $f(x) \rightarrow \frac{1}{2}x^2, x \in \mathbb{R}, x \geq 0.$
 9 $f(x) \rightarrow \ln(x - 1), x \in \mathbb{R}, x \geq 1.$
 10 $f(x) \rightarrow e^{x-1}, x \in \mathbb{R}, x \geq 0.$

FUNCTIONS – PAST EXAMINATION QUESTIONS

- 1 Given that $f(x) = 3x^2 - 4$, $x > 0$, and $g(x) = x + 4$, $x \in \mathbb{R}$, find
 (i) $f^{-1}(x)$, $x > -4$, (ii) $fg(x)$, $x > -4$.

[Ans: (i) $\sqrt{\frac{4+x}{3}}$ (ii) $3x^2 + 24x + 44$]

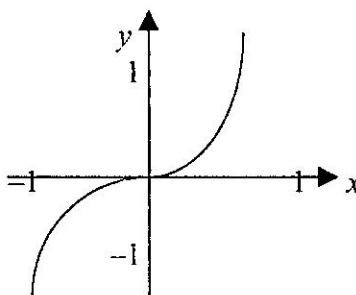
- 2 The functions f and g are defined by $f: x \rightarrow e^{2x}$, $x \in \mathbb{R}$, $g: x \rightarrow x^2$, $x \in \mathbb{R}$.
 Find $f^{-1}(x)$ and $gf(x)$. Sketch the graph of $y = f^{-1}(x-2)$.

[Ans: $\frac{1}{2} \ln x$, e^{4x}]

- 3 Functions f and g are defined for $x \in \mathbb{R}$ by $f: x \rightarrow e^{-2x}$, $g: x \rightarrow 1-x$. Find
 (a) $fg(x)$, (b) $f^{-1}(x)$.

[Ans: (a) $e^{2(x-1)}$ (b) $-\frac{1}{2} \ln x$]

- 4 (a) The function f is defined, for real values of x , by $f(x) = 1 + e^{-x}$.
 Express $f^{-1}(x)$ in terms of x , where $x > 1$. [Ans: $-\ln(x-1)$]
 (b) The diagram shows the graph of $y = g(x)$. Sketch the graph of $y = g^{-1}(x)$.



- 5(a) Functions g and h are defined by $g: x \rightarrow \ln x$, $x \in \mathbb{R}$, $x > 0$, $h: x \rightarrow 1+x$, $x \in \mathbb{R}$. The function f is defined by $f: x \rightarrow gh(x)$, $x \in \mathbb{R}$, $x > -1$.

- (i) Sketch the graph of $y = f(x)$
 (ii) Write down expressions for $g^{-1}(x)$ and $h^{-1}(x)$
 (iii) Write down an expression for $g^{-1}h^{-1}(x)$
 (iv) Sketch the graph of $y = g^{-1}h^{-1}(x)$.

[Ans: (ii) e^x , $x-1$ (iii) e^{x-1}]

- (b) The function q is defined by $q: x \rightarrow x^2 - 4x$, $x \in \mathbb{R}$, $|x| \leq 1$. Show, by means of a graphical argument or otherwise, that q is one-one, and find an expression for $q^{-1}(x)$.

[Ans: $2 - \sqrt{4+x}$]

- 6 The functions g and h are defined by $g: x \rightarrow 3x^2 + 2$, $x \in \mathbb{R}$, $h: x \rightarrow \sqrt{\frac{x-2}{3}}$, $x \geq 2$.
 Find $hg(x)$ for (a) $x \geq 0$, (b) $x < 0$. [Ans: (a) x (b) $-x$]

- 7 Functions f and g , each with domain $\mathbf{R}$, are defined by $f: x \rightarrow x+1$, $g: x \rightarrow |x|$.
- (a) Write down an expression for $f^{-1}(x)$, and state, giving a reason, whether g has an inverse.
- (b) Sketch the graph of $y = gf(x)$.
- (c) Find the solution set of the equation $gf(x) = fg(x)$.
- [Ans : (a) $x - 1$, g is not 1-1 $\therefore$ it has no inverse (c) $x \geq 0$]
- 8 The functions f and g are defined for $x \in \mathbf{R}$ by $f: x \rightarrow x^3$, $g: x \rightarrow 2-3x$. Find
- (a) $fg(x)$, (b) $(fg)^{-1}(x)$.
- [Ans : (a) $(2-3x)^3$ (b) $\frac{1}{3}(2-\sqrt[3]{x})$]
- 9 Given that $f(x) = 2x^{\frac{3}{5}} + 7$, $x \in \mathbf{R}$, find $f^{-1}(x)$. [Ans : $\left(\frac{x-7}{2}\right)^{\frac{5}{3}}$]
- 10 The functions f and g are given by $f(x) = 1 + \frac{1}{x}$, $g(x) = 1 - \frac{1}{x^2}$, and each function has domain the set of all real numbers excluding 0.
- (i) Find the range of f and the range of g .
- (ii) Find $f^{-1}(x)$.
- (iii) Explain why g does not have an inverse.
- [Ans : (i) $y \in \mathbf{R}$, $y \neq 1$; $y \in \mathbf{R}$, $y > 1$ (ii) $\frac{1}{1-x}$ (iii) g is not 1-1]
- 11 The function h is defined by $h: x \rightarrow \frac{1}{1+\sqrt{x}}$, $x \geq 0$. Find an expression for $h^{-1}(x)$ and state the domain and range of h^{-1} .
- [Ans : $\left(\frac{1}{x} - 1\right)^2$; $0 < x \leq 1$; $y \geq 0$]

CHAPTER 3

COORDINATE GEOMETRY

In this section, we expect the reader to be familiar with terms such as gradient, midpoint, distance formula, equations of line either in the form $y = mx + c$ or $(y - y_1) = m(x - x_1)$. We also expect the reader to be familiar with parallel and perpendicular lines.

Example 1

Find the equation of the line passing through A(2,0) and through the midpoint of B(1,3) and C(-3,5).

Solution

$$\text{Midpoint of BC, } M = \left(\frac{1+(-3)}{2}, \frac{3+5}{2} \right) = (-1, 4)$$

$$\text{Gradient of AM} = \frac{4-0}{-1-2} = -\frac{4}{3}$$

Equation of required line is given by

$$\frac{y-0}{x-2} = -\frac{4}{3}$$

$$3y = -4(x-2)$$

$$3y = -4x + 8 \text{ (Ans)}$$

Example 2

Show that $2x + 3y = 1$ and $2y = 3x - 5$ are perpendicular.

Solution

Two lines are perpendicular if the product of their gradients $= -1$.

$$\text{Now } 2x + 3y = 1$$

$$3y = -2x + 1$$

$$y = -\frac{2}{3}x + \frac{1}{3}$$

$$\therefore m_1 = -\frac{2}{3}$$

$$\text{Now } 2y = 3x - 5$$

$$y = \frac{3}{2}x - \frac{5}{2}$$

$$\therefore m_2 = \frac{3}{2}$$

$$m_1 m_2 = -\frac{2}{3} \times \frac{3}{2} = -1 \text{ (shown)}$$

Example 3

Find the length of the line joining the origin to the point of intersection of the line $y = x$ and the curve $y = \frac{1}{x^2}$.

Solution

To find the point of intersection, we solve simultaneously

$$y = x \quad \dots\dots(1)$$

$$y = \frac{1}{x^2} \quad \dots\dots(2)$$

$$\Rightarrow x = \frac{1}{x^2}$$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow x = 1$$

When $x = 1$, $y = 1$

So, the point of intersection is $(1, 1)$.

We have to find the distance between the points $(0, 0)$ and $(1, 1)$.

Using $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we have

$$d = \sqrt{(1-0)^2 + (1-0)^2}$$

$$d = \sqrt{2} \text{ (Ans)}$$

Example 4

Find the equation of the line passing through $(-1, 2)$ and parallel to $2x - 3y = 1$.

Solution

$$2x - 3y = 1$$

$$-3y = -2x + 1$$

$$y = \frac{2}{3}x - \frac{1}{3}$$

So the gradient of the required line is $\frac{2}{3}$.

Equation of required line is given by

$$\frac{y-2}{x-(-1)} = \frac{2}{3}$$

$$3(y-2) = 2(x+1)$$

$$3y - 6 = 2x + 2$$

$$3y = 2x + 8 \text{ (Ans)}$$

COORDINATE GEOMETRY – PAST EXAMINATION QUESTIONS

- 1 The points P, Q, R have coordinates (3, -1), (2, 5), (8, 3) respectively. Find the equation of the straight line joining P to the mid-point of QR.
[Ans : $5x - 2y - 17 = 0$]

- 2 The vertices of the triangle ABC are A(-3, 1), B(10, -8) and C(1, 4). Find an equation of the line passing through A and B, giving your answer in the form $px + qy + r = 0$, where p, q and r are integers. Show by calculation that CA and CB are perpendicular.
[Ans : $9x + 13y + 14 = 0$]

- 3 The curve $y = \frac{1}{4}x^2 - 1$ and the line $2y = x + 10$ intersect at the points A and B, and O is the origin. Calculate the coordinates of A and B, and hence show that OA and OB are perpendicular.
[Ans : (-4, 3) ; (6, 8)]

- 4 The straight line L passes through the point (3, -2) and is perpendicular to the line $x + 2y + 1 = 0$. Find the equation of L, giving your answer in the form $y = mx + c$.
[Ans : $y = 2x - 8$]

- 5 The straight line $y = 20 - 3x$ meets the circle $x^2 + y^2 - 2x - 14y = 0$ at the points A and B. Calculate the exact length of the chord AB.
[Ans : $4\sqrt{10}$]

- 6 The points A, B, C have coordinates (3, -5), (4, -6), (11, 1) respectively.
(a) Show that AB is perpendicular to BC.
(b) Find the length of AC.
(c) Find the coordinates of the midpoint of AC.
[Ans : (b) 10 (c) (7, -2)]

- 7 The points A and B have coordinates (3, -1) and (7, 1) respectively. Find the equation of the perpendicular bisector of AB, giving your answer in the form $ax + by = c$.
[Ans : $2x + y = 10$]

- 8 The straight line l has equation $2y - x + 7 = 0$. The straight line l^1 passes through the point P (-1, 6) and is perpendicular to l .
(i) Find the equation of l^1 , giving your answer in the form $ax + by + c = 0$.
(ii) Find the coordinates of the point of intersection of l and l^1 .
(iii) Show that the perpendicular distance from P to l is $4\sqrt{5}$.
(iv) It is given that the points Q(-7, -7) and R(9, 1) lie on l . Find the exact area of the triangle PQR.
[Ans : (i) $2x + y - 4 = 0$ (ii) (3, -2) (iv) 80]

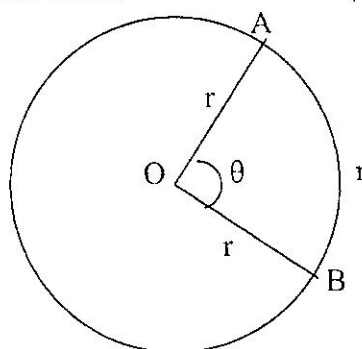
CHAPTER 4

CIRCULAR MEASURE

In angular measurement, the most common system is to measure angles in terms of degrees. A measurement of 360° has been traditionally chosen to devote one complete revolution. However, in some branches of mathematics, angular measurement cannot be conveniently done in degrees. Thus a new unit, the **radian**, is introduced to describe the magnitude of an angle. This system of angular measurements, known as circular measure, is applied especially to those branches of Mathematics which involve the differentiation and integration of trigonometrical ratios.

The Radian (rad)

Consider a circle with centre O and radius r units and an arc AB. Suppose the length of the arc is equal to the radius. The angle θ , subtended by the arc AB at the centre O, is then defined as one **radian**. A radian is the angle subtended at the centre of a circle by an arc equal in length of the radius of the circle.



When $AB = 2\pi r$, it means that the arm OA has made one complete revolution, that is, OA has moved through an angle of 360° .

$$\text{Hence } 2\pi \text{ rad} = 360^\circ$$

$$\therefore \pi \text{ rad} = 180^\circ$$

$$\text{and } 1 \text{ rad} = \frac{180}{\pi} \approx 57.3^\circ$$

$$\text{Also } 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad.}$$

Example 1

Convert the following angles from radian to degrees.

(a) $\frac{\pi}{2}$ (b) 2π (c) $\frac{3\pi}{5}$

Solution

To convert radian to degrees, we have to replace π by 180° .

(a) $\frac{\pi}{2} = \frac{180}{2} = 90^\circ$

(b) $2\pi = 2(180^\circ) = 360^\circ$

(c) $\frac{3\pi}{5} = \frac{3(180^\circ)}{5} = 108^\circ$.

Example 2

Convert the following angles from degrees to radians.

(a) 60° (b) 360°

Solution

Since $180^\circ = \pi$ radian $\Rightarrow 1^\circ = \frac{\pi}{180}$

Thus to convert degrees to radians, we have to multiply by $\frac{\pi}{180}$.

(a) $60^\circ = 60^\circ \times \frac{\pi}{180} = \frac{\pi}{3}$

(b) $360^\circ = 360^\circ \times \frac{\pi}{180} = 2\pi$

Exercise 4.1

1 Convert the following angles from degrees to radians.

(a) 45° (b) 120° (c) 720° (d) 150° (e) 240°

(f) 48° (g) 135° (h) 72° (i) 100°

[Ans : (a) $\frac{\pi}{4}$; (b) $\frac{2\pi}{3}$; (c) 4π ; (d) $\frac{5\pi}{6}$; (e) $\frac{4\pi}{3}$; (f) $\frac{4\pi}{15}$; (g) $\frac{3\pi}{4}$; (h) $\frac{2\pi}{5}$; (i) $\frac{5\pi}{9}$]

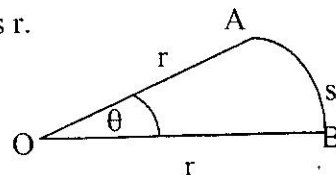
2 Convert the following angles from radian to degrees.

(a) $\frac{5\pi}{4}$ (b) $\frac{5\pi}{3}$ (c) 3π (d) $\frac{\pi}{6}$ (e) $\frac{7\pi}{3}$ (f) $\frac{8\pi}{15}$

[Ans : (a) 225° ; (b) 150° ; (c) 540° ; (d) 30° ; (e) 420° ; (f) 96°]

4.2.1 Length of arc and area of sector

Consider a sector of a circle having centre O and radius r.
Suppose the length of arc AB is s.



