Polynomials

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SYLLABUS (6042) REQUIREMENTS

- Carry out polynomial operations of addition, subtraction, multiplication and division
- Complete the square of a quadratic polynomial
- Deriving factor and remainder theorems
- use the remainder and factor theorems in solving problems

POLYNOMIALS

Definition: A polynomial of degree n is any function that is defined by an equation of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $(a_n, a_{n-1}, \ldots a_1, a_0) \in \Re$.

- The constants $(a_n, a_{n-1}, \ldots a_1, a_0)$ are sometimes called the coefficients of the polynomial
- a_0 is called the constant/ independent term
- The leading term is the term involving the highest power of x, here $a_n x^n$.
- The degree is the power of x in the leading term.
- Polynomials are often written in order from the highest term degree to the lowest (Descending Order)
- For small degree polynomials, we use the following names:
  - (i) a polynomial of degree 1 is called linear e.g. $3x + 1$
  - (ii) a polynomial of degree 2 is called a quadratic e.g. $x^2 - 2x + 7$
  - (iii) a polynomial of degree 3 is called a cubic e.g. $2x^3 + 5x - 1$
  - (iv) a polynomial of degree 4 is called a quartic e.g. $x^4 + x^3 - 6x^2 + 2x + 4$
  - (v) a polynomial of degree 5 is called a quintic e.g. $6x^5 + 14x^4 - 8x^3 - 11$

NB: When powers of x are negative or fractions then it is NOT a polynomial e.g. $x^2 - \sqrt{x}$ or $5x^{-3} + 2x$

POLYNOMIAL OPERATIONS

ADDITION AND SUBTRACTION

- Adding and subtracting polynomials is the same as the procedure used in combining like terms.
- When adding polynomials, simply drop the parenthesis and combine like terms. When subtracting polynomials, distribute the negative first, and then combine like terms.
- Polynomials can be added or subtracted simply by adding or subtracting the corresponding terms using the distributive law

Example

If $a = 2x^2 - 3x + 2$ and $b = 5x^2 + 8x$, find:

(i) $b - 2a$

(ii) $a + 2b - 1$

Suggested Solution

(i) $b - 2a = (5x^2 + 8x) - 2(2x^2 - 3x + 2)$

$= 5x^2 + 8x - 4x^2 + 6x - 4$

$= x^2 + 14x - 4$

(ii) $a + 2b - 1 = (2x^2 - 3x + 2) + 2(5x^2 + 8x) - 1$

$= 2x^2 - 3x + 2 + 10x^2 + 16x - 1$

$= 12x^2 + 13x + 1$

MULTIPLICATION

- To multiply two polynomials, every term of the first polynomial must be multiplied by every term of the second.
- NB: (i) A monomial is a one-term polynomial.

  (ii) A binomial is a two-term polynomial.

Case 1

Monomial times Monomial: To multiply a monomial times a monomial, just multiply the numbers then multiply the variables using the rules for exponents. Use the distributive property.

Example

If $a = 2x^2$ and $b = 8x$, find:

(i) ab

(ii) $-2ab^2$

Suggested Solution

(i) $ab = (2x^2)(8x)$

$= 16x^3$

(ii) $-2ab^2 = -2(2x^2)(8x^2)$

$= -2(16x^4)$

$= -32x^4$

Case 2

Monomial times Polynomial: Simply use the distributive property to multiply a monomial times a polynomial.

Example

If $a = 2x$ and $b = x^2 - 2x + 1$, find:

(i) ab

(ii) a^2b

Suggested Solution

(i) $ab = 2x(x^2 - 2x + 1\ )$

$= 2x^3 - 4x^2 + 2x$

(ii) $a^2b = (2x)^2(x^2 - 2x + 1\ )$

$= 4x^2(x^2 - 2x + 1\ )$

$= 4x^4 - 8x^3 + 4x^2$

Case 3

Binomial times a Binomial: To multiply two binomials, use the FOIL method (First times first, Outside times outside, Inside times inside, and Last times last).

NB: Special Products: The following formulas may be used in these special cases as a short cut to the FOIL method.

(i) Difference of Squares: $x^2 - y^2 = (x - y)(x + y)$

(ii) Perfect Squares: $(x + y)^2 = x^2 + 2xy + y^2$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(First\ term)^2 + 2 \times First\ term \times Last\ term + (Last\ term)^2$$

Example

If $a = x + 2,\ b = x - 2$ and $c = 2x + 1$, find:

(i) ac

(ii) ab

(iii) c^2

Suggested Solution

(i) $ac = (x + 2)(2x + 1)$

$= x(2x + 1) + 2(2x + 1)$

$= 2x^2 + x + 4x + 2$

$= 2x^2 + 5x + 2$

(ii) $ab = (x + 2)(x - 2)$

$= (x)^2 - (2)^2$

$= x^2 - 4$

(iii) $c^2 = (2x + 1)^2$

$= (2x)^2 + 2(2x)(1) + (1)^2$

$= 4x^2 + 4x + 1$

Case 4

Polynomial times polynomial: To multiply two polynomials where at least one has more than two terms, distribute each term in the first polynomial to each term in the second.

Example

If $a = 2x^2 - 3x + 2$ and $b = 5x^2 + 8x + 1$, find $2ba$

Suggested Solution

$$2ba = 2(5x^2 + 8x + 1)(2x^2 - 3x + 2)$$
$$= 2[5x^2(2x^2 - 3x + 2) + 8x(2x^2 - 3x + 2) + 1(2x^2 - 3x + 2)]$$
$$= 2(10x^4 - 15x^3 + 10x^2 + 16x^3 - 24x^2 + 16x + 2x^2 - 3x + 2)$$
$$= 2(10x^4 + x^3 - 12x^2 + 13x + 2)$$
$$= 20x^4 + 2x^3 - 24x^2 + 26x + 4$$

DIVISION

Case 1

Division by Monomial: Each term of the polynomial is divided by the monomial and it is simplified as individual fractions.