Then

$$\frac{\text{Arc length AB}}{\text{Circumference of circle}} = \frac{\text{angle subtended by AB at O}}{\text{angle subtended by the circle at O}}$$

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

$$s = r\theta$$

Hence the formula used for the length of arc of a circle is $r\theta$, where θ is measured in radians.

Similarly

$$\frac{\text{Area of sector OAB}}{\text{Area of circle}} = \frac{\text{angle subtended by AB at O}}{\text{angle subtended by the circle at O}}$$

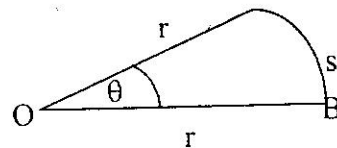
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \frac{1}{2} r^2 \theta$$

Thus the area of a sector is given by the formula $A = \frac{1}{2} r^2 \theta$, where θ is measured in radians.

4.2.2 Area of segment (A)

$$\begin{aligned} \text{Area of the shaded region} &= \text{Area of sector OAB} - \text{Area of } \triangle OAB \\ &= \frac{1}{2} r^2 \theta - \frac{1}{2} r \cdot r \cdot \sin \theta \\ &= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta \\ A &= \frac{1}{2} r^2 (\theta - \sin \theta) \end{aligned}$$



Thus to find the area of a segment
we use the formula $\frac{1}{2} r^2 (\theta - \sin \theta)$,
where θ is measured in radians.

Example 1

OAB is a sector of a circle centre O and having radius 10 cm . If angle AOB = 1.2 radians
Calculate

- (a) the length of arc AB,
(b) area of sector AOB.

Solution

(a) the length of arc AB $\approx r\theta$

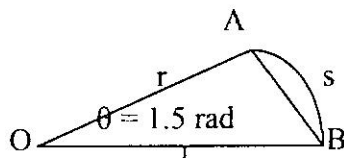
$$\approx 10 (1.2) \approx 12 \text{ cm}$$

(b) area of the sector AOB $\approx \frac{1}{2} r^2 \theta$

$$\approx \frac{1}{2} (10)^2 (1.2) \approx 60 \text{ cm}^2$$

Example 2

Find the area of the shaded region.

Solution

$$\begin{aligned} \text{Area of shaded region} &= \frac{1}{2} r^2 (\theta - \sin \theta) \\ &= \frac{1}{2} (10)^2 (1.5 - \sin 1.5) \\ &\approx 25.1 \text{ cm}^2. \end{aligned}$$

Example 3

The diagram shows a circle of radius 6 cm, divided into 2 sectors. The length of the arc APB is 24 cm.

- Find (a) the angle in radians,
(b) the area of the minor sector.(C)

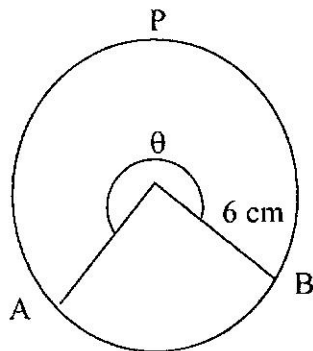
Solution

(a) since length of arc $r\theta$

$$\therefore r\theta = 24$$

$$6\theta = 24$$

$$\theta = 4 \text{ rad.}$$

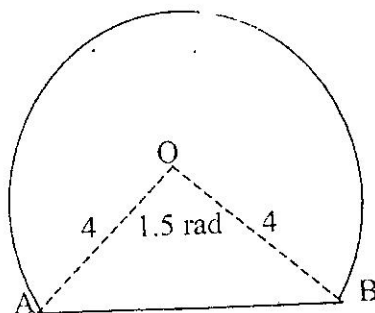


(b) minor angle AOB $= 2\pi - \theta$

$$\begin{aligned} \therefore \text{Area of minor sector} &= \frac{1}{2} r^2 (2\pi - \theta) \\ &= \frac{1}{2} (6)^2 (2\pi - 4) \\ &= 41.1 \text{ cm}^2. \end{aligned}$$

$\therefore$ The area of the minor sector is 41.1 cm^2 correct to 3 s.f.

Example 4



The diagram shows the cross-section of a tunnel. The cross-section has the shape of a major segment of a circle, and the point O is the centre of the circle. The radius of the circle is 4m, and the size of the angle AOB is 1.5 radians. Calculate the perimeter of the cross section. (C)

Solution

Perimeter = AB + length of major arc AB.

To find AB, we use cosine rule

$$AB^2 = 4^2 + 4^2 - 2 \times 4 \times 4 \times \cos 1.5$$

$$AB^2 = 29.7$$

$$AB = 5.45$$

$$\text{Reflex angle AOB} = 2\pi - 1.5$$

$$\text{Length of major arc AB} = r\theta = 4 \times (2\pi - 1.5) = 19.13$$

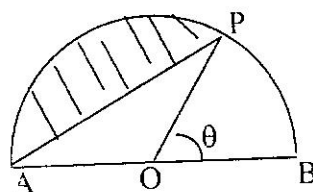
$$\text{Hence perimeter of the tunnel} = 5.45 + 19.13 = 24.6 \text{ (3 s.f.)}$$

Example 5

The diagram shows a semicircle APB on AB as diameter. The midpoint of AB is O. The point P on the semicircle is such that the area of the sector POB is equal to twice the area of the shaded segment.

Given that angle POB is θ radians, show that

$$3\theta = 2(\pi - \sin \theta) \text{ (C)}$$



Solution

Let r be the radius of the semicircle

$$\text{Area of sector POB} = \frac{1}{2} r^2 \theta$$

$$\text{Area AOP} = \pi - \theta$$

$$\text{Area of shaded region} = \frac{1}{2} r^2 [(\theta - \pi) - \sin(\pi - \theta)]$$

$$= \frac{1}{2} r^2 [\pi - \theta - \sin \theta] \quad (\because \sin(\pi - \theta) = \sin \theta)$$

Given area of sector POB = 2 Area of shaded region

$$\frac{1}{2} r^2 \theta = 2 \times \frac{1}{2} r^2 [\pi - \theta - \sin \theta]$$

$$\theta = 2[\pi - \theta - \sin \theta]$$

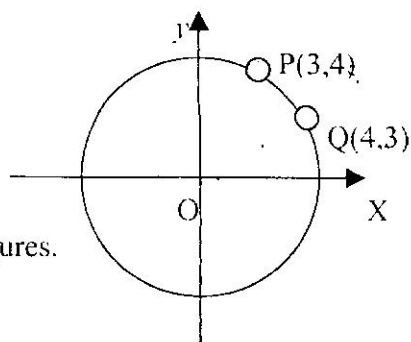
$$\theta = 2\pi - 2\theta - 2\sin \theta$$

$$3\theta = 2(\pi - \sin \theta) \text{ (shown)}$$

Exercise 4.2

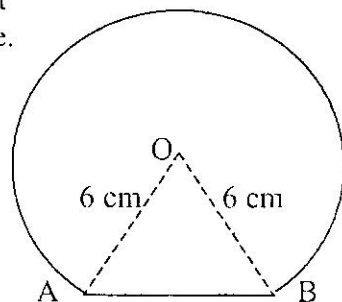
- 1 The perimeter of a sector of a circle, centre O is 16 cm. Given that the radius of the circle is 5 cm, find, in radians the angle θ subtended at the centre.
[Ans : $6/5$]
- 2 The length of arc of a sector of a circle is 15 cm and the area of the sector of the circle is 75 cm^2 . Find, in radians, the angle subtended at the centre.
[Ans : 1.5]
- 3 A piece of wire 20 cm long is bent to form the shape of a sector. If the arc has a length of 8 cm, calculate the angle of the sector and the area enclosed by this sector.
[Ans : $4/3$; 24]
- 4 A chord PQ of a circle, radius 5 cm, subtends an angle of 2 radians at the centre of the circle. Calculate
(a) the length of the major arc PQ,
(b) the area enclosed by the chord PQ and the minor arc PQ.
[Ans : (a) 21.4 (b) 13.6]
- 5 A piece of wire is in the form of an arc of a circle, centre O and radius 10 cm. Given that the area of the sector AOB is 20 cm^2 , find the length of the wire.
[Ans : 24]

- 6 The diagram shows the circle $x^2 + y^2 = 25$.
Find
(a) angle POX
(b) angle QOX
(c) the length of the minor arc between the points P(3,4) and Q(4,3),
giving your answer correct to 3 significant figures.
[Ans: 1.42 cm]



- 7 The diagram shows the cross-section of a tunnel.
The cross-section has the shape of a major segment of a circle, and the point O is the centre of the circle.
The radius of the circle is 6 cm, and the size of the angle AOB is 2 radians.
Calculate the perimeter of the cross-section.

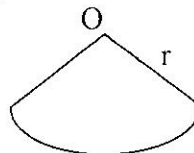
[Ans : 35.8]



CIRCULAR MEASURE – PAST EXAMINATION QUESTIONS

- 1 A sector OAB of a circle, of radius a and centre O, has $\angle AOB = \theta$ radians. Given that the area of the sector OAB is twice the square of the length of the arc AB, find θ .
[Ans : $\frac{1}{4}$]

- 2 The diagram shows a sector of a circle, with centre O and radius r . The length of the arc is equal to half the perimeter of the sector. Find the area of the sector in terms of r .
[Ans : r^2]

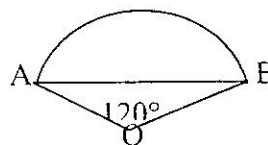


- 3 The diagram shows OAB of a circle, with centre O, in which angle $AOB = 120^\circ$.

(a) Show that $\frac{\text{Area of sector OAB}}{\text{Area of triangle OAB}} = \frac{4\pi}{3\sqrt{3}}$.

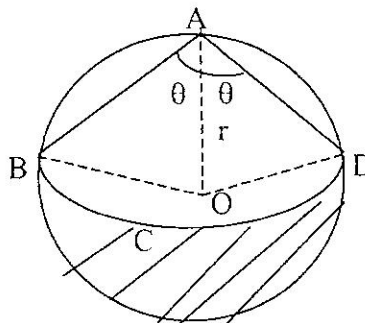
(b) Find the exact value of $\frac{\text{Length of arc AB}}{\text{Length of chord AB}}$.

[Ans : (b) $\frac{2\pi}{3\sqrt{3}}$]



- 4 A and B are two points on a circle with centre O and radius 5 cm. Given that the length of the minor arc AB is 4 cm, calculate the area of the sector AOB.
[Ans : 10 cm^2]

- 5 The diagram shows a circle with centre O and radius r . The three points A, B, D on the circle are such that $\angle BAO = \angle DAO = \theta$ radian, and BCD is an arc of the circle with centre A and radius AB. The point E lies on the arc AD. Find, in terms of r and θ , expressions for



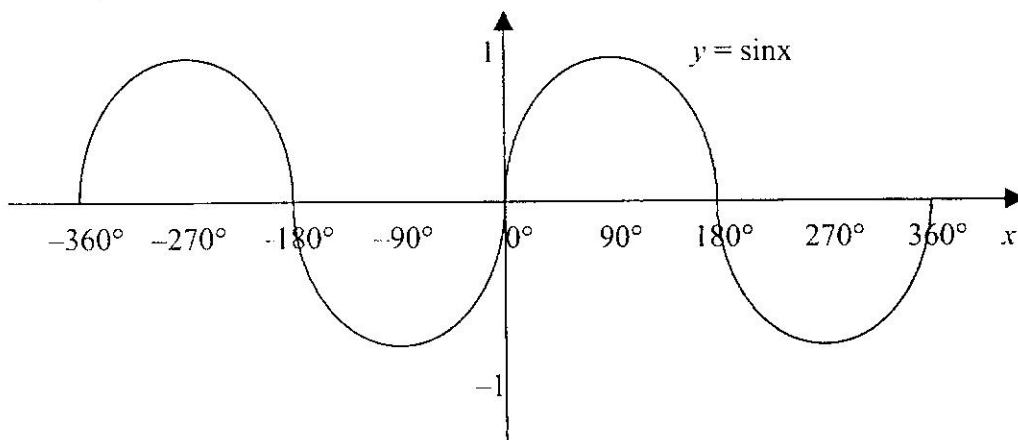
- the length of the straight line AD.
- the area of the sector of the circle (centre A) bounded by the lines AB and AD and the arc BCD.
- the area of the triangle OAD.
- the area of the segment ADE.

[Ans : (a) $2r \cos \theta$ (b) $4r^2 \cos^2 \theta$ (c) $\frac{1}{2} r^2 \sin 2\theta$ (d) $r^2 \cos \theta - \frac{1}{2} r^2 \sin 2\theta$]

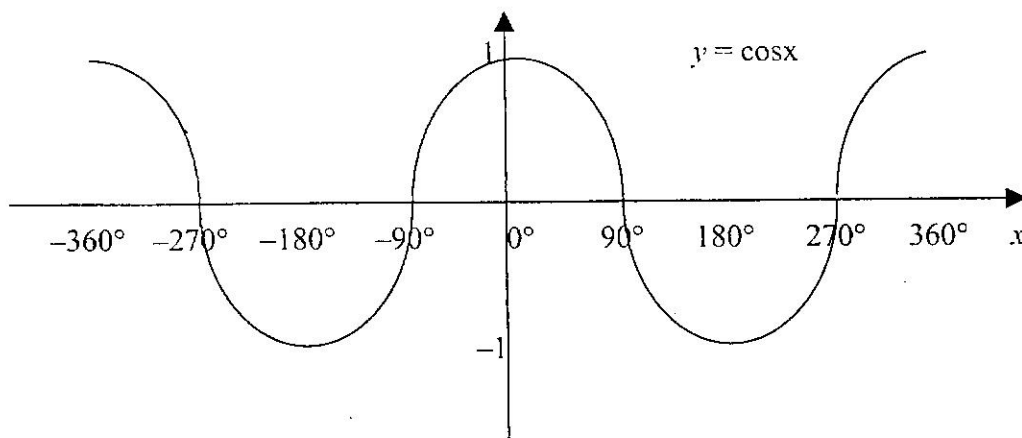
CHAPTER 5

TRIGONOMETRY 1

5.1.1 Graph of sine

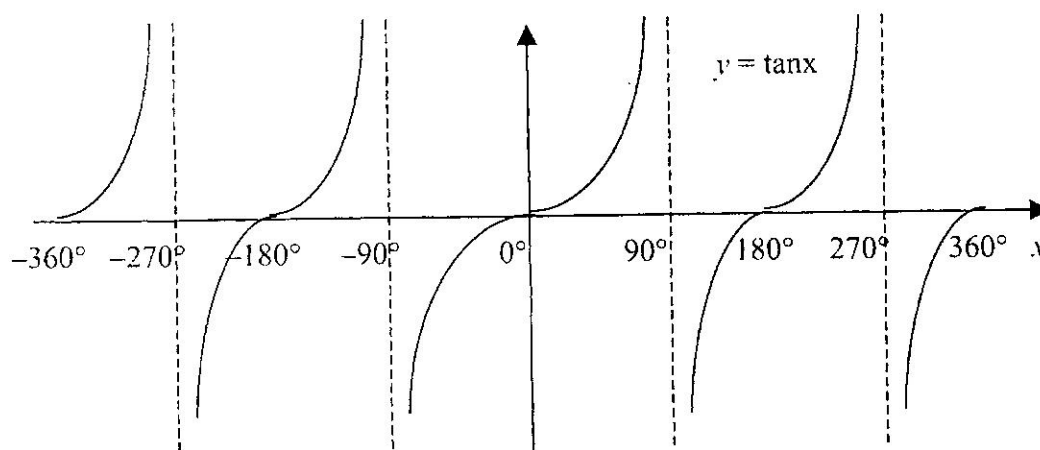


5.1.2 Graph of cosine



The graphs of $y = \sin x$ and $y = \cos x$ show that the curves are similar in shape. There is a complete cycle for $0^\circ \leq x \leq 360^\circ$ in both the sine and cosine graphs. The only difference is that the cosine curve is lagging 90° behind the sine curve. This 90° difference is called the **phase difference**. For angles greater than 360° or less than 0° the curves repeat themselves in **periodic cycles**. Such functions are called **periodic functions**. The sine and cosine curves are useful especially in the study of waves and electricity.

5.1.3 Graph of tangent



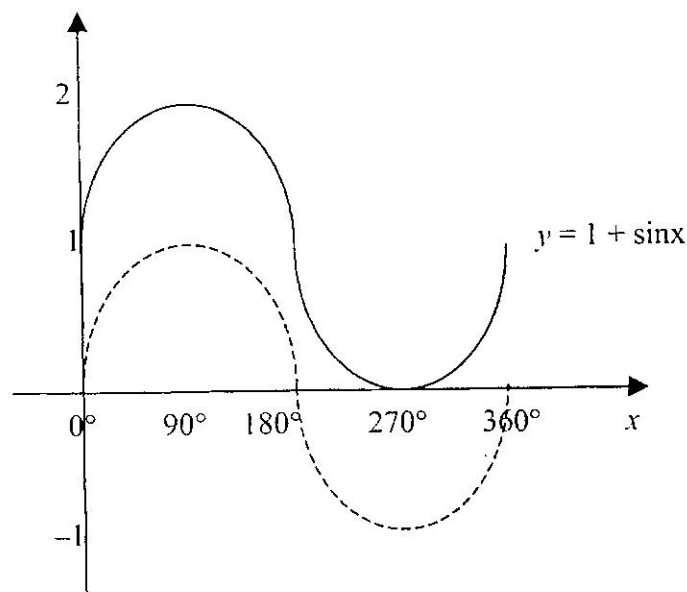
The tangent curve approaches very close to the vertical lines drawn through -270° , -90° , 90° , 270° etc. as x approaches these values. But the graph never really touches these lines. Such vertical lines are called **asymptotes**, and we say that the graph is nearing them asymptotically.

Example 1

Sketch the graph of $y = 1 + \sin x$ for $0^\circ \leq x \leq 360^\circ$.

Solution

First we sketch the graph of $y = \sin x$. Then the whole curve is moved 1 unit up to obtain $y = 1 + \sin x$.

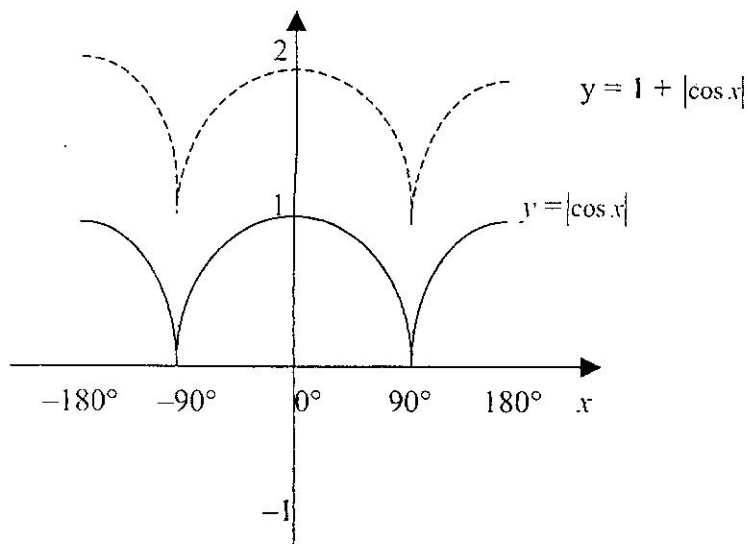


Example 2

Sketch the graph of $y = 1 + |\cos x|$ for $-180^\circ \leq x \leq 180^\circ$.

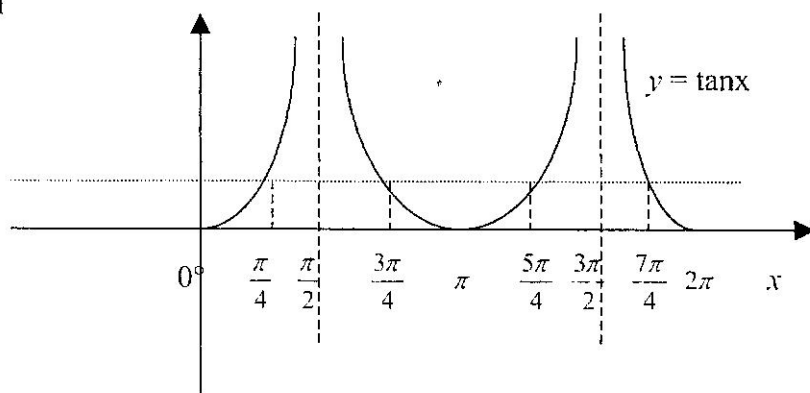
Solution

First we sketch the graph of $y = |\cos x|$. Then the whole curve is moved 1 unit up to obtain $y = 1 + |\cos x|$.

**Example 3**

Sketch the graph of $y = |\tan x|$ for $0 \leq x \leq 2\pi$ showing clearly the position of the asymptotes.

Solve the inequality $|\tan x| < 1$ for $0 \leq x \leq 2\pi$. (C)

Solution

To solve the inequality $|\tan x| < 1$, we draw the line $y = 1$.

Next we solve $\tan x = 1$

We have $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

From the graph we can see that $|\tan x| < 1$ for $0 < x < \frac{\pi}{4}, \frac{3\pi}{4} < x < \frac{5\pi}{4}, \frac{7\pi}{4} < x < 2\pi$.

Exercise 5.1

- 1 Sketch the graph of $y = 2 + \sin x$ for $0^\circ \leq x \leq 360^\circ$.
- 2 Sketch the graph of $y = 1 + |\sin x|$ for $-180^\circ \leq x \leq 180^\circ$.
- 3 Sketch the graph of $y = \sin x - 1$ for $0^\circ \leq x \leq 360^\circ$.
- 4 Sketch the graph of $y = 1 + \cos x$ for $0^\circ \leq x \leq 360^\circ$.
- 5 Sketch the graph of $y = |\cos x| - \frac{1}{2}$ for $-\pi \leq x \leq \pi$.
- 6 Sketch the graph of $y = \sin 2x$ for $0 \leq x \leq \pi$.
- 7 Sketch the graph of $y = \cos 2x$ for $0 \leq x \leq \pi$.
- 8 Sketch the graph of $y = \tan 2x$ for $0 \leq x \leq \pi$.
- 9 Sketch the graph of $y = 3 + \tan x$ for $0^\circ \leq x \leq 360^\circ$.
- 10 Sketch the graph of $y = 2 \cos x$ for $0^\circ \leq x \leq 360^\circ$.
- 11 Sketch the graph of $y = 3 \sin x$ for $-180^\circ \leq x \leq 180^\circ$.
- 12 Sketch the graph of $y = 1 + 2 \cos x$ for $-180^\circ \leq x \leq 180^\circ$.
- 13 Sketch the graph of $y = 2 + 3 \sin x$ for $-0^\circ \leq x \leq 360^\circ$.
- 14 Sketch the graph of $y = 2 - \cos x$ for $-180^\circ \leq x \leq 180^\circ$.
- 15 Sketch the graph of $y = 1 - \sin x$ for $0^\circ \leq x \leq 360^\circ$.
- 16 Sketch the graph of $y = \sin(x - 30^\circ)$ for $0^\circ \leq x \leq 180^\circ$.
- 17 Sketch the graph of $y = \cos(x - 60^\circ)$ for $0^\circ \leq x \leq 360^\circ$.
- 18 Sketch the graph of $y = \tan(x + 45^\circ)$ for $0^\circ \leq x \leq 360^\circ$.
- 19 Sketch the graph of $y = |1 + 3\cos x|$ for $0^\circ \leq x \leq 180^\circ$.
- 20 Sketch the graph of $y = |1 + 2\sin x|$ for $0^\circ \leq x \leq 180^\circ$.

5.2 Exact values of trigonometry functions

In this section we are going to recall the exact values of sine, cosine and tangent of 30° , 45° , 60° and related angles, example 120° or 150° .

The following table will help the reader to remember the exact values of trigonometric functions very easily.

	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
X	30°	45°	60°
Sinx	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
Cosx	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tanx	$\frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

To find the exact values of sine 30° , 45° and 60° , we take the square root of the numbers found in the first row.

Similarly to find the exact values of cosine 30° , 45° and 60° , we reverse the values obtained for sin x.

And we know that $\tan x = \frac{\sin x}{\cos x}$. So, to derive the exact values of tan x, we have to divide the third row by the fourth row.

Example 1

Without using calculator or tables, find the exact value of $\tan 105^\circ$.

Solution

$$\begin{aligned}\tan 105^\circ &= \tan (45^\circ + 60^\circ) = \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ} \\ &= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}\end{aligned}$$

Rationalising, we have

$$\begin{aligned}\frac{1 + \sqrt{3}}{1 - \sqrt{3}} &= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \times \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \\ &= \frac{1 + 2\sqrt{3} + 3}{1 - 3} \\ &= \frac{4 + 2\sqrt{3}}{-2} = -(2 + \sqrt{3})\end{aligned}$$

Example 2

Without using calculator or tables, find the exact value of $\sin 15^\circ$.

Solution

$$\begin{aligned}
 \sin 15^\circ &= \sin (45^\circ - 30^\circ) \\
 &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\
 &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} \\
 &= \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

Example 3

Without using calculator or tables, find the exact value of $\cos 150^\circ$.

Solution

$$\begin{aligned}
 \cos 150^\circ &= \cos (90^\circ + 60^\circ) \\
 &= \cos 90^\circ \cos 60^\circ - \sin 90^\circ \sin 60^\circ \\
 &= \left(0 \times \frac{1}{2}\right) - \left(1 \times \frac{\sqrt{3}}{2}\right) \\
 &= -\frac{\sqrt{3}}{2}
 \end{aligned}$$

Exercise 5.2

Without using calculator or tables, find the exact value of each of the following:

1 $\cos 15^\circ$ [Ans : $\frac{\sqrt{6} + \sqrt{2}}{4}$]

8 $\sin 120^\circ$ [Ans : $\frac{\sqrt{3}}{2}$]

2 $\sin 150^\circ$ [Ans : $\frac{1}{2}$]

9 $\cos 120^\circ$ [Ans : $-\frac{1}{2}$]

3 $\tan 75^\circ$ [Ans : $\frac{3 + \sqrt{3}}{3 - \sqrt{3}}$]

4 $\sin 105^\circ$ [Ans : $\frac{\sqrt{6} + \sqrt{2}}{4}$]

5 $\cos 75^\circ$ [Ans : $\frac{\sqrt{6} - \sqrt{2}}{4}$]

6 $\tan 15^\circ$ [Ans : $\frac{3 - \sqrt{3}}{3 + \sqrt{3}}$]

7 $\sin 75^\circ$ [Ans : $\frac{\sqrt{6} + \sqrt{2}}{4}$]

5.3 Proof of identities

If we recall the identities learnt in lower classes, we have

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

In addition, the reader is expected to be familiar with the following identities

$$\frac{\sin \theta}{\cos \theta} = \tan \theta,$$

$$\frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} = \cot \theta,$$

$$\frac{1}{\sin \theta} = \operatorname{cosec} \theta,$$

$$\frac{1}{\cos \theta} = \sec \theta.$$

To prove an identity, the following steps will prove useful.