Example

If $a = 4x^2 + 6x$ and $b = 2x$, find:

(i) $\frac{a}{b}$

(ii) $\frac{b^2}{3}$

Suggested Solution

(i) $\frac{a}{b} = \frac{4x^2+6x}{2x}$

$$= \frac{4x^2}{2x} + \frac{6x}{2x}$$
$$= 2x + 3$$

(ii) $\frac{b^2}{3} = \frac{(2x)^2}{3}$

$= \frac{4}{3}x^2$

Case 2

Division by Binomial or Larger Polynomial

a) If the degree of the numerator and the denominator are the same we can use the algebraic manipulation algorithm i.e.

Example 1

Simplify $\frac{x+3}{x+5}$.

Suggested Solution

$$\frac{x+3}{x+5} = \frac{x+5-2}{x+5}$$

$$= \frac{x+5}{x+5} - \frac{2}{x+5}$$

$$= 1 - \frac{2}{x+5}$$

Example 2

Simplify $\frac{x^2-2x}{x^2+8x+15}$

Suggested Solution

$$\frac{x^2-2x}{x^2+8x+15} \equiv \frac{x^2+8x-10x+15-15}{x^2+8x+15}$$

$$\equiv \frac{x^2 + 8x + 15 \ - 10x - 15}{x^2 + 8x + 15}$$

$$\equiv \frac{x^2 + 8x + 15}{x^2 + 8x + 15} - \frac{(10x + 15)}{x^2 + 8x + 15}$$

$$\equiv 1 - \frac{(10x + 15)}{x^2 + 8x + 15}$$

b) If degrees are different it's wise to use the long division

Long Division of Polynomials – Steps

(i) Arrange the terms of both the dividend and the divisor in descending powers of any variable.

(ii) Divide the first term in the dividend by the first term in the divisor. The result is the first term of the quotient.

(iii) Multiply every term in the divisor by the first term in the quotient. Write the resulting product beneath the dividend with like terms lined up.

(iv) Subtract the product from the dividend.

(v) Bring down the next term in the original dividend and write it next to the remainder to form a new dividend.

(vi) Use this new expression as the dividend and repeat this process until the remainder can no longer be divided. This will occur when the degree of the remainder (the highest exponent on a variable in the remainder) is less than the degree of the divisor.

Solved Problems

Example

Simplify $\frac{x+3}{x+5}$.

Suggested Solution

Divisor

Quotient

$$
\begin{array}{r|l}
 & \quad 1 \\ \hline
x+5 & \quad x+3 \\
 & -\ \ x+5 \\ \hline
 & \quad\ -2
\end{array}
$$

Remainder

$$\therefore \frac{x+3}{x+5} = 1 - \frac{2}{x+5}$$

Example 2

Simplify $\frac{3x^3-2x^2-16x+20}{x^2-4}$.

Suggested Solution

Divisor

Quotient

$$
\begin{array}{r|l}
 & \quad 3x - 2 \\ \hline
x^2-4 & 3x^3 - 2x^2 - 16x + 20 \\
- & 3x^3 \qquad\quad - 12x \\ \hline
 & \quad\ -2x^2 - 4x + 20 \\
- & \quad\ - 2x^2 \qquad + 8 \\ \hline
 & \qquad\quad -4x + 12
\end{array}
$$

Remainder

$$\therefore \frac{3x^3 - 2x^2 - 16x + 20}{x^2 - 4} \equiv 3x - 2 + \frac{(12 - 4x)}{x^2 - 4}.$$

NB: Any polynomial can be written in the following form:

$$Polynomial = Divisor \times Quotient + Remainder$$

Remainder Theorem and Factor Theorem

Remainder Theorem

When a polynomial $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$.

Example

Simplify $\frac{2x^2-16x+18}{x+2}$.

Suggested Solution

Divisor

Quotient

$$\begin{array}{r|l} & \quad 2x - 20 \\ \hline x + 2 & 2x^2 - 16x + 18 \\ & -\ 2x^2 + 4x \\ \hline & \qquad -20x + 18 \\ & -\quad -20x - 40 \\ \hline & \qquad\qquad 58 \end{array}$$

Remainder

NOTE: $f(x) = 2x^2 - 16x + 18$ and the divisor is $(x + 2)$

Now:

$$f(-2) = 2(-2)^2 - 16(-2) + 18$$
$$= 2(4) + 32 + 18$$
$$= 8 + 50$$
$$= 58 \text{ (Remainder)}$$

Remember that any polynomial can be written in the following form:

$$Polynomial = Divisor \times Quotient + Remainder$$

i.e. $f(x) = (x + 2) \times (2x - 20) + 58$

$$= (x + 2)(2x - 20) + 58$$
$$= 2x^2 - 20x + 4x - 40 + 58$$
$$= 2x^2 - 16x + 18$$

Factor Theorem

If the remainder is zero, we say that the polynomial is exactly divisible by the divisor, or that the divisor is a factor of the polynomial. When a polynomial $f(x)$ is divided by $(x - a)$, and if $(x - a)$ is a factor of $f(x)$ then $f(c) = 0$.

NB: The Factor theorem is useful in factorisation of polynomials

Example

Simplify $\frac{2x^2+x-6}{x+2}$.

Suggested Solution

Divisor Quotient

$$
\begin{array}{r|l}
 & 2x - 3 \\
\hline
x + 2 & 2x^2 + x - 6 \\
 & -\ 2x^2 + 4x \\
\hline
 & \quad -3x - 6 \\
 & -\quad -3x - 6 \\
\hline
 & \qquad\quad 0
\end{array}
$$

Remainder

NOTE: $f(x) = 2x^2 + x - 6$ and the divisor is $(x + 2)$

Now:

$f(-2) = 2(-2)^2 + (-2) - 6$

$= 2(4) - 2 - 6$

$= 8 - 8$

$= 0$ (Remainder)

Remember that any polynomial can be written in the following form:

$$Polynomial = Divisor \times Quotient + Remainder$$

i.e. $f(x) = (x + 2) \times (2x - 3) + 0$

$= (x + 2)(2x - 3) + 0$

$= 2x^2 - 3x + 4x - 6$

$= 2x^2 + x - 6$

Solved Past Examination Questions

Question 1

UCLES NOVEMBER 1997 PAPER 1

The cubic polynomial $x^3 + ax^2 - 3x - 2$, where a is a constant, is denoted by $f(x)$. Given that $(x + 1)$ is a factor of $f(x)$,

(a) find the value of a, **[2]**

(b) factorise $f(x)$ completely. **[2]**

Suggested Solution

(i) $f(x) = x^3 + ax^2 - 3x - 2$ and the divisor is $(x + 1)$

Now: $f(-1) = 0$

$\Rightarrow (-1)^3 + a(-1)^2 - 3(-1) - 2 = 0$

$-1 + a + 3 - 2 = 0$

$-3 + a + 3 = 0$

$\therefore a = 0$

(ii)

$$
\begin{array}{r|l}
 & x^2 - x - 2 \\
\hline
x + 1 & x^3 \qquad\quad - 3x - 2 \\
 & -\ (x^3 + x^2) \\
\hline
 & \quad -x^2 - 3x - 2 \\
 & -\ (-x^2 - x) \\
\hline
 & \qquad\quad -2x - 2 \\
 & -\ (-2x - 2) \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

Now:

$f(x) = (x + 1)(x^2 - x - 2)$

$= (x+1)(x^2 - 2x + x - 2)$

$= (x+1)(x+1)(x-2)$

Question2

UCLES NOVEMBER 2001 PAPER 1

The cubic polynomial $2x^3 + 5x^2 + ax - 6$, where a is a constant, is denoted by $f(x)$.

Given that $(x+2)$ is a factor of $f(x)$, find the value of a. **[2]**

When a has this value, find all the roots of the equation $f(x) = 0$. **[2]**

Suggested Solution

$f(x) = 2x^3 + 5x^2 + ax - 6$ and the divisor is $(x+2)$

Now: $f(-2) = 0$

$\Rightarrow 2(-2)^3 + 5(-2)^2 + a(-2) - 6 = 0$

$-16 + 20 - 2a - 6 = 0$

$4 - 2a - 6 = 0$

$-2a = 2$

$\therefore a = -1$

$$
\begin{array}{r|l}
 & 2x^2 + x - 3 \\
\hline
x+2 & 2x^3 + 5x^2 - x - 6 \\
- & \underline{2x^3 + 4x^2} \\
 & x^2 - x - 6 \\
- & \underline{x^2 + 2x} \\
 & -3x - 6 \\
- & \underline{-3x - 6} \\
 & 0
\end{array}
$$

Now:

$f(x) = (x+2)(2x^2 + x - 3) = 0$

$\Rightarrow (x+2)(2x^2 + 3x - 2x - 3) = 0$

$(x+2)(x-1)(2x+3) = 0$

$\therefore x = -2; 1 \; or -\dfrac{3}{2}$

Question 3

ZIMSEC NOVEMBER 2014 PAPER 1

The remainder when $4x^4 - 5x^2 - 13x + 3$ is divided by $x + b$ is equal to the square of the remainder when $2x^2 - 3$ is divided by $x + b$.

Calculate the possible values of b, correct to 2 decimal places. **[4]**

Suggested Solution

$f(x) = 4x^4 - 5x^2 - 13x + 3$ and the divisor is $(x + b)$

Now the remainder is $f(-b)$.

$\Rightarrow f(-b) = 4(-b)^4 - 5(-b)^2 - 13(-b) + 3$

$\quad\quad\quad = 4b^4 - 5b^2 + 13b + 3$

and

$g(x) = 2x^2 - 3$ and the divisor is $(x + b)$

Now the remainder is $g(-b)$.

$\Rightarrow g(-b) = 2(-b)^2 - 3$

$\quad\quad\quad = 2b^2 - 3$

Also

$[g(-b)]^2 = (2b^2 - 3)^2$

$\quad\quad\quad = 4b^4 - 12b^2 + 9$

We are told that $[g(-b)]^2 = f(-b)$

$\Rightarrow 4b^4 - 12b^2 + 9 \; = 4b^4 - 5b^2 + 13b + 3$

$\Rightarrow 12b^2 - 5b^2 + 13b + 3 - 9 \; = 0$

$\Rightarrow 7b^2 + 13b - 6 \; = 0$

$$\Rightarrow b^2 + \frac{13}{7}b = \frac{6}{7}$$

$$\Rightarrow \left(b + \frac{13}{14}\right)^2 = \frac{6}{7} + \left(\frac{13}{14}\right)^2$$

$$\Rightarrow \left(b + \frac{13}{14}\right)^2 = \frac{337}{196}$$

$$\Rightarrow b + \frac{13}{14} = \pm\sqrt{\frac{337}{196}}$$

$$\Rightarrow b = -\frac{13}{14} \pm \sqrt{\frac{337}{196}}$$

$$\Rightarrow b = -\frac{13}{14} + \sqrt{\frac{337}{196}} \; or \; -\frac{13}{14} - \sqrt{\frac{337}{196}}$$

$$\Rightarrow b = 0.382682839 \; or -2.239825696$$

$$\therefore b = 0.38 \; or -2.24 \;\; (to\ 2\ decimal\ places)$$

Question 4

ZIMSEC NOVEMBER 2015 PAPER 1

When a polynomial $f(x)$ is divided by $(x-3)$ the remainder is -9 and when divided by $(2x-1)$, the remainder is -6.