Step1 Convert all trigonometric ratios to sine and cosine ratios

Step2 Take L.C.M and simplify

Step3 Use the identity $\sin^2 \theta + \cos^2 \theta = 1$

Example 1

Show that $\sec \theta \operatorname{cosec} \theta - \cot \theta = \tan \theta$

Proof

$$\begin{aligned} \text{L.H.S} \quad \sec \theta \operatorname{cosec} \theta - \cot \theta &= \frac{1}{\cos \theta} \times \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \\ &= \frac{1}{\cos \theta \sin \theta} - \frac{\cos \theta}{\sin \theta} \\ &= \frac{1 - \cos^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{\sin^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{\sin \theta}{\cos \theta} = \tan \theta \text{ (R.H.S)} \end{aligned}$$

Example 2

Show that $\tan \theta + \cot \theta = \frac{2}{\sin 2\theta}$. (C)

Proof

$$\begin{aligned} \text{L.H.S} \quad \tan \theta + \cot \theta &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{1}{\cos \theta \sin \theta} \\ &= \frac{2}{2 \cos \theta \sin \theta} = \frac{2}{\sin 2\theta} \text{ (R.H.S)} \end{aligned}$$

Exercise 5.3

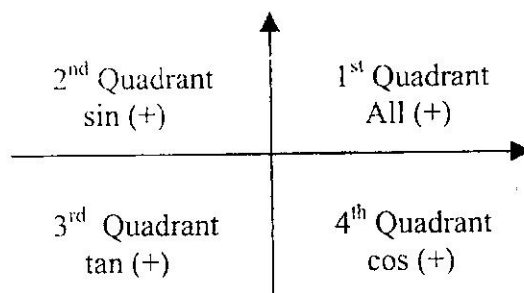
Prove the following identities

- 1 $\operatorname{cosec} A - \sin A = \cos A \cot A$
- 2 $\sec^2 x + \operatorname{cosec}^2 x = \operatorname{cosec}^2 x \sec^2 x$
- 3 $\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = \frac{2}{\cos x}$
- 4 $\frac{(\sin x + \cos x)^2}{\sin x \cos x} = 2 + \sec x \operatorname{cosec} x$
- 5 $\cos A \operatorname{cosec} A + \sin A \sec A = \operatorname{cosec} A \sec A$
- 6 $(\sin x + \cos x)(1 - \sin x \cos x) = \sin^3 x + \cos^3 x.$
- 7 $(\sec x - \tan x)(\operatorname{cosec} x + 1) = \cot x$
- 8 $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$
- 9 $(\sin A - \cos A)(\tan A + \cot A) = \sec A - \operatorname{cosec} A$
- 10 $\cot A + \tan A = \sec A \operatorname{cosec} A$
- 11 $(1 - \cos \theta)(1 + \sec \theta) = \sin \theta \tan \theta$
- 12 $\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \operatorname{cosec} A$
- 13 $\tan^2 y + \cot^2 y = \sec^2 y + \operatorname{cosec}^2 y - 2$
- 14 $(\cot \theta - \tan \theta) \sin \theta = 2 \cos \theta - \sec \theta$
- 15 $\operatorname{cosec} \theta - \frac{\sin \theta}{1 + \cos \theta} = \cot \theta$
- 16 $\cos^4 \theta + \sin^2 \theta = \sin^4 \theta + \cos^2 \theta$
- 17 $\operatorname{cosec} A - \frac{\sin A}{1 + \cos A} = \cot A.$

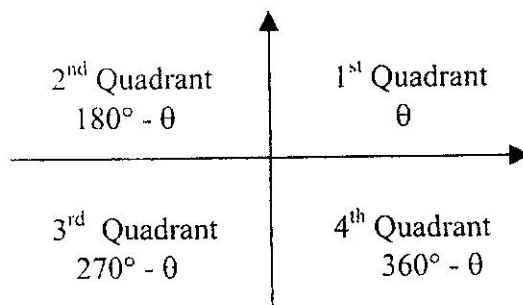
5.4 Solving trigonometric equations

In this part of the chapter, the reader is expected to be familiar with the four quadrants and the sign of the different trigonometric ratios in each of the four quadrants.

The following will help the reader to solve trigonometric equations more confidently.



Once the principal values (key angles) of the trigonometric equations are known, the following can be used to find the required angles.



Example 1

Find all values of x for which $0^\circ < x < 360^\circ$ that satisfy the equation $\sin\left(\frac{1}{2}x\right) = \frac{1}{4}$. (C)

Solution

$$\sin\left(\frac{1}{2}x\right) = \frac{1}{4} \Rightarrow \frac{1}{2}x = \sin^{-1}\left(\frac{1}{4}\right)$$

$$\frac{1}{2}x = 14.5^\circ, 165.5^\circ$$

$$x = 29.0^\circ, 331.0^\circ$$

Example 2

Solve the equation $2 \cos^2 \theta + \cos \theta - 1 = 0$ for $0^\circ < \theta < 360^\circ$.

Solution

$$2 \cos^2 \theta + \cos \theta - 1 = 0$$

$$(2 \cos \theta - 1)(\cos \theta + 1) = 0$$

Either $\cos \theta = \frac{1}{2}$ or $\cos \theta = -1$

$$\theta = 60^\circ, 300^\circ \text{ or } \theta = 180^\circ$$

Hence $\theta = 60^\circ, 180^\circ, 300^\circ$. (Ans)

Example 3

Solve the equation $2 \sin x = \sec x$, for $0^\circ < \theta < 360^\circ$.

Solution

$$2 \sin x = \sec x$$

$$2 \sin x = \frac{1}{\cos x}$$

$$2 \sin x \cos x = 1$$

$$\sin 2x = 1$$

$$x = 90^\circ, 450^\circ$$

Hence $x = 45^\circ, 225^\circ$. (Ans)

Example 4

Solve, for $0^\circ < \theta < 360^\circ$, the equation $3 \sec \theta + \tan^2 \theta + 2 = 0$. (C)

Solution

$$3 \sec \theta + \tan^2 \theta + 2 = 0.$$

$$3 \sec \theta + \sec^2 \theta - 1 + 2 = 0.$$

$$\sec^2 \theta + 3 \sec \theta + 1 = 0.$$

Using the quadratic formula, we have

$$\sec \theta = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

$$\text{Either } \sec \theta = -0.38 \quad \text{or } \sec \theta = -2.62$$

$$\text{Either } \cos \theta = -2.62 \quad \text{or } \cos \theta = -0.38$$

$$\cos \theta = -2.62 \text{ (No Solution)}$$

$$\cos \theta = -0.38$$

$$\text{Key angle} = 67.5^\circ$$

$$\theta = 180^\circ - 67.5^\circ, 180^\circ + 67.5^\circ$$

Hence $\theta = 112.5^\circ, 247.5^\circ$.

Exercise 5.4

Find all the angles between 0° and 360° inclusive which satisfy the following equations:

- 1 $\sin (x - 10^\circ) = \frac{\sqrt{2}}{2}$ [Ans: $55^\circ, 145^\circ$]
- 2 $\cos (x + 15^\circ) = \frac{\sqrt{3}}{2}$ [Ans: $15^\circ, 315^\circ$]
- 3 $\sin 2x = 0.5$ [Ans: $15^\circ, 75^\circ, 195^\circ, 255^\circ$]
- 4 $\cos 2x = 0.5$ [Ans: $30^\circ, 150^\circ, 210^\circ, 330^\circ$]
- 5 $\tan 3x = 1$ [Ans: $15^\circ, 75^\circ, 135^\circ, 195^\circ, 255^\circ, 315^\circ$]
- 6 $2 \tan y = 3 \sin y$ [Ans: $0^\circ, 48.2^\circ, 180^\circ, 311.8^\circ, 360^\circ$]
- 7 $2 \cos^2 x + \sin x = 1$ [Ans: $90^\circ, 210^\circ, 330^\circ$]
- 8 $2 \cos^2 x - 7 \cos x + 3 = 0$ [Ans: $60^\circ, 300^\circ$]
- 9 $3 \sin^2 x - 7 \sin x + 2 = 0$ [Ans: $19.5^\circ, 160.5^\circ$]
- 10 $\sin^2 x + 2 \sin x \cos x = 0$ [Ans: $0^\circ, 116.6^\circ, 180^\circ, 243.4^\circ, 360^\circ$]
- 11 $3 \cos^2 \theta = 2 \cos \theta$ [Ans: $48.2^\circ, 90^\circ, 270^\circ, 311.8^\circ$]
- 12 $\tan 2x = \cot 70^\circ$ [Ans: $10^\circ, 100^\circ, 190^\circ, 280^\circ$]
- 13 $3 (\sin \theta - \cos \theta) = 2 (\sin \theta + \cos \theta)$ [Ans: $78.7^\circ, 258.7^\circ$]
- 14 $5 \cos^2 \theta + 2 \sin^2 \theta = 3$ [Ans: $54.7^\circ, 125.3^\circ, 243.7^\circ, 305.3^\circ$]
- 15 $\tan \frac{1}{2} x = \frac{1}{2}$ [Ans: 53.2°]
- 16 $3 \cos \theta + \cot \theta = 0$ [Ans: $90^\circ, 199.5^\circ, 270^\circ, 340.5^\circ$]

TRIGONOMETRY 1 – PAST EXAMINATION QUESTIONS

- 1 Solve for x , where $-180^\circ < x \leq 180^\circ$, the equation $\sin(x + 60^\circ) = \cos x$.
[Ans : $15^\circ, -165^\circ$]
- 2 Find the solution for x , where $0^\circ < x \leq 360^\circ$, of the equation $\sin 2x = \cos 36^\circ$.
[Ans : $27^\circ, 63^\circ, 207^\circ, 243^\circ$]
- 3 Using the fact that $75^\circ = 30^\circ + 45^\circ$, find the value of $\sin 75^\circ$ in surd form, simplifying your answer. [Ans : $\frac{1}{4}(\sqrt{6} + \sqrt{2})$]
- 4 By completing the square, find the greatest and least values, as θ varies, of $\cos^2\theta - \cos\theta + 6$. [Ans : $8, 5\frac{1}{4}$]
- 5 Given that $\operatorname{cosec} A = \frac{7}{2}$, where $90^\circ < A < 180^\circ$, find the exact value of $\sec A$.
[Ans : $-\frac{7\sqrt{5}}{15}$]
- 6 Show that the equation $2\sin x \cos x + 4\cos^2 x = 1$ may be written in the form $\tan^2 x - 2\tan x - 3 = 0$.
Hence solve the equation $2\sin x \cos x + 4\cos^2 x = 1$, giving solutions for $0^\circ < x \leq 360^\circ$.
[Ans : $71.6^\circ, 251.6^\circ, 135^\circ, 315^\circ$]
- 7 Given that $7\tan\theta + \cot\theta = 5\sec\theta$, derive a quadratic equation for $\sin\theta$. Hence, or otherwise, find all values of θ in the interval $0^\circ \leq x \leq 180^\circ$ which satisfy the equation, giving your answers to the nearest 0.1° , where necessary.
[Ans : $6\sin^2\theta - 5\sin\theta + 1 = 0$; $30^\circ, 150^\circ, 19.5^\circ, 160.5^\circ$]
- 8 Solve the equation $\sec\theta \tan\theta = 2$, giving solutions for $0^\circ \leq x < 360^\circ$.
[Ans : $51.3^\circ, 128.7^\circ$]

CHAPTER 6

VECTORS

6.1 Standard notations for vectors

A vector quantity can be represented by a line segment with the direction specified, e.g. $\overrightarrow{AB}$.

Normally, a vector is represented by a single bold letter, say **a**.

Vectors can be represented horizontally, e.g. $2\mathbf{i} + 3\mathbf{j}$ or in column vector form, e.g. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Vectors can be represented in two-dimensions e.g. $3\mathbf{i} - 4\mathbf{j}$ or $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

Vectors can also be represented in three-dimensions, e.g. $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ or $\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$.

Class Activity

Express the following in column vector form.

(a) $4\hat{i} + \hat{j} + 3\hat{k}$

(b) $2\hat{i} - 5\hat{j} + 7\hat{k}$

(c) $\hat{i} - 2\hat{j} + 3\hat{k}$

(d) $-\hat{i} + 6\hat{j} + \hat{k}$

(e) $\hat{i} - \hat{j} + \hat{k}$

(f) $\hat{i} + 4\hat{k}$

(g) $5\hat{i} + \hat{j}$

(h) $2\hat{i} - 3\hat{k}$

(i) $7\hat{i}$

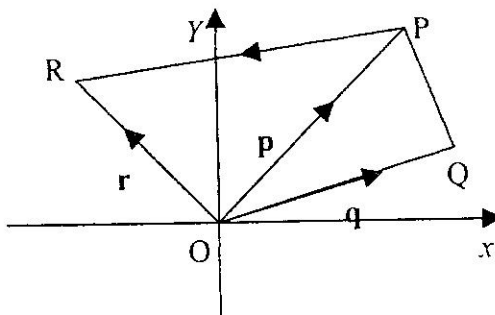
(j) $6\hat{j}$

(k) $5\hat{k}$

(l) $4\hat{i} + 3\hat{k}$

6.2 Position vectors and Displacement vectors

The position of the points P, Q and R relative to the origin O are denoted by $\vec{OP}$, $\vec{OQ}$ and $\vec{OR}$. The vectors $\vec{OP}$, $\vec{OQ}$ and $\vec{OR}$ are called position vectors of P, Q and R respectively. $\vec{OP}$, $\vec{OQ}$ and $\vec{OR}$ can be represented by $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively.



From the diagram

$$\vec{OP} = \mathbf{p}$$

$$\vec{OQ} = \mathbf{q}$$

$$\vec{OP} + \vec{OQ} = \mathbf{p} + \mathbf{q}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \mathbf{q} - \mathbf{p}$$

Similarly

$$\vec{PR} = \vec{OR} - \vec{OP} = \mathbf{r} - \mathbf{p}$$

$$\vec{QR} = \vec{OR} - \vec{OQ} = \mathbf{r} - \mathbf{q}$$

In general, a vector between two points A and B can be expressed in terms of their position vectors.

$$\vec{AB} = \vec{OB} - \vec{OA}$$

Example 1

The position vectors of A and B relative to an origin O are $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$ respectively. If M is the midpoint of AB. Find the position vector of M.

Solution

$$\vec{OM} = \frac{1}{2} (\vec{OA} + \vec{OB})$$

$$\text{Now } \vec{OA} + \vec{OB} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\vec{OM} = \frac{1}{2} \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Exercise 6.2

- 1 The position vectors of A, B and C relative to a fixed origin O are

$$\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \text{ respectively. Find the displacement vectors}$$

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{AC}$ [Ans : (a) (1, -4, -2) (b) (-1, -5, 1)]

- 2 The position vectors of P, Q and R relative to a fixed origin O are

$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ respectively. Find the displacement vectors}$$

(a) $\overrightarrow{PQ}$ (b) $\overrightarrow{QR}$ [Ans : (a) (-2, 1, -1) (b) (1, 2, 3)]

- 3 The position vectors of A and B relative to a fixed origin O are

$$\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \text{ respectively.}$$

If M is the midpoint of AB, find the position vector of M. [Ans : (3, 2, 2)]

- 4 The position vectors of P and Q relative to a fixed origin O are

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \text{ respectively.}$$

If N is the midpoint of PQ, find the position vector of N. [Ans : (0, 1, 5/2)]

- 5 The position vectors of A and B relative to a fixed origin O are $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

respectively. If R is a point on AB such that $AR : RB = 1 : 3$, find the position vector of R. [Ans : (5/4, 0, -3/4)]

- 6 The position vectors of A and B relative to a fixed origin O are $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$

respectively. If R is a point on AB such that $AR : RB = 1 : 2$, find the position vector of R. [Ans : (2, -1, 1)]

6.3 Magnitude of a vector

The magnitude of a vector $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, denoted by $|\mathbf{a}|$, is given by

$$|\mathbf{a}| = \sqrt{a^2 + b^2 + c^2}$$

The magnitude of a vector represents the length or modulus of that vector.