Find the remainder when $f(x)$ is divided by $(x-3)(2x-1)$. **[4]**

Suggested Solution

Any polynomial can be written in the following form:

$$Polynomial = Divisor \times Quotient + Remainder$$

Let:

- The remainder be a linear function i.e. $ax + b$
- The quotient is $Q(x)$ and
- The divisor is $(x-3)(2x-1)$

Now the function is given by:

$f(x) = (x-3)(2x-1) \times Q(x) + (ax+b)$

Also

$f(3) = -9$

$\Rightarrow (3-3)(2\times 3 - 1) \times Q(3) + (3a+b) = -9$

$\Rightarrow 3a + b = -9$ (i) and

$f\left(\frac{1}{2}\right) = -6$

$\Rightarrow (x-3)\left(2\times\frac{1}{2} - 1\right) \times Q\left(\frac{1}{2}\right) + \left(\frac{1}{2}a + b\right) = -6$

$\Rightarrow \frac{1}{2}a + b = -6$

$\Rightarrow a + 2b = -12$ (ii)

$\Rightarrow 3a + 6b = -36$ (ii)

Solving (i) and (ii) simultaneously:

$3a + b = -9$ (i)

$\underline{3a + 6b = -36}$ (ii)

$-5b = 27$

$\therefore b = -\frac{27}{5}$

$3a + b = -9$ (i)

$3a - \frac{27}{5} = -9$

$3a = -\frac{18}{5}$

$\therefore a = -\frac{6}{5}$

$\therefore$ The remainder is equal to $-\frac{6}{5}x - \frac{27}{5}$

Question 5

ZIMSEC JUNE 2018 PAPER 1

The polynomials $f(x) = 2x^3 + 7x^2 + ax + b$ and $g(x) = x^3 + ax^2 - 5x + 2b$ have a common factor $(x + 3)$.

Find

(a) the value of a and the value of b, **[4]**

(b) a second common factor. **[3]**

Suggested Solution

(a) $f(x) = 2x^3 + 7x^2 + ax + b$ and the divisor is $(x + 3)$

Now: $f(-3) = 0$

$\Rightarrow 2(-3)^3 + 7(-3)^2 + a(-3) + b = 0$

$-54 + 63 - 3a + b = 0$

$-3a + b = -9$ (i)

Also

$g(x) = x^3 + ax^2 - 5x + 2b$ and the divisor is $(x + 3)$

Now: $g(-3) = 0$

$\Rightarrow (-3)^3 + a(-3)^2 - 5(-3) + 2b = 0$

$-27 + 9a + 15 + 2b = 0$

$9a + 2b = 12$ (ii)

Now:

$-3a + b = -9$ (i) $\times 3$

$\Rightarrow -9a + 3b = -27$ (i)

$\Rightarrow 9a + 2b = 12$ (ii)

Solving (i) and (ii) simultaneously:

$-9a + 3b = -27$ (i)

$\underline{9a + 2b = 12}$ (ii)

$5b = -15$

$\therefore b = -3$

$9a + 2b = 12$ (ii)

$9a + 2(-3) = 12$

$9a = 18$

$\therefore a = 2$

(b)

$f(x) = 2x^3 + 7x^2 + 2x - 3$

$$
\begin{array}{r|l}
 & 2x^2 + x - 1 \\
\hline
x + 3 & 2x^3 + 7x^2 + 2x - 3 \\
 & -\ \underline{2x^3 + 6x^2} \\
 & \qquad x^2 + 2x - 3 \\
 & -\ \underline{\qquad x^2 + 3x} \\
 & \qquad\qquad -x - 3 \\
 & -\ \underline{\qquad\qquad -x - 3} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Now:

$f(x) = (x+3)(2x^2 + x - 1)$

$= (x+3)(2x^2 + 2x - x - 1)$

$(x+2)(x+1)(2x-1)$

and

$g(x) = x^3 + 2x^2 - 5x - 6$

$$
\begin{array}{r|l}
 & x^2 - x - 2 \\
\hline
x + 3 & x^3 + 2x^2 - 5x - 6 \\
 & -\ \ x^3 + 3x^2 \\
\hline
 & \quad -x^2 - 5x - 6 \\
 & -\quad -x^2 - 3x \\
\hline
 & \qquad -2x - 6 \\
 & -\qquad -2x - 6 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

Now:

$g(x) = (x + 3)(x^2 - x - 2)$

$\quad = (x + 3)(x^2 - 2x + x - 2)$

$\quad = (x + 3)(x + 1)(x - 2)$

$\therefore$The other common factor is $(x + 1)$

Past Examination Question

UCLES JUNE 1997 PAPER 1

One root of the equation $x^3 + kx + 11 = 0$, where k is a constant, is 1. Find the value of k.

[2]

Hence find the other two roots of the equation, giving your answers in exact form. **[4]**

UCLES JUNE 1998 PAPER 1

The polynomial $f(x)$ is given by $f(x) = x^3 - \boldsymbol{k}x^2 + k\boldsymbol{x} - 1 = 0$, where k is a constant.

(i) Use factor theorem to obtain a linear factor of $f(x)$. **[2]**

Hence obtain a quadratic factor of $f(x)$, and find the set of values of k for which all the roots of the equation $f(x) = 0$ are real. **[5]**

(ii) For the case $k = 3$, the graph of $y = f(x)$ is a transformation of the graph of $y = x^3$. Describe this transformation fully. **[2]**

UCLES JUNE 2000 PAPER 1

Show that $x + 3$ is a factor of $x^3 + 2x^2 + \boldsymbol{x} + 12$. **[2]**

UCLES NOVEMBER 2000 PAPER 1

Use factor theorem to show that $x + 2$ is a factor of $x^3 + ax^2 + 2ax + 8$ for all values

of the constant a. **[2]**

Hence solve the equation

$$x^3 + 7x^2 + 14x + 8 = 0.$$ **[5]**

ZIMSEC NOVEMBER 2004 PAPER 1

The cubic polynomial $f(x)$ is given by $f(x) = 2x^3 + 6x^2 + kx + 12$, where k is a constant,. The function $f(x)$ leaves a remainder of 6 when divided by $x + 2$.

Find

(a) the value of a, **[2]**

(b) the solution of the equation $f(x) = 9$, given that it has only one real root. **[5]**

ZIMSEC JUNE 2006 PAPER 1

Given that the function $f(x) = 2x^3 + ax^2 - 11x + 6$ is divisible by $(x + 2)$, find the value of a. **[2]**

With this value of a, show that $f(x)$ is also divisible by $(2x - 1)$ and $(x - 3)$. **[2]**

Hence, or otherwise find $f(x - 2)$ in factor form. **[2]**

ZIMSEC JUNE 2013 PAPER 1

When a polynomial $f(x) = x^3 + ax^2 + bx + c$ is divided by $x^2 - 4$ the remainder is $2x + 11$. If $x + 1$ is a factor of $f(x)$, find the values of a, b and c. **[5]**

ZIMSEC JUNE 2015 PAPER 1

a) Given that $f(x) = -x^3 + 2x^2 + 3x - 6$,

(i) factorise $f(x)$ completely,

(ii) sketch the curve $y = f(x)$ showing all intersections with the axes.