Example 1

Calculate the magnitude of the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Solution

$$|\mathbf{a}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

Example 2

If $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, find $|\overrightarrow{AB}|$.

Solution

$$\text{Now } \overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{3^2 + (-4)^2 + 3^2} = \sqrt{34}$$

Exercise 6.3

- 1 The position vectors of A, B and C relative to a fixed origin O are

$$\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} \text{ respectively. Find}$$

$$(a) |\overline{AB}| \quad (b) |\overline{AC}| \quad [\text{Ans : (a) } \sqrt{3} \quad (b) 7]$$

- 2 The position vectors of P, Q and R relative to a fixed origin O are

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ respectively. Find}$$

$$(a) |\overline{PQ}| \quad (b) |\overline{QR}| \quad [\text{Ans : (a) } \sqrt{8} \quad (b) 2\sqrt{3}]$$

- 3 The position vectors of A and B relative to a fixed origin O are

$$\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \text{ respectively. Find } |\overline{AM}|, \text{ where M is the midpoint of AB.}$$

$$[\text{Ans : } \sqrt{6}]$$

- 4 The position vectors of P and Q relative to a fixed origin O are

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \text{ respectively. Find } |\overline{PM}|, \text{ where M is the midpoint of PQ.}$$

$$[\text{Ans : } \sqrt{3}]$$

- 5 Relative to the origin O, the points L and N have position vectors $12\mathbf{i} + 5\mathbf{k}$ and $9\mathbf{i} + \mathbf{j}$. Calculate the magnitudes of OL and LN.

$$[\text{Ans : } 13, \sqrt{35}]$$

- 6 Relative to the origin O, the points P, Q and R have position vectors $6\mathbf{i} + 8\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j}$ and $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Calculate the magnitudes of OP and QR.

$$[\text{Ans : } 10, \sqrt{52}]$$

6.4 Unit vectors

A unit vector is a vector whose magnitude or length is 1.

The unit vector in the direction of $\mathbf{a}$ denoted by $\hat{\mathbf{a}}$ is given by

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example 1

Find the unit vector in the direction of $12\mathbf{i} + 5\mathbf{j}$.

Solution

Let $\mathbf{a} = 12\mathbf{i} + 5\mathbf{j}$.

$$\begin{aligned}\text{Then } \hat{\mathbf{a}} &= \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{12\hat{\mathbf{i}} + 5\hat{\mathbf{j}}}{\sqrt{12^2 + 5^2}} \\ &= \frac{12\hat{\mathbf{i}} + 5\hat{\mathbf{j}}}{\sqrt{144 + 25}} \\ &= \frac{12\hat{\mathbf{i}} + 5\hat{\mathbf{j}}}{\sqrt{169}} \\ &= \frac{1}{13} (12\mathbf{i} + 5\mathbf{j})\end{aligned}$$

Example 2

Find the unit vector in the direction of $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Solution

Let $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$\begin{aligned}\text{Then } \hat{\mathbf{a}} &= \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{2^2 + 2^2 + 1^2}} \\ &= \frac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{9}} \\ &= \frac{1}{3} (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\end{aligned}$$

Example 3

If $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, find a unit vector in the direction of $\overrightarrow{AB}$.

Solution

$$\text{Now } \overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \text{The unit vector in the direction of } \overrightarrow{AB} &= \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\hat{i} - 5\hat{j} + 3\hat{k}}{\sqrt{1^2 + (-5)^2 + 3^2}} \\ &= \frac{\hat{i} - 5\hat{j} + 3\hat{k}}{\sqrt{35}} \quad (\text{Ans}) \end{aligned}$$

Exercise 6.4

- 1 The position vectors of A, B and C relative to a fixed origin O are

$$\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ -8 \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} \text{ respectively.}$$

Find unit vectors in the direction of **OA**, **OB** and **BC**.

$$[\text{Ans : } \frac{3\hat{i} + 4\hat{j}}{5}, \frac{6\hat{i} - 8\hat{k}}{10}, \frac{-\hat{i} + 2\hat{j} + 14\hat{k}}{\sqrt{201}}]$$

- 2 The position vectors of P, Q and R relative to a fixed origin O are

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \text{ respectively.}$$

Find unit vectors in the direction of **OP** and **QR**.

$$[\text{Ans : } \frac{2\hat{i} + 3\hat{j} + 4\hat{k}}{\sqrt{29}}, \frac{-\hat{i} + 5\hat{j} + 3\hat{k}}{\sqrt{35}}]$$

- 3 Relative to the origin O, the points L and N have position vectors $12\hat{i} + 5\hat{k}$ and $8\hat{i} + 5\hat{j}$. Find unit vectors in the direction of **OL** and **LN**.

$$[\text{Ans : } \frac{12\hat{i} + 5\hat{k}}{13}, \frac{-4\hat{i} + 5\hat{j} - 5\hat{k}}{\sqrt{66}}]$$

- 4 Relative to the origin O, the points P and Q have position vectors $2\hat{i} + 3\hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j}$. Find unit vectors in the direction of **OP** and **PQ**.

$$[\text{Ans : } \frac{2\hat{i} + 3\hat{j} + \hat{k}}{\sqrt{14}}, \frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}]$$

6.5 Scalar product (dot product) of two vectors

The scalar product of two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, written as $\mathbf{a} \cdot \mathbf{b}$ is given by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

Example 1

The position vectors of A, B and C relative to a fixed origin O are

$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ respectively.

Find (i) $\mathbf{OA} \cdot \mathbf{OB}$ (ii) $\mathbf{AB} \cdot \mathbf{BC}$

Solution

$$(i) \quad \mathbf{OA} \cdot \mathbf{OB} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = 6 + 0 + 6 = 12$$

$$(ii) \quad \mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\mathbf{BC} = \mathbf{OC} - \mathbf{OB} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$$

$$\mathbf{AB} \cdot \mathbf{BC} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$$

$$= -2 + -4 + -3$$

$$= -9$$

Exercise 6.5

- 1 The position vectors of A, B and C relative to a fixed origin O are $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$ respectively. Find
 (a) $\overrightarrow{OA} \cdot \overrightarrow{OB}$ (b) $\overrightarrow{OB} \cdot \overrightarrow{AC}$ [Ans : (a) 14 (b) -7]

- 2 The position vectors of P, Q and R relative to a fixed origin O are $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ respectively. Find
 (a) $\overrightarrow{OP} \cdot \overrightarrow{OR}$ (b) $\overrightarrow{OR} \cdot \overrightarrow{PQ}$ [Ans : (a) 14 (b) -5]

- 3 The position vectors of A and B relative to a fixed origin O are $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ respectively. Find
 (a) $\overrightarrow{OA} \cdot \overrightarrow{OB}$ (b) $\overrightarrow{OB} \cdot \overrightarrow{AB}$ [Ans : (a) 11 (b) 6]

- 4 The position vectors of L and M relative to a fixed origin O are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ respectively. Find
 (a) $\overrightarrow{OL} \cdot \overrightarrow{OM}$ (b) $\overrightarrow{OL} \cdot \overrightarrow{LM}$ [Ans : (a) 6 (b) -8]

- 5 Relative to the origin O, the points P and Q have position vectors $10\mathbf{i} + 5\mathbf{k}$ and $7\mathbf{i} + 5\mathbf{j}$. Find
 (a) $\overrightarrow{OP} \cdot \overrightarrow{OQ}$ (b) $\overrightarrow{OQ} \cdot \overrightarrow{QP}$ [Ans : (a) 70 (b) -4]

- 6 Relative to the origin O, the points A and B have position vectors $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j}$. Find
 (a) $\overrightarrow{OA} \cdot \overrightarrow{OB}$ (b) $\overrightarrow{OB} \cdot \overrightarrow{BA}$ [Ans : (a) 18 (b) 5]

6.6 Angle between two directions

To find the angle between two vectors a and b , we start by drawing a and b from a common point. Then the angle θ between the two directions is given by

$$a \cdot b = |a| |b| \cos \theta$$
$$\text{or } \cos \theta = \frac{a \cdot b}{|a||b|}$$

Example 1

Find the acute angle between the vectors $a = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $b = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

Solution

$$\text{Now } a \cdot b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 2 + 3 + 4 = 9$$

$$|a| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|b| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$\text{Thus } \cos \theta = \frac{9}{\sqrt{3}\sqrt{29}}$$

$$\text{Hence } \theta = 15.2^\circ$$

Example 2

Given that $\mathbf{OA} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{OC} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$, find $\cos \angle \text{BAC}$, giving your answer in exact form.

Solution

$$\cos \angle \text{BAC} = \frac{\mathbf{AB} \cdot \mathbf{AC}}{|\mathbf{AB}||\mathbf{AC}|}$$

$$\text{Now } \mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{AC} = \mathbf{OC} - \mathbf{OA} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$$

$$\text{Also } \mathbf{AB} \cdot \mathbf{AC} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = -11$$

$$|\mathbf{AB}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$|\mathbf{AC}| = \sqrt{(-1)^2 + (-2)^2 + 4^2} = \sqrt{21}$$

$$\begin{aligned} \text{Hence } \cos \angle \text{BAC} &= \frac{-11}{\sqrt{6}\sqrt{21}} \\ &= \frac{-11}{\sqrt{2} \times \sqrt{3} \times \sqrt{7} \times \sqrt{3}} = \frac{-11}{3\sqrt{14}} \quad (\text{Answer}) \end{aligned}$$

Exercise 6.6

- 1 The position vectors of A and B relative to a fixed origin O are $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ respectively. Find $\angle \text{AOB}$. [Ans : 45.6°]
- 2 The position vectors of P and Q relative to a fixed origin O are $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ respectively. Find $\angle \text{POQ}$. [Ans : 42.5°]
- 3 The position vectors of A and B relative to a fixed origin O are $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ respectively. Find $\angle \text{ABO}$. [Ans : 90°]
- 4 The position vectors of P, Q and R relative to a fixed origin O are $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ respectively. Find $\angle \text{QPR}$. [Ans : 70.5°]
- 5 Relative to the origin O, the points L and N have position vectors $4\mathbf{i} + 3\mathbf{k}$ and $2\mathbf{i} + 5\mathbf{j}$. Calculate $\angle \text{LON}$. [Ans : 72.7°]
- 6 Relative to the origin O, the points P, Q and R have position vectors $6\mathbf{i} + 8\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j}$ and $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Calculate $\angle \text{PQR}$. [Ans : 42.7°]

6.7 Perpendicular vectors

Two vectors a and b are perpendicular to each other if $a \cdot b = 0$.

Example 1

Show that $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ are perpendicular.

Solution

Let $a = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $b = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ are perpendicular.

$$\text{Then } a \cdot b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 2 - 3 + 1 = 0$$

Since $a \cdot b = 0$, a and b are perpendicular.

Example 2

Find the value of p for which $2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ is perpendicular to $3\mathbf{i} - p\mathbf{j} + 2\mathbf{k}$.

Solution

Let $a = 2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $b = 3\mathbf{i} - p\mathbf{j} + 2\mathbf{k}$.

Since a and b are perpendicular $a \cdot b = 0$

$$\begin{aligned} \text{Now } a \cdot b &= \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -p \\ 2 \end{pmatrix} = 0 \\ 6 - 4p + 10 &= 0 \\ 16 - 4p &= 0 \\ p &= 4 \end{aligned}$$

Example 3

The position vectors of A and B relative to a fixed origin O are $\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}$ respectively.

Find the value of k for which $k\mathbf{a} + 2\mathbf{b}$ is perpendicular to $\mathbf{b}$.

Solution

$$k\mathbf{a} + 2\mathbf{b} = k \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3k+10 \\ 5k-4 \\ 0 \end{pmatrix}$$

$$\text{Now } k\mathbf{a} + 2\mathbf{b} \cdot \mathbf{b} = \begin{pmatrix} 3k+10 \\ 5k-4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} = 15k + 50 - 10k + 8 + 0 = 5k + 58$$

$$\text{Hence } k = -\frac{58}{5}.$$

Exercise 6.7

- 1 The position vectors of A and B relative to an origin O are $3\mathbf{i} + 4\mathbf{j}$ and $2\mathbf{i} + p\mathbf{j}$ respectively. Find the value of p for which OA is perpendicular to AB.

[Ans : $\frac{19}{4}$]

- 2 The position vectors of A, B and C relative to a fixed origin O are $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ p \\ 1 \end{pmatrix}$ respectively. Find the value of p for which OA is perpendicular to BC.

[Ans : 6]

- 3 The position vectors of P, Q and R relative to a fixed origin O are $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 1 \\ p \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$ respectively. Find the value of p for which OR is perpendicular to PQ.

[Ans : $\frac{13}{3}$]

- 4 The position vectors of P and Q relative to a fixed origin O are $\begin{pmatrix} 2 \\ 3 \\ p \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ respectively.

Find the value of p for which OQ is perpendicular to QP. [Ans : -7]

- 5 The position vectors of P and Q relative to a fixed origin O are $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$ respectively.

Find the value of k for which $k\mathbf{p} + \mathbf{q}$ is perpendicular to $\mathbf{p}$. [Ans : $-\frac{4}{5}$]

- 6 The position vectors of P, Q, R and S relative to a fixed origin O are $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{q} = 5\mathbf{i}$, $\mathbf{r} = \frac{1}{4}(\mathbf{p} + \mathbf{q})$ and $\mathbf{s} = \frac{1}{2}(\mathbf{p} - \mathbf{q})$. Show that $\mathbf{r}$ is perpendicular to $\mathbf{s}$.