[6]

[You need not to find the turning points]

b) Hence or otherwise, write down the solution of the inequality $f(x) > 0$. **[2]**

ZIMSEC NOVEMBER 2016 PAPER 1

The polynomial $2x^3 + ax^2 - bx + 12$, where a and b are constants, is denoted by $f(x)$. The result of differentiating $f(x)$ with respect to x is denoted by $f'(x)$. It is given that $(x - 1)$ is a factor of $f(x)$ and when $f'(x)$ is divided by $(x - 1)$ the remainder is -5.

(a) Find the value of a and b. **[5]**

(b) When a and b have these values in (**i**), solve the equation $f(x) = 0$. **[4]**

ZIMSEC JUNE 2017 PAPER 1

If $f(x) = 3x^3 + x^2 - 8x + 4$, factorise $f(x)$ completely. **[4]**

ZIMSEC NOVEMBER 2017 PAPER 1

It is give that $g(x) = 3x^4 + bx^3 + cx^2 - 7x - 4$ has factors $(x + 1)$ and $(x - 1)$.

(i) Find the value of b and the value of c. **[2]**

(ii) Factorise $g(x)$ completely. **[3]**

ZIMSEC NOVEMBER 2019 PAPER 1

The equation $x^4 + ax^3 + bx^2 - 2x - 4 = 0$, has roots -2 and 1.

(i) Find the values of a and b. **[4]**

(ii) Hence find the other roots. **[4]**

ANALYSIS OF QUADRATIC POLYNOMIALS

- A quadratic polynomial is of the form $f(x) = ax^2 + bx + c$
- $ax^2 + bx + c = 0$ is called a quadratic equation.
- A quadratic equation is solved using any of the following methods: Factorisation
  Completion of Square
  Quadratic Equation

FACTORISATION

Definition: The operation of resolving a quantity into factors

Steps

For a given function $ax^2 + bx + c$:

- Take the product of 1st and last terms i.e. acx^2
- Write down the factors of that product
- Find the two factor which when added/ subtracted (depending with the sign of the product) together will give the middle term i.e. b
- Rewrite the middle term with those numbers
- Factor the first two and last two terms separately.
- If we've done this correctly, our two new terms should have a clearly visible common factor

Example

Solve the equation $2x^2 - 5x - 3 = 0$

Suggested Solution

$2x^2 - 5x - 3 = 0$

Aside

$$2x^2 - 5x - 3 \qquad\qquad -6x^2$$

$$2x^2 - 6x + 1x - 3 \qquad\qquad -6x \text{ and } +1x$$

$$2x(x-3) + 1(x-3) \qquad\qquad 3x \text{ and } 2x$$

$$(x-3)(2x+1)$$

Now

$$2x^2 + 4x - 3 = 0$$

$$\Rightarrow (x-3)(2x+1) = 0$$

Either $(x-3) = 0$ *or* $(2x+1) = 0$

$$\therefore x = 3 \ \ or \ \ x = -\frac{1}{2}$$

COMPLETION OF SQUARE

Definition: It is the method of converting a quadratic equation which is not a perfect square into the sum or difference of a perfect square and a constant by adding or subtracting the suitable constant terms.

- To complete the square we need to write the quadratic function in the form $(x+a)^2 = b$, where a and b are real numbers

Steps

For a given equation $ax^2 + bx + c = 0$:

- Separate variable terms from non-variable terms
- Make sure that the coefficient of x^2 , a is 1. If $a \neq 1$, then divide both sides by a
- Halve the coefficient of x and square this value
- Add this value to each side of the equation to force one side to be a perfect square.

- Express the terms in the left side of the equation as a square
- Simplify the terms in the right side of the equation
- Equate the results of prior two steps
- Solve the final equation to obtain the required roots

Example

Solve the equation $2x^2 - 5x - 3 = 0$

Suggested Solution

$$2x^2 - 5x - 3 = 0$$

$$2x^2 - 5x = 3$$

$$x^2 - \frac{5}{2}x = \frac{3}{2}$$

$$x^2 + 2x + \left(-\frac{5}{4}\right)^2 = \frac{3}{2} + \left(-\frac{5}{4}\right)^2$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{3}{2} + \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{49}{16}$$

$$x - \frac{5}{4} = \pm\sqrt{\frac{49}{16}}$$

$$x - \frac{5}{4} = \pm\frac{7}{4}$$

$$x = \frac{5}{4} \pm \frac{7}{4}$$

$$x = \frac{5}{4} + \frac{7}{4} \; or \; x = \frac{5}{4} - \frac{7}{4}$$

$x = \frac{12}{4}$ or $x = -\frac{2}{4}$

$\therefore x = 3$ or $x = -\frac{1}{2}$

Quadratic Formula

If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Derivation

The quadratic formula is derived using completion of square method i.e.:

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Solve the equation $2x^2 - 5x - 3 = 0$

Suggested Solution

$$2x^2 - 5x - 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} : a = 2; b = -4 \text{ and } c = -3$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$x = \frac{5 \pm \sqrt{49}}{4}$$

$$x = \frac{5 \pm 7}{4}$$

$$x = \frac{5 + 7}{4} \; or \; x = \frac{5 - 7}{4}$$

$$x = \frac{12}{4} \; or \; x = -\frac{2}{4}$$

$$\therefore x = 3 \; or \; x = -\frac{1}{2}$$

Analysis of the Quadratic formula

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

The Sum of Roots

The separate roots are $-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$ and $-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$.

Now:

$$\left(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)+\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)=\frac{-b}{2a}+\frac{-b}{2a}$$

$$=-\frac{2b}{2a}$$

$$=-\frac{b}{a}$$

Thus the sum of the roots $=-\frac{b}{a}$

NB: This fact is very useful as a check on the accuracy of roots that have been calculated.