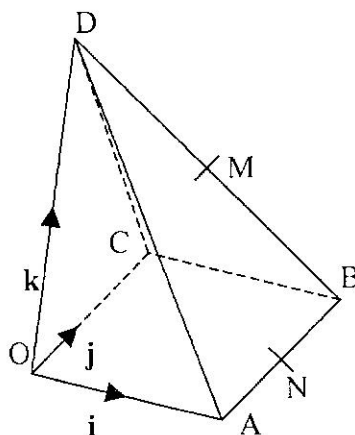
VECTORS I – PAST EXAMINATION QUESTIONS

- 1 The points A, B and C have position vectors given respectively by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$. Calculate angle ABC, giving your answer to the nearest degree.
[Ans : 24°]

- 2 Relative to a fixed origin O, the points A, B and C have position vectors given respectively by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find
 - (a) the length of AB, correct to 3 significant figures,
 - (b) angle BAC, correct to the nearest degree,
 - (c) the area of triangle ABC, correct to 3 s.f. [Ans : (a) $5\sqrt{2}$ (b) 56° (c) 8.75]

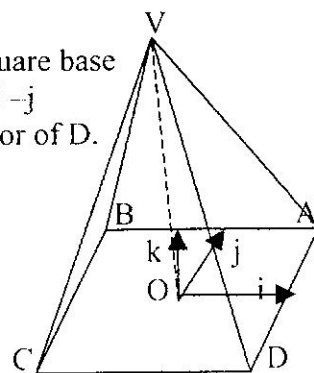
- 3 The angle between the vector $\lambda\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and the vector $\mathbf{i}$ is 120° . Find the exact value of the constant λ .
[Ans : $-\sqrt{15}$]

- 4 The diagram shows a pyramid DOABC. Taking unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as shown, the position vectors of A, B, C, D are given by $\overrightarrow{OA} = 4\mathbf{i}$, $\overrightarrow{OB} = 4\mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OC} = 2\mathbf{j}$, $\overrightarrow{OD} = 6\mathbf{k}$. The mid-points of BD and AB are M and N respectively.
 - (i) Find the vector $\overrightarrow{MN}$ and the angle between the directions of $\overrightarrow{MN}$ and $\overrightarrow{OB}$.
 - (ii) The point P lying on OD has position vector $p\mathbf{k}$ and is such that angle PMN is a right angle. Find the value of p.



[Ans : (i) $(2, 0, -3)$; 60° (ii) $5/3$]

- 5 With respect to the origin O, the corners A, B, C of the square base ABCD of a pyramid have position vectors $\mathbf{i} + \mathbf{j}$, $-\mathbf{i} + \mathbf{j}$, $-\mathbf{i} - \mathbf{j}$ respectively (see diagram). Write down the position vector of D. The vertex V of the pyramid has position vector $(\sqrt{6})\mathbf{k}$.
 - (i) Express the vectors $\overrightarrow{AV}$ and $\overrightarrow{CV}$ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.
 - (ii) Use a scalar product to show that angle AVC = 60° .
 [Ans : $\mathbf{i} - \mathbf{j}$; $-\mathbf{i} - \mathbf{j} + \sqrt{6}\mathbf{k}$; $\mathbf{i} + \mathbf{j} + \sqrt{6}\mathbf{k}$]



CHAPTER 7

SERIES

7.1 Expansion of $(a + b)^n$, where n is a positive integer

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + b^n.$$

Example 1

Expand $(2 + 3x)^4$.

Solution

$$\begin{aligned} (2 + 3x)^4 &= 2^4 + \binom{4}{1} 2^3 (3x)^1 + \binom{4}{2} 2^2 (3x)^2 + \binom{4}{3} 2^1 (3x)^3 + (3x)^4 \\ &= 2^4 + 4(8)(3x) + 6(4)(9x^2) + 4(2)(27x^3) + 81x^4 \\ &= 16 + 96x + 216x^2 + 216x^3 + 81x^4. \end{aligned}$$

Example 2

Obtain the first four terms in the expansion of $\left(2 + \frac{x}{4}\right)^8$ in ascending powers of x . Find the value of $(2.0025)^8$ correct to three decimal places. (C)

Solution

$$\begin{aligned} \left(2 + \frac{x}{4}\right)^8 &= 2^8 + \binom{8}{1} (2)^7 \left(\frac{x}{4}\right) + \binom{8}{2} (2)^6 \left(\frac{x}{4}\right)^2 + \binom{8}{3} (2)^5 \left(\frac{x}{4}\right)^3 \\ &\quad + \binom{8}{4} (2)^4 \left(\frac{x}{4}\right)^4 + \dots \\ &= 256 + 8(128) \left(\frac{x}{4}\right) + 28(64) \left(\frac{x^2}{16}\right) + 56(32) \left(\frac{x^3}{64}\right) + \dots \\ &= 256 + 256x + 112x^2 + 28x^3 + \dots \end{aligned}$$

Let $2 + \frac{x}{4} = 2.0025$

$$\frac{x}{4} = 2.0025$$

$$x = 0.01$$

So we replace $x = 0.01$ in the above expansion,

$$\begin{aligned} (2.0025)^8 &= 256 + 256(0.01) + 112(0.01)^2 + 28(0.01)^3 \\ &\quad + \dots \\ &= 258.571 \text{ (3 d.p.)} \end{aligned}$$

Exercise 7.1

1 Write down the expansion of

(a) $(2 + 3x)^5$ [Ans : $32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$]

(b) $(2 + x)^6$ [Ans : $64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$]

(c) $(1 + 2x)^4$ [Ans : $1 + 8x + 24x^2 + 32x^3 + 16x^4$]

(d) $(2 - 3x)^3$ [Ans : $8 - 36x + 54x^2 - 27x^3$]

2 Write down the first four terms in each of the following expansions.

(a) $(2 + x)^{10}$ [Ans : $1024 + 5120x + 11520x^2 + 15360x^3$]

(b) $(x + 2y)^{12}$ [Ans : $x^{12} + 24x^{11}y + 240x^{10}y^2 + 1760x^9y^3$]

(c) $(2 - x)^9$ [Ans : $512 - 2304x + 4608x^2 - 5376x^3$]

(d) $(x + \frac{1}{x})^{10}$ [Ans : $x^{10} + 10x^8 + 45x^6 + 120x^4$]

3 Expand $(2 + x)^6$ up to terms in x^2 . Hence find an approximate value of $(2.01)^6$.
[Ans : $64 + 192x + 240x^2$; 65.944]

4 Find the first three terms in the expansion of $(2 - 3x)^7$. Use the expansion to find $(1.997)^7$ correct to the nearest whole number.
[Ans : $128 - 1344x + 6048x^2$; 127]

5 Expand $(2 - \frac{x}{2})^5$ in ascending powers of x . Use the first four terms of the expansion to find an approximate value of $(1.99)^5$.

[Ans : $32 - 40x + 20x^2 - 5x^3 + \frac{5}{8}x^4 - \frac{1}{32}x^5$; 31.20796]

6 Find, in ascending powers of x , the first three terms in the expansion of

(i) $(2 - x)^6$ [Ans : $64 - 192x + 240x^2$]

(ii) $(2 + x)(2 - x)^6$ [Ans : $128 - 320x + 288x^2$]

7.2 Finding the n th term of an expansion

In the expansion of $(a + b)^n$, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

For example, $T_9 = \binom{n}{8} a^{n-8} b^8$ or $T_{12} = \binom{n}{11} a^{n-11} b^{11}$

Example

In each of the following expansions, find the indicated term.

(a) $(2 + x)^9$; 5th term

(b) $\left(2 + \frac{1}{x}\right)^{10}$; 7th term

Solution

(a) $T_5 = \binom{9}{4} 2^5 x^4 = 4032x^4$

(b) $T_7 = \binom{10}{6} 2^4 \left(\frac{1}{x}\right)^6 = \frac{3360}{x^6}$

Exercise 7.2

In each of the following expansions, find the indicated term.

1 $(2 + x^2)^9$; 5th term [Ans : $13440x^8$]

2 $\left(3 + \frac{2}{x}\right)^{12}$; 9th term [Ans: $\frac{10264320}{x^2}$]

3 $(3 - x)^{11}$; 7th term [Ans : $112266x^6$]

4 $\left(2 + \frac{x}{4}\right)^9$; 6th term [Ans: $\frac{63}{32}x^5$]

5 $(2 + 5x)^{10}$; 4th term [Ans : $192000x^3$]

6 $\left(1 + \frac{1}{2x}\right)^9$; 5th term [Ans: $\frac{63}{8x^4}$]

7 $(3 - x^2)^8$; 6th term [Ans : $-1512x^{10}$]

7.3 Finding the coefficient of a given term

To find the coefficient of a given term, we have to use of the formula $T_{r+1} = \binom{n}{r} a^{n-r} b^r$.

Example

Evaluate the coefficient of x^5 in the expansion of $\left(x^2 - \frac{2}{x}\right)^7$.

Solution

For $\left(x^2 - \frac{2}{x}\right)^7$, $T_{r+1} = \binom{7}{r} (x^2)^{7-r} \left(-\frac{2}{x}\right)^r$

Power of x in $T_{r+1} = 2(7-r) - r$

Let $2(7-r) - r = 5$

$$14 - 2r - r = 5$$

$$-3r = -9$$

$$r = 3$$

The term in $x^5 = \binom{7}{3} (x^2)^4 \left(-\frac{2}{x}\right)^3 = -280$

Exercise 7.3

- 1 In the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{10}$, find the coefficient of x^{10} . [Ans : 3360]
- 2 Evaluate the coefficient of x^3 in the expansion of $\left(3 - \frac{x}{2}\right)^8$. [Ans : 7]
- 3 Evaluate the coefficient of x^3 in the expansion of $(2 + 3x)\left(1 - \frac{x}{2}\right)^8$. [Ans : -1701]
- 4 Evaluate the coefficient of x^3 in the expansion of $(9 + 8x)\left(1 - \frac{x}{3}\right)^9$. [Ans : 4]
- 5 Find, in its simplest form, the coefficient of x^4 in the expansion of
 - (a) $(1 + 3x)^6$ [Ans : 1215]
 - (b) $\left(x^3 + \frac{3}{x}\right)^5$ [Ans : 90]
- 6 Find, in its simplest form, the coefficient of x^4 in the expansion of
 - (a) $(2 + x)^6$ [Ans : 60]
 - (b) $\left(x - \frac{5}{x}\right)^{10}$ [Ans : -15000]

7.4 Arithmetic progressions

Consider the sequence 2, 5, 8, 11, 14,

Clearly to obtain each term we have to add 3 to the preceding term.

The next three terms of the above sequence is 17, 20 and 23.

A sequence of the above type is called an **Arithmetic progression (AP)**.

The difference between any two successive terms is called the **common difference (d)**.

We usually use the letter "**a**" to denote the first term.

The following are examples of AP's.

1, 3, 5, 7, 9,	$a = 1, d = 2$
45, 40, 35, 30, 25,	$a = 45, d = -5$
6, 6.1, 6.2, 6.3,	$a = 6, d = 0.1$

Now

$$\begin{aligned}T_1 &= a \\T_2 &= T_1 + d = a + d \\T_3 &= T_2 + d = a + d + d = a + 2d \\T_4 &= T_3 + d = a + 2d + d = a + 3d \\T_5 &= T_4 + d = a + 3d + d = a + 4d\end{aligned}$$

Similarly $T_{10} = a + 9d$

In general the n th term is given by $T_n = a + (n - 1)d$

7.4.1(a) Use of the formula $T_n = a + (n - 1)d$ to determine the n th term

We can make use of the formula $T_n = a + (n - 1)d$ to determine the n th term of a given sequence.

Example

Find the tenth term of the AP 2, 5, 8, 11,

Solution

Here $a = 2$, $d = 3$

The tenth term, $T_{10} = a + 9d = 2 + 9(3) = 29$

Exercise 7.4.1(a)

- | | | |
|---|--|---------------|
| 1 | Find the tenth term of the AP 3, 5, 7, 9, | [Ans : 21] |
| 2 | Find the sixth term of the AP 4, 9, 14, 19, | [Ans : 29] |
| 3 | Find the seventh term of the AP 5, 11, 17, 23, | [Ans : 41] |
| 4 | Find the ninth term of the AP 20, 17, 14, 11, | [Ans : -4] |
| 5 | Find the eighth term of the AP 3, 3.2, 3.4, 3.6, | [Ans : 4.4] |
| 6 | Find the fifth term of the AP 2, 5, 8, 11, | [Ans : 14] |
| 7 | Find the eighteenth term of the AP 10, 8.5, 7, 5.5, | [Ans : -15.5] |
| 8 | Find the twenty fifth term of the AP 4, 7, 10, 13, | [Ans : 76] |

7.4.1(b) Using $T_n = a + (n - 1)d$ to determine the value of n

We can make use of the formula $T_n = a + (n - 1)d$ to determine the number of terms of a given sequence.

Example 1

Find the number of terms in the sequence 4, 7, 10, 13,, 1000.

Solution

Here $a = 4$, $d = 3$

$T_1 = 4$, $T_2 = 7$, $T_3 = 10$

Let $T_n = 1000$

Using $T_n = a + (n - 1)d$

$$1000 = 4 + (n - 1) 3$$

$$996 = 3n - 3$$

$$3n = 999$$

$$n = 333 \text{ (Ans)}$$

Hence the sequence 4, 7, 10, 13,, 1000 has 333 terms.

Example 2

Find the number of terms in the sequence 1, 1.1, 1.2, 1.3,, 9.

Solution

Here $a = 1$, $d = 0.1$

$T_1 = 1$, $T_2 = 1.1$, $T_3 = 1.2$

Let $T_n = 9$

Using $T_n = a + (n - 1)d$

$$9 = 1 + (n - 1) 0.1$$

$$8 = 0.1n - 0.1$$

$$0.1n = 8.1$$

$$n = 81 \text{ (Ans)}$$

Hence the sequence 1, 1.1, 1.2, 1.3,, 9 has 81 terms.

Exercise 7.4.1(b)

- 1 Find the number of terms in the sequence 2, 5, 8,, 68. [Ans : 23]
- 2 Find the number of terms in the sequence 1, 5, 9,, 129. [Ans : 33]
- 3 Find the number of terms in the sequence 5, 5.2, 5.4,, 25. [Ans : 101]
- 4 Find the number of terms in the sequence 60, 55, 50,, -40. [Ans : 21]
- 5 Find the number of terms in the sequence 8, 11, 14,, 80. [Ans : 25]
- 6 Find the number of terms in the sequence 4, 4.5, 5, 5.5,, 50. [Ans : 93]

7.4.1(c) Using $T_n = a + (n-1)d$ to determine the value of a and d

We can make use of the formula $T_n = a + (n-1)d$ to determine the value of the first term and the common difference.

Example 1

For a given AP, the n th term is given by $T_n = 2n + 5$.

Find the value of the first term and the common difference.

Solution

$$T_n = 2n + 5$$

$$T_1 = 2(1) + 5 = 7$$

$$T_2 = 2(2) + 5 = 9$$

$$T_3 = 2(3) + 5 = 11$$

Clearly, $a = T_1 = 7$

and $d = 9 - 7 = 2$

Example 2

For a given AP, the n th term is given by $T_n = 3n - 1$. Find the value of the first term and the common difference. Also calculate the hundredth term.