The Nature of Roots

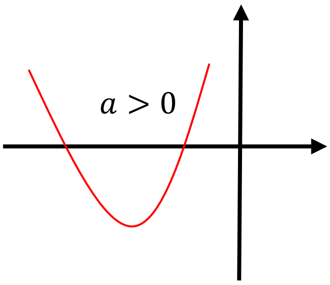
- It is called the discriminant of the equation
- The discriminant is the part of the quadratic formula underneath the square root symbol i.e.:

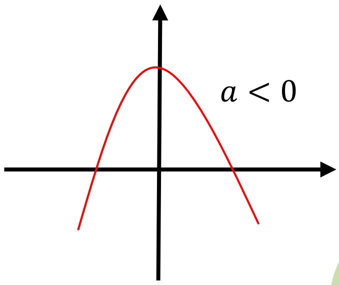
$$b^2-4ac$$

(i) The discriminant tell us the nature of the root

Case 1

When it is positive i.e. $b^2 - 4ac > 0$ we get two Real and Distinct roots

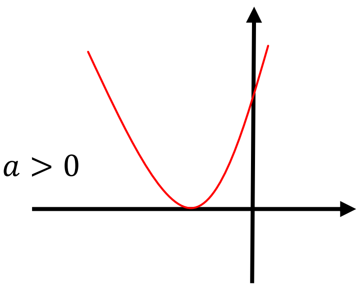


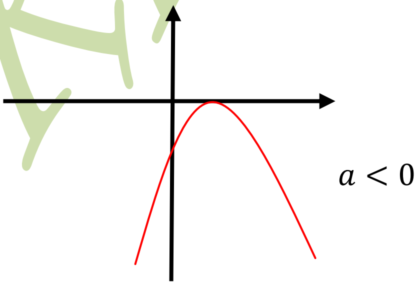


NB: The graph of $y = f(x)$, will intersect the x-axis twice.

Case 2

When it is zero i.e. $b^2 - 4ac = 0$ we get one Real and Repeated root (both answers are the same)

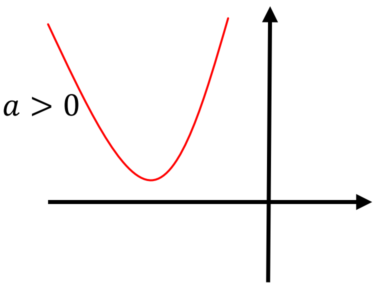


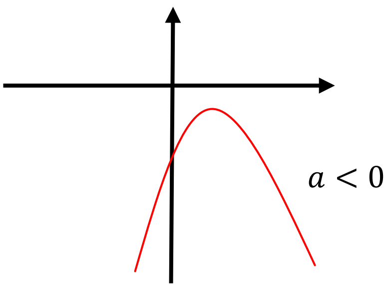


NB: The graph of $y = f(x)$, will intersect the x-axis once i.e. it touches the x-axis hence the x-axis is a tangent to the parabola.

Case 3

When it is negative i.e. $b^2 - 4ac < 0$ we get a pair of Complex roots or Non-Real roots





NB: The graph of $y = f(x)$, neither touch nor intersect the x-axis.

NB: When $b^2 - 4ac \geq 0$, we get real roots

Worked Problems

Example 1

Find the value(s) of k for which the equation $3x^2 + kx + 7 = 0$ has

(i) only one solution

(ii) two distinct roots

(iii) no real roots

Suggested Solution

$3x^2 + kx + 7 = 0$

$a = 3; b = k$ and $c = 7$

(i) only one solution

$b^2 - 4ac = 0$

$k^2 - 4(3)(7) = 0$

$k^2 - 84 = 0$

$k^2 = 84$

$\therefore k = \pm\sqrt{84}$

(ii) two distinct roots

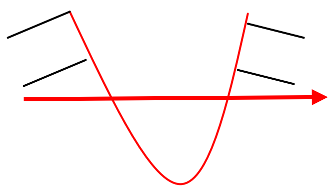
$b^2 - 4ac > 0$

$k^2 - 4(3)(7) > 0$

$k^2 - 84 > 0$

$\left(k - \sqrt{84}\right)\left(k - \sqrt{84}\right) > 0$

The critical values are $\pm\sqrt{84}$



From the graph the solution set is: $x \in \Re: k < -\sqrt{84} \cup k > \sqrt{84}$

(iii) no real roots

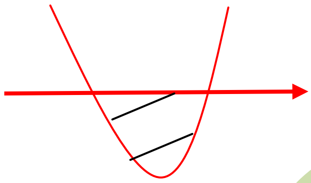
$b^2 - 4ac < 0$

$k^2 - 4(3)(7) < 0$

$k^2 - 84 < 0$

$(k - \sqrt{84})(k - \sqrt{84}) < 0$

The critical values are $\pm\sqrt{84}$



From the graph the solution set is: $x \in \Re: -\sqrt{84} < k < \sqrt{84}$

Example 2

Determine the nature of the roots of the equation $x^2 - 6x + 1 = 0$.

Suggested Solution

$x^2 - 6x + 1 = 0 \qquad a = 1; b = -6$ and $c = 1$

Now:

$b^2 - 4ac = (-6)^2 - 4(1)(1)$

$= 36 - 4$

$= 36$

Since $b^2 - 4ac > 0$ $\therefore$ the roots are real and distinct.