Solution

$$T_n = 3n - 1$$

$$T_1 = 3(1) - 1 = 2$$

$$T_2 = 3(2) - 1 = 5$$

$$T_3 = 3(3) - 1 = 8$$

Clearly, $a = T_1 = 2$

and $d = 5 - 2 = 3$

Hence $T_{100} = a + 99d = 2 + 99(3) = 299$

Exercise 7.4.1(c)

- 1 For a given AP, the n th term is given by $T_n = 4n + 3$. Find the value of the first term and the common difference. [Ans : $a = 7$, $d = 4$]
- 2 For a given AP, the n th term is given by $T_n = 3n + 5$. Find the value of the first term and the common difference. [Ans : $a = 8$, $d = 3$]
- 3 For a given AP, the n th term is given by $T_n = 2n - 7$. Find the value of a and d . Also calculate the fiftieth term. [Ans : $a = -5$, $d = 2$; 93]
- 4 For a given AP, the n th term is given by $T_n = 10 - 3n$. Find the value of a and d . Also calculate the twentieth term. [Ans : $a = 7$, $d = -3$; -50]
- 5 For a given AP, the n th term is given by $T_n = 5n + 3$. Find the value of a and d . Also calculate the fourteenth term. [Ans : $a = 8$, $d = 5$; 73]

7.4.1(d) Using $T_n = a + (n-1)d$ to determine the value of a and d

We can make use of the formula $T_n = a + (n-1)d$ to determine the value of the first term and the common difference.

Example 1

For a given AP, the fourth term is 11 and the sixth term is 17. Find the value of the first term and the common difference.

Solution

$$\begin{aligned}T_4 = 11 &\Rightarrow a + 3d = 11 \\T_6 = 17 &\Rightarrow a + 5d = 17 \\ \text{Subtracting} &\quad -2d = -6 \\ &\Rightarrow d = 3 \\ &a = 11 - 3(3) = 2\end{aligned}$$

Example 2

For a given AP, the third term is 17 and the fifth term is 9. Find the hundredth term.

Solution

$$\begin{aligned}T_3 = 17 &\Rightarrow a + 2d = 17 \\T_5 = 9 &\Rightarrow a + 4d = 9 \\ \text{Subtracting} &\quad -2d = 8 \\ &\Rightarrow d = -4 \\ &\Rightarrow a = 17 - 2(-4) = 25 \\ \text{Hence } T_{100} = a + 99d &= 25 + 99(-4) = -371.\end{aligned}$$

Exercise 7.4.1(d)

- 1 For a given AP, the fifth term is 18 and the eighth term is 27. Find the value of the first term and the common difference. [Ans : $a = 6$, $d = 3$]
- 2 The fourth term of an AP is 28 and the sixth term is 40. Find the value of the first term and the common difference. [Ans : $a = 10$, $d = 6$]
- 3 For a given AP, the third term is 5.2 and the eleventh term is 6. Find the twenty fifth term. [Ans : 7.4]
- 4 The second term of an AP is 9.2 and the fifth term is 12.8. Find the thirteenth term. [Ans : 22.4]
- 5 For a given AP, the third term is $\frac{3}{4}$ and the eighth term is 2. Find the eighteenth term. [Ans : 4.5]
- 6 The third term of an AP is 12.6 and the seventh term is 13.8. Find the fifteenth term. [Ans : 16.2]

7.4.2 Sum of an AP

For the sequence $T_1, T_2, T_3, T_4, T_5, \dots, T_n, \dots$ if we add up the terms, we will obtain a series $T_1 + T_2 + T_3 + T_4 + T_5 + \dots$

If S_n denotes the sum of the first n terms, we have

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}[a + l], \text{ where } l \text{ is the last term}$$

Example 1

For the AP 2, 5, 8, 11, Find the sum of the first 25 terms.

Solution

Here $a = 2$ and $d = 3$

Using $S_n = \frac{n}{2}[2a + (n-1)d]$, we have

$$\begin{aligned} S_{25} &= \frac{25}{2}[2(2) + 24(3)] \\ &= \frac{25}{2}[4 + 72] = 950 \text{ (Ans)} \end{aligned}$$

Example 2

Find the sum of the AP 1, 4, 7, 10, 13,, 1000.

Solution

First we have to find the number of terms of the above AP.

Here $a = 1$ and $d = 3$

Using $T_n = a + (n-1)d$, we have

$$1000 = 1 + (n-1).3$$

$$1000 = 1 + 3n - 3$$

$$3n = 1002$$

Hence $n = 334$

Since the last term is known, we use $S_n = \frac{n}{2}[a + l]$

$$\therefore S_{334} = \frac{334}{2}[1 + 1000] = 167167 \text{ (Ans)}$$

Example 3

n th term of an AP is given by $T_n = 3n + 10$. Find the sum of the first 15 terms of the AP.

Solution

Now $S_{15} = T_1 + T_2 + T_3 + \dots + T_{15}$

$$T_1 = 3(1) + 10 = 13$$

$$T_{15} = 3(15) + 10 = 55$$

$$\text{Using } S_n = \frac{n}{2}[a+l] \quad \therefore S_{15} = \frac{15}{2}[13+55] = 510 \text{ (Ans)}$$

Example 4

Find the sum of all even integers between 20 and 100 inclusive.

Solution

The integers are 20, 22, 24,, 100.

Here $a = 20$ and $d = 2$

To find the number of terms of the above AP, we use

$$T_n = a + (n-1)d,$$

$$100 = 20 + (n-1)2$$

$$80 = 2(n-1)$$

$$40 = n-1$$

$$\text{Hence } n = 41$$

$$\text{Using } S_n = \frac{n}{2}[a+l]$$

$$\therefore S_{41} = \frac{41}{2} [20+100] = 2460 \text{ (Ans)}$$

Example 5

Find the sum of all integers between 50 and 100 inclusive which are **not** multiples of 3.

Solution

Integers from 50 to 100 are 50, 51, 52, 53,, 100.

Clearly there are 51 terms.

$$\text{Using } S_n = \frac{n}{2}[a+l]$$

$$\therefore S_{51} = \frac{51}{2}[50+100] = 3825$$

Consider those integers which are multiples of 3

The integers are 51, 54, 57,, 99

Using $T_n = a + (n-1)d$, we have

$$99 = 51 + (n-1)3$$

$$48 = 3(n-1)$$

$$16 = n-1$$

$$\text{Hence } n = 17$$

$$\text{Using } S_n = \frac{n}{2}[a+l]$$

$$\therefore S_{17} = \frac{17}{2}[51+99] = 1275$$

Hence the required sum $= 3825 - 1275 = 2550 \text{ (Ans)}$

Exercise 7.4.2

- 1 For the AP 2, 6, 10, find the sum of the first 50 terms. [Ans : 5000]
- 2 For the AP 5, 8, 11, 14.... find the sum of the first 60 terms. [Ans : 5610]
- 3 The n th term of an AP is given by $T_n = 2n + 3$. Find the sum of the first twenty terms. [Ans : 480]
- 4 The n th term of an AP is given by $T_n = 3n + 5$. Find the sum of the first forty terms. [Ans : 2660]
- 5 Find the sum of the AP 1, 1.1, 1.2,, 50. [Ans:12520.5]
- 6 Find the sum of the AP 4, 7, 10, 13,, 499. [Ans: 41749]
- 7 Find the sum of all odd numbers between 11 and 51 inclusive. [Ans : 651]
- 8 Find the sum of all even numbers between 50 and 100 inclusive. [Ans : 1950]
- 9 The second term of an AP is 4 and the fifth term is 13. Find the sum of the first hundred term . [Ans : 14950]
- 10 Given that the first and second term of an AP are 12 and 6 respectively, find the sum of the first hundred terms. (C) [Ans : -28500]
- 11 The first few terms of an AP are 60, 55, 50, Find the sum of the positive terms of the sequence. [Ans : 390]
- 12 Show that the sum of the positive integers less than 100 which do not contain the digit 7 is 3762. (C)
- 13 An arithmetic progression has 241 terms and a common difference of 0.1. Given that the sum of all the terms is 964, find the first term. (C) [Ans : -8]
- 14 An arithmetic series is such that the ninth term is zero and the sum of the first 25 terms is 50. Find the first term of the series. (C) [Ans : -4]
- 15 An AP is such that the sum of the first 8 terms is 156, and the sum of the first 10 terms is 215. Find the fifth term. (C) [Ans : 20.5]
- 16 The sum of the first 6 terms of an AP 55.5 and the sum of the next 6 terms is 145.5. Find the common difference and the first term. (C) [Ans : 20.5]

7.5 Geometric progressions

Consider the sequence 2, 6, 18, 54,

Clearly to obtain each term we have to multiply the preceding term by 3.

The next two terms of the above sequence is 162 and 486.

A sequence of the above type is called an **Geometric progression (GP)**.

The ratio between any two successive terms is called the **common ratio (r)**.

As usual, we use the letter "**a**" to denote the first term.

The following are examples of GP's.

$$1, 3, 9, 27, \dots \quad a = 1, r = 3$$

$$100, 20, 4, 0.8, \dots \quad a = 100, r = \frac{1}{5}$$

$$2, 20, 200, 2000, \dots \quad a = 2, r = 10$$

Now

$$T_1 = a$$

$$T_2 = T_1 \times r = ar$$

$$T_3 = T_2 \times r = ar \times r = ar^2$$

$$T_4 = T_3 \times r = ar^2 \times r = ar^3$$

$$T_5 = T_4 \times r = ar^3 \times r = ar^4$$

Similarly

$$T_{10} = ar^9$$

In general the nth term is given by $T_n = ar^{n-1}$

7.5.1 Use of the formula $T_n = ar^{n-1}$ to determine the nth term

We can make use of the formula $T_n = ar^{n-1}$ to determine the nth term of a given sequence.

Example

Find the eighth term of the GP 2, 4, 8, 16, 32,

Solution

Here $a = 2$, $r = 2$

The tenth term, $T_8 = ar^7 = 2(2)^7 = 256$

Exercise 7.5.1

- | | | |
|---|--|----------------|
| 1 | Find the tenth term of the GP 3, 6, 12, 24, | [Ans : 1536] |
| 2 | Find the sixth term of the GP 4, 12, 36, 108, | [Ans : 972] |
| 3 | Find the seventh term of the GP 100, 50, 25, | [Ans : 1.5625] |
| 4 | Find the ninth term of the GP 1, 3, 9, 27, | [Ans : 6561] |
| 5 | Find the eighth term of the GP 1, $\frac{1}{2}$, $\frac{1}{4}$, | [Ans : 1/128] |
| 6 | Find the fifth term of the GP 10, 20, 40, , | [Ans : 160] |

7.5.2 Using $T_n = ar^{n-1}$ to determine the value of n

We can make use of the formula $T_n = ar^{n-1}$ to determine the number of terms of a given sequence.

Example

Find the number of terms in the sequence 1, 2, 4, 8,, 1024.

Solution

Here $a = 1$, $r = 2$

$T_1 = 1$, $T_2 = 2$, $T_3 = 4$

Let $T_n = 1024$

Using $T_n = ar^{n-1}$

$$1024 = 1(2)^{n-1}$$

$$2^{n-1} = 1024$$

Taking $\ln$ on both sides

$$\ln 2^{n-1} = \ln 1024$$

$$(n-1) \ln 2 = \ln 1024$$

$$n-1 = \frac{\ln 1024}{\ln 2} = 10$$

$$n = 11$$

Hence the sequence 1, 2, 4, 8,, 1024 has 11 terms.

Exercise 7.5.2

- 1 Find the number of terms in the sequence 2, 6, 18,, 1458. [Ans : 7]
- 2 Find the number of terms in the sequence 1, 3, 9, 81,, 2187. [Ans : 8]
- 3 Find the number of terms in the sequence 8, 4, 2, 1,, $\frac{1}{512}$. [Ans : 13]
- 4 Find the number of terms in the sequence 100, 25, $\frac{25}{4}$, ..., $\frac{25}{1024}$. [Ans : 7]
- 5 Find the number of terms in the sequence 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ..., $\frac{1}{2048}$. [Ans : 12]
- 6 Find the number of terms in the sequence 6, 2, $\frac{2}{3}$,, $\frac{2}{6561}$. [Ans : 10]
- 7 Find the number of terms in the sequence 5, 10, 20, 40, ..., 640. [Ans : 8]

7.5.3 Using $T_n = ar^{n-1}$ to determine the value of a and r

We can make use of the formula $T_n = ar^{n-1}$ to determine the value of the first term and the common ratio.

Example 1

For a given GP, the n th term is given by $T_n = 2^{n+1}$.

Find the value of the first term and the common ratio.

Solution

$$T_n = 2^{n+1}$$

$$T_1 = 2^2 = 4$$

$$T_2 = 2^3 = 8$$

Clearly, $a = T_1 = 4$ and $r = \frac{8}{4} = 2$

Example 2

For a given GP, the n th term is given by $T_n = 2(3^{n-1})$. Find the value of the first term and the common ratio. Also calculate the tenth term.

Solution

$$T_n = 2(3^{n-1})$$

$$T_1 = 2(3^0) = 2$$

$$T_2 = 2(3^1) = 6$$

Clearly, $a = T_1 = 2$ and $r = \frac{6}{2} = 3$

Hence $T_{10} = ar^9 = 2(3^9) = 39366$ (Ans)

Exercise 7.5.3

For a given GP, the n th term is given by $T_n = 3^{n+1}$. Find the value of the first term and the common ratio. [Ans : $a = 9, r = 3$]

- 2 The n th term of a GP is given by $T_n = 5(2^{n-1})$. Find the value of the first term and the common ratio. [Ans : $a = 5, r = 2$]

- 3 For a given GP, the n th term is given by $T_n = 3\left(\frac{1}{2}\right)^n$. Find the value of a and r .

Also calculate the tenth term. [Ans : $a = \frac{3}{2}, r = \frac{1}{2}; \frac{3}{1024}$]

- 4 The n th term of a GP is given by $T_n = 2\left(\frac{1}{3}\right)^{n+1}$.

Find the value of the seventh term. [Ans : $a = \frac{2}{9}, r = \frac{1}{3}; \frac{2}{6561}$]

7.5.4 Using $T_n = ar^{n-1}$ to determine the value of a and d

We can make use of the formula $T_n = ar^{n-1}$ to determine the value of the first term and the common ratio.

Example 1

For a given GP, the third term is 18 and the sixth term is 486. Find the value of the first term and the common ratio.

Solution

$$T_3 = 18 \Rightarrow ar^2 = 18$$

$$T_6 = 486 \Rightarrow ar^5 = 486$$

$$\text{Dividing} \quad r^3 = 27$$

$$r = 3$$

$$a(3)^2 = 18$$

$$a = 2$$

Example 2

For a given GP, the fourth term is 8 and the seventh term is 1. Find the fifteenth term.

Solution

$$T_4 = 8 \Rightarrow ar^3 = 8$$

$$T_7 = 1 \Rightarrow ar^6 = 1$$

$$\text{Dividing} \quad r^3 = \frac{1}{8}$$

$$r = \frac{1}{2}$$

$$a\left(\frac{1}{2}\right)^3 = 8$$

$$a = 64$$

$$\text{Hence } T_{15} = 64\left(\frac{1}{2}\right)^{14} = \frac{1}{256}$$

Exercise 7.5.4

- For a given GP, the third term is 16 and the fifth term is 256. Find the value of the first term and the common ratio. [Ans : $a = 1, r = 4$]
- The fourth term of a GP is $\frac{1}{2}$ and the sixth term is $\frac{1}{8}$. Find the value of the first term and the common ratio. [Ans : $a = 4, r = \frac{1}{2}$]
- In a certain GP, the second term is 20 and the fourth term is 500. Show that the first term is 4.
- The first term of a GP is 20 and the third term is 28.8. Find the tenth term. [Ans : 103.2]

7.6 Sum of a GP

For the sequence $T_1, T_2, T_3, T_4, T_5, \dots, T_n, \dots$ if we add up the terms, we will obtain a series $T_1 + T_2 + T_3 + T_4 + T_5 + \dots + T_n$.

If S_n denotes the sum of the first n terms, we have

$$S_n = a \left(\frac{r^n - 1}{r - 1} \right) \rightarrow \text{This formula is used when the common ratio exceeds 1 (i.e. } r > 1)$$

$$\text{or } S_n = a \left(\frac{1 - r^n}{1 - r} \right) \rightarrow \text{This formula is used when the common ratio is less than 1 (i.e. } r < 1)$$

Example 1

For the GP 3, 6, 12, 24, ... find the sum of the first 10 terms.

Solution

Here $a = 3$ $r = 2$

$$\begin{aligned} \text{Since } r > 1, \text{ we use } S_n &= a \left(\frac{r^n - 1}{r - 1} \right) \\ &= 3 \left(\frac{2^{10} - 1}{2 - 1} \right) = 3069 \text{ (Ans)} \end{aligned}$$

Example 2

For the GP 64, 16, 4, $\frac{1}{256}$.

Solution

Here $a = 64$ $r = \frac{1}{4}$

Using $T_n = ar^{n-1}$ to find the number of terms

$$\frac{1}{256} = 64 \left(\frac{1}{4} \right)^{n-1}$$

$$\frac{1}{16384} = \left(\frac{1}{4} \right)^{n-1}$$

$$\ln \frac{1}{16384} = \ln \left(\frac{1}{4} \right)^{n-1}$$

$$n - 1 = \frac{\ln \frac{1}{16384}}{\ln \frac{1}{4}} = 7$$

$$n = 8$$

$$\text{Since } r < 1, \text{ we use } S_n = a \left(\frac{1 - r^n}{1 - r} \right) \Rightarrow S_8 = 64 \left(\frac{1 - \left(\frac{1}{4} \right)^8}{1 - \frac{1}{4}} \right) = 85.3 \text{ (Ans)}$$

Example 3

A GP has first term 6 and common ratio of 1.5. Find the least number of terms which must be taken so that the sum of these terms exceeds 10 000.

Solution

Since $r > 1$, we use $S_n = a \left(\frac{r^n - 1}{r - 1} \right)$

$$S_n = 6 \left(\frac{1.5^n - 1}{1.5 - 1} \right) > 10000$$

$$12(1.5^n - 1) > 10000$$

$$1.5^n > 833.3$$

Taking $\ln$ on both sides

$$\ln 1.5^n > \ln 833.3$$

$$n \ln 1.5 > \ln 833.3$$

$$n > 16.6$$

Hence $n = 17$ (Ans)

Exercise 7.6

- 1 For the GP 4, 8, 16, 32, find the sum of the first 12 terms. [Ans : 16380]
- 2 For the GP 50, 25, 12.5, find the sum of the first 9 terms. [Ans : 99.8]
- 3 The n th term of an GP is given by $T_n = 3^{n+2}$. Find the sum of the first seven terms. [Ans : 29511]
- 4 Find the sum of the GP 2, 1.2, 0.72,, 0.15552. [Ans: 4.76672]
- 5 The fourth term of a GP is 24 and the sixth term is 96. Find the sum of the first ten term. [Ans : 3069]
- 6 The sum of the first three terms of a GP is 117 and the sum of the next three terms is $\frac{13}{3}$. Calculate the third term. [Ans : 9]
- 7 Find the least number of terms of the GP 4, 8, 16, which must be taken for their sum to exceed 500. [Ans : 7]
- 8 A GP has first term 5 and last term 2560. If the sum of all its terms is 5115, how many terms are there? [Ans : 10]
- 9 The sum of the first 4 terms of a GP is 40. Its third term exceeds the first term by 8. If the first term is greater than 1, calculate the sum of the first six term. [Ans : 168]

7.7 Sum to infinity (S_{∞}) of a GP

Consider the formula $S_n = a \left(\frac{1-r^n}{1-r} \right)$. Basically it is being used when the value of r is such that $-1 < r < 1$. As $n \rightarrow \infty$, $r^n \rightarrow 0$.

Thus we have $S_{\infty} = \frac{a}{1-r}$

Hence the sum to infinity of a GP exists provided $-1 < r < 1$.

Example 1

For the GP 4, 2, 1, ... find the sum to infinity.

Solution

Here $a = 4$, $r = \frac{1}{2}$

$$S_{\infty} = \frac{a}{1-r} = \frac{4}{1-\frac{1}{2}} = 8$$

Example 2

A GP has first term a and a common ratio of $\frac{1}{\sqrt{2}}$. Show that the sum to infinity of the series is given by $a(2 + \sqrt{2})$. (C)

Solution

Here $a = a$, $r = \frac{1}{\sqrt{2}}$

$$\begin{aligned} S_{\infty} &= \frac{a}{1-r} = \frac{a}{1-\frac{1}{\sqrt{2}}} = \frac{a}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\sqrt{2}a}{\sqrt{2}-1} \\ &= \frac{\sqrt{2}a}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} \\ &= \frac{a(2+\sqrt{2})}{1} = a(2 + \sqrt{2}) \text{ (shown)} \end{aligned}$$

Example 3

The second term of a GP is 2 while its sum to infinity is 8. Find the sum of the first 8 terms.

Solution

$$\text{Given } T_2 = ar = 2 \quad \dots (1)$$

$$\text{and } S_{\infty} = \frac{a}{1-r} = 8 \quad \dots (2)$$

$$\text{From (1)} \quad a = \frac{2}{r}$$

$$\text{Substituting in (2), we have } \frac{2/r}{1-r} = 8$$

$$\Rightarrow \frac{2}{r} = 8(1-r)$$

$$\begin{aligned} &\Rightarrow 1 - 4r - 4r^2 \\ &\rightarrow 4r^2 - 4r + 1 = 0 \\ &\rightarrow (2r - 1)(2r - 1) = 0 \\ &\therefore r = \frac{1}{2} \text{ and } a = \frac{2}{1/2} = 4 \end{aligned}$$

$$S_8 = 4 \left(\frac{1 - \left(\frac{1}{2}\right)^8}{1 - \frac{1}{2}} \right) = 7 \frac{31}{32} \text{ (Ans)}$$

Exercise 7.7

- 1 Find the sum to infinity for the following GP.
 - (a) $1, \frac{1}{2}, \frac{1}{4}, \dots$ [Ans : 2]
 - (b) $6, 2, \frac{2}{3}, \frac{2}{9}$ [Ans : 9]
 - (c) $16, 4, 1, \frac{1}{4}$ [Ans : $\frac{64}{3}$]
 - (d) $25, 5, 1, \frac{1}{5}$ [Ans : $\frac{125}{4}$]
- 2 Find the common ratio of a GP whose first term is 18 and whose sum to infinity is 42. [Ans : $\frac{4}{7}$]
- 3 The first term of a GP is a and the common ratio is r . Given that $a + 96r = 0$ and that the sum to infinity is 32, find the eighth term. [Ans : -0.375]
- 4 The second term of a GP is 24 and its sum to infinity is 100. Find the possible values of the common ratio. [Ans : $\frac{2}{5}, \frac{3}{5}$]
- 5 An infinite GP has a finite sum of 256. The sum of the first three terms is 224. What is the value of the third term? [Ans : 32]
- 6 In a certain GP, the second term is 32 and the fifth term is 4. The sum of the first 4 terms is $x\%$ of the sum to infinity. Find the value of x . [Ans : 93.75]
- 7 Find the least value of n for which the sum to n terms of the geometric series $1 + 0.98 + (0.98)^2 + \dots$ is greater than half the sum to infinity. [Ans: 35]

CHAPTER 8

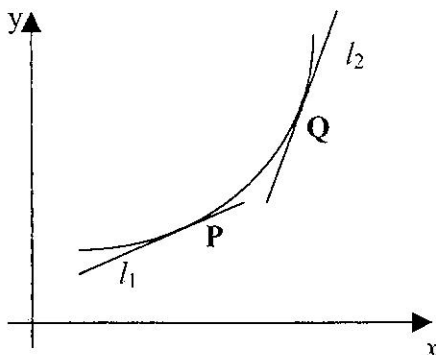
DIFFERENTIATION 1

8.1 Gradient of a curve

We know that the gradient of a line from a point $P(x_1, y_1)$ to $Q(x_2, y_2)$ is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient of a straight line is constant throughout and is equal to the tangent of the angle which the line makes with the positive direction of the x -axis.



When a line touches a curve without cutting it, the line is said to be the tangent to the curve.

In the figure, the line l_1 is said to be the tangent to the curve at the point P and the line l_2 is said to be the tangent to the curve at the point Q.

A curve is not a straight line and its gradient is **not constant**. In fact the gradient of a curve changes continuously.

We define the gradient of a curve at a point P to be the gradient of the tangent drawn at the point P.

The gradient of a point P is found by drawing a tangent to the curve at the point P. The gradient at Q is found by drawing a tangent to the curve at the point Q. However, the gradients obtained by drawing tangents are only **approximate** values.

We can obtain the exact value of gradients by using the method of **differentiation**.

8.2 Derivative of ax^n

If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$, where a is a constant and n is an integer or a rational number.

Example 1

Find the derivative of the following

(a) $5x^3 + 7x^2 + 1$ (b) $12x^2 - \frac{6}{x^3}$

Solution

(a) Let $y = 5x^3 + 7x^2 + 1$

$$\frac{dy}{dx} = 15x^2 + 14x$$

(b) Let $y = 12x^2 - \frac{6}{x^3} = 12x^2 - 6x^{-3}$

$$\frac{dy}{dx} = 24x + 18x^{-4} = 24x + \frac{18}{x^4}$$

Example 2

Find the derivative of the following

(a) $\frac{4}{\sqrt{x}}$ (b) $\frac{6x^5 + 5x}{x^4}$

Solution

(a) Let $y = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = -2x^{-\frac{3}{2}} = -\frac{2}{x^{\frac{3}{2}}}$$

(b) Let $y = \frac{6x^5 + 5x}{x^4} = 6x + 5x^{-3}$

$$\frac{dy}{dx} = 6 - 15x^{-4} = 6 - \frac{15}{x^4}$$

Example 3

Given that $f(x) = x + \frac{4}{x^2}$, find $f'(x)$ and $f''(x)$.

Solution

$$f(x) = x + \frac{4}{x^2} = x + 4x^{-2}$$

$$f'(x) = 1 - 8x^{-3} = 1 - \frac{8}{x^3}$$

and $f''(x) = 24x^{-4} = \frac{24}{x^4}$.

Example 4

If $y = 2x^2 + 3\sqrt{x} - \frac{1}{x^3}$, find $\frac{d^2y}{dx^2}$.

Solution

$$y = 2x^2 + 3\sqrt{x} - \frac{1}{x^3}$$

$$= 2x^2 + 3x^{\frac{1}{2}} - x^{-3}$$

$$\therefore \frac{dy}{dx} = 4x + \frac{3}{2}x^{-\frac{1}{2}} + 3x^{-4}$$

$$= 4x + \frac{3}{2\sqrt{x}} + \frac{3}{x^4}$$

$$\begin{aligned}\text{Now } \frac{d^2y}{dx^2} &= 4 - \frac{3}{4}x^{-\frac{3}{2}} - 12x^{-5} \\ &= 4 - \frac{3}{4x^{\frac{3}{2}}} - \frac{12}{x^5} \quad (\text{Ans})\end{aligned}$$

Exercise 8.2

Differentiate the following with respect to x .

1 $3x^2 + 6x$ [Ans : $6x + 6$]

2 $8x^3 + 7x^2 - 3x$ [Ans : $24x^2 + 14x - 3$]

3 $x^2 + \frac{1}{x}$ [Ans : $2x - \frac{1}{x^2}$]

4 $\frac{4x^4 + 3x^3}{x^2}$ [Ans : $8x + 3$]

5 $\frac{6x^5 - 4x^3}{2x^3}$ [Ans : $6x$]

6 $2x + \sqrt{x}$ [Ans : $2 + \frac{1}{2\sqrt{x}}$]

7 $x^3 - \frac{2}{\sqrt{x}}$ [Ans : $3x^2 + \frac{1}{x^{\frac{3}{2}}}$]

8 $4x^2 + 3\sqrt{x}$ [Ans : $8x + \frac{3}{2\sqrt{x}}$]

9 $\sqrt[3]{8x}$ [Ans : $\frac{2}{3x^{\frac{2}{3}}}$]

10 Given that $f(x) = 5x^3 - \frac{3}{x} + 2$, find $f'(x)$ and $f''(x)$. [Ans : $15x^2 + \frac{3}{x^2}$; $30x - \frac{6}{x^3}$]

11 Given that $f(x) = 4x + \frac{16}{x^2}$, find $f'(x)$ and $f''(x)$. [Ans : $4 - \frac{32}{x^3}$; $\frac{96}{x^4}$]

12 If $y = \frac{2x^2 + 3}{x}$, prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = y$.

13 $y = 3x^2 + 4x$, prove that $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$.