Example 3

ZIMSEC NOVEMBER 2003 PAPER 1

Given that $f(x) = mx^3 + (m + 2n)\,x^2 + (m + n)x + 2$, is exactly divisible by $(x + 1)$,

express n in terms of m. **[2]**

Show that if in this case, the equation $f(x) = 0$ has exactly one real root, then

$m^2 - 6m + 4 < 0$ **[4]**

Suggested Solution

$f(x) = mx^3 + (m + 2n)\, x^2 + (m + n)x + 2$, and the divisor is $(x + 1)$

Now: $f(-1) = 0$

$\Rightarrow m(-1)^3 + (m + 2n)\,(-1)^2 + (m + n)(-1) + 2 = 0$

$-m + (m + 2n) - m - n + 2 = 0$

$-m + n + 2 = 0$

$\therefore n = m - 2$

Now

$f(x) = mx^3 + [m + 2(m - 2)]\, x^2 + (m + m - 2)x + 2$

$\quad\;\; = mx^3 + (3m - 4)\, x^2 + (2m - 2)x + 2$

$$
\begin{array}{r|l}
 & mx^2 + (2m - 4)x + 2 \\
\hline
x + 1 & mx^3 + (3m - 4)x^2 + (2m - 2)x + 2 \\
- & mx^3 + mx^2 \\
\cline{2-2}
 & \qquad (2m - 4)x^2 + (2m - 2)x + 2 \\
- & \qquad (2m - 4)x^2 + (2m - 4)x + 2 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad 2x + 2 \\
- & \qquad\qquad\qquad\qquad 2x + 2 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Now:

$f(x) = (x + 2)[mx^2 + (2m - 4)x + 2] = 0$

Since $f(x) = 0$ has exactly one real root then $mx^2 + (2m - 4)x + 2 = 0$ has no real roots

$\Rightarrow b^2 - 4ac < 0$

$\Rightarrow (2m - 4)^2 - 4(m)(2) < 0$

$4m^2 - 16m + 16 - 8m < 0$

$4m^2 - 24m + 16 < 0$

$m^2 - 6m + 4 < 0 \quad (as\ required)$

PAST EXAMINATION QUESTIONS

ZIMSEC NOVEMBER 2002 PAPER 1

Find the set of values of k, for which the equation

$kx^2 - 3x = k - 3$ has real roots. **[2]**

ZIMSEC JUNE 2006 PAPER 1

Show that the line $y = x - 2m$ always meets the curve $xy = 8$, for all the values of

m, where $m \in \Re$. **[3]**

ZIMSEC NOVEMBER 2016 PAPER 1

The equation of a circle $x^2 + y^2 = 4$ and the equation of its tangent is $y = 2mx + c$.

Show that $16m^2 - c + 4 = 0$. **[4]**

ZIMSEC NOVEMBER 2019 PAPER 1

Find the two possible values of k for which the line $4y + 3x = k$ is a tangent to the

curve $x^2 + y^2 - 4x - 21 = 0$. **[7]**

SKETCHING POLYNOMIAL FUNCTIONS

- We sketch the graph of a quadratic function by
  - (i) locating the intercepts
  - (ii) finding the vertex.
- The vertex is an example of a turning point.
- For polynomials of degree greater than 2, finding turning points is not an elementary procedure and usually requires the use of calculus, however:
  - (i) To find the y-intercept, we put $x = 0$.
  - (ii) To find the x-intercepts, we put $y = 0$ and solve the corresponding polynomial equation, if possible.
- A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the standard form
- The standard form of a quadratic function is $a(x-h)^2 + k$ and is obtained by completing the square.
- The graph of $f(x)$ is a parabola with vertex (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.
- The extreme value (maximum or minimum) value of $f(x)$ occurs at $x = h$ i.e. $(x-h)^2 = 0$
- The line of symmetry is $x = h$

Solved Problems

Example 1

ZIMSEC JUNE 2007 PAPER 1

Express $2x^2 - 6x + 3$ in the form $a(x+b)^2 + c$. **[3]**

a) Hence state

(i) the coordinates of the turning point of the graph of $y = 2x^2 - 6x + 3$, **[2]**

(ii) a sequence of three transformations by which the graph of $y = x^2$ is transformed into the graph of $y = 2x^2 - 6x + 3$. **[3]**

b) Find the values of k for which the equation $2x^2 - 6x + 3 = kx + 1$ has distinct real roots. **[4]**

Suggested Solution

Express $2x^2 - 6x + 3$ in the form $a(x+b)^2 + c$.

$2x^2 - 6x + 3 \equiv a(x+b)^2 + c$

Expanding RHS

$a(x+b)^2 + c = a(x^2 + 2xb + b^2) + c$

$= (ax^2 + 2abx + ab^2) + c$

$= ax^2 + 2abx + ab^2 + c$

$2x^2 - 6x + 3 \equiv ax^2 + 2abx + ab^2 + c$

Comparing coefficients of

$x^2: a = 2$

$x: 2ab = -6$

$4b = -6$

$\therefore b = -\frac{3}{2}$

$x^0: ab^2 + c = 3$

$(2)\left(-\frac{3}{2}\right)^2 + c = 7$

$c = 7 - \frac{9}{2}$

$\therefore c = \frac{5}{2}$

$2x^2 - 6x + 3 \equiv 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$

NB: We can also use the completion of square method

a) Hence

(i) the coordinates of the turning point of the graph of $y = 2x^2 - 6x + 3$ are $\left(\frac{3}{2}; \frac{5}{2}\right)$

(ii) Translation, x-axis moving $\frac{3}{2}$ units rightwards or Translation vector $\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$

Stretch, y-axis with factor 2 or multiply every y-coordinate by 2.

Translation, y-axis moving $\frac{5}{2}$ units upwards or Translation vector $\begin{pmatrix} 0 \\ \frac{5}{2} \end{pmatrix}$

b) $2x^2 - 6x + 3 = kx + 1$

$2x^2 - 6x - kx + 3 - 1 = 0$

$2x^2 - (k + 6)x + 2 = 0$

Now: $a = 2;\quad b = -(k + 6)\quad$ and $c = 2$

$b^2 - 4ac > 0$

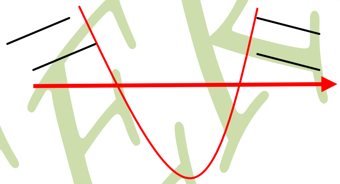
$[-(k + 6)]^2 - 4(2)(2) > 0$

$k^2 + 12k + 36 - 16 > 0$

$k^2 + 12k + 20 > 0$

$(k + 10)(k + 2) > 0$

The critical values are -2 or -10



From the graph the solution set is: $x \in \Re: k < -10 \cup k > -2$

Example 2

ZIMSEC NOVEMBER 2017 PAPER 1

(i) Express $2x^2 - 3x + 7$ in the form $p(x + q)^2 + r$, where p, q and r are constants. **[3]**

(ii) Hence or otherwise, write down the coordinates of the turning point of $y = 2x^2 - 3x + 7$. **[1]**

Suggested Solution

(i) $2x^2 - 3x + 7 \equiv p(x + q)^2 + r$

Expanding RHS

$p(x + q)^2 + r = p(x^2 + 2xq + q^2) + r$

$= (px^2 + 2pqx + pq^2) + r$

$= px^2 + 2pqx + pq^2 + r$

$2x^2 - 3x + 7 \equiv px^2 + 2pqx + pq^2 + r$

Comparing coefficients of

$x^2: p = 2$ $\quad$ $x: 2pq = -3$ $\quad$ $x^0: pq^2 + r = 7$

$4q = -3$

$\therefore q = -\frac{3}{4}$

$(2)\left(-\frac{3}{4}\right)^2 + r = 7$

$r = 7 - \frac{9}{8}$

$\therefore r = \frac{47}{8}$

$$2x^2 - 3x + 7 \equiv 2\left(x - \frac{3}{4}\right)^2 + \frac{47}{8}$$

NB: We can also use the completion of square method

(ii) The coordinates of the turning points are $\left(\frac{3}{4}; \frac{47}{8}\right)$

PAST EXAMINATION QUESTIONS

ZIMSEC NOVEMBER 2009 PAPER 1

Express $12x - 3x^2 + 15$ in the form $a + b(x + c)^2$. **[2]**

Hence, or otherwise, sketch the graph of state $y = 12x - 3x^2 + 15$ clearly labelling the coordinates of the turning point. **[2]**

On the same axes sketch the graph of $y = e^{-x}$ and state the number of real roots of the equation

$$e^{-x} + 3x^2 - 12x - 15 = 0$$ **[2]**

Taking $x_1 = -0.8$, use the Newton-Raphson method to obtain the value of a root of the equation correct to 4 decimal places. **[5]**

ZIMSEC NOVEMBER 2010 PAPER 1

(i) Express $12x - 4x^2$ in the form $c - a(x+b)^2$, where a , b and c are constants to be found. **[2]**

(ii) Find the equation of the tangent to the curve $y = 12x - 4x^2$, at the point $y = 8$ and $x < \frac{3}{2}$. **[6]**

ZIMSEC NOVEMBER 2011 PAPER 1

The function $h(x) = 2x^2 - 6x + 11,\ x \in \Re$

Express

a) $h(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants. **[2]**

b) Write down the range of h. **[1]**

c) Explain why h does not have an inverse. **[1]**

ZIMSEC JUNE 2017 PAPER 1

It is given that $f(x) = x^2 - 8x + 10$.

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are constants. **[2]**

b) Hence or otherwise, find the least value of the function, $f(x)$ and write down the equation of the line of symmetry of the graph $y = f(x)$. **[2]**

c) State the geometrical transformations which map the graph of $y = x^2$ onto the graph of $y = x^2 - 8x + 10$. **[3]**

ZIMSEC NOVEMBER 2019 PAPER 1

(i) Express $4x^2 - 8x - 5$ in the form $a(x+b)^2 + c$, where a, b and c are constants. **[2]**

(ii) Find the coordinates of the stationary point of $4x^2 - 8x - 5$ and determine its nature. **[2]**

POLYNOMIAL IDENTITIES

Two functions are identical if and only if they have the same values of coefficients e.g.

$$2x^2 - 3x + 5 \equiv 5 - 3x + 2x^2$$

Worked Examples

Example 1

Find the values a, b and c such that

$$2x^4 + 2x^3 + 5x^2 + 3x + 3 \equiv (x^2 + x + 1)(ax^2 + bx + c)$$

Suggested Solution

$2x^4 + 2x^3 + 5x^2 + 3x + 3 \equiv (x^2 + x + 1)(ax^2 + bx + c)$

RHS

$$(x^2 + x + 1)(ax^2 + bx + c) \equiv x^2(ax^2 + bx + c) + x(ax^2 + bx + c) + 1(ax^2 + bx + c)$$

$$= ax^4 + bx^3 + cx^2 + ax^3 + bx^2 + cx + ax^2 + bx + c$$

$$= ax^4 + (a + b)x^3 + (a + b + c)x^2 + (b + c)x + c$$

Now

$2x^4 + 2x^3 + 5x^2 + 3x + 3 \equiv ax^4 + (a + b)x^3 + (a + b + c)x^2 + (b + c)x + c$

Comparing coefficients of

$x^4: a = 2$

$x^3: a + b = 2$

$2 + b = 2$

$\therefore b = 0$

$x^2: a + b + c = 5$

$2 + 0 + c = 5$

$\therefore c = 3$

Example 2

Given that

$$x^2 + x + a \equiv (x + a)(x - 1) + b$$

find the values of a and b.

Suggested Solution

$x^2 + x + a \equiv (x + a)(x - 1) + b$

RHS

$(x + a)(x - 1) + b \equiv x(x - 1) + a(x - 1) + b$

$= x^2 - x + ax - a + b$

$= x^2 + (a - 1)x + b - a$

Now

$x^2 + x + a \equiv x^2 + (a - 1)x + b - a$

Comparing coefficients of

$x: a - 1 = 1$

$a = 1 + 1$

$\therefore a = 2$

$x^0: b - a = a$

$b = 2a$

$\therefore b = 4$

PAST EXAMINATION QUESTIONS

UCLES NOVEMBER 1996 PAPER 1

Given that

$$x^3 - 3x^2 + ax + 2 \equiv (x + 1)(x^2 + bx + 2)$$

find the values of a and b. **[3]**

ZIMSEC NOVEMBER 2007 PAPER 1

Find the values of a, b and c such that

$$2x^4 + 6x^3 + 7x^2 + 15x + 5 \equiv (x^2 + 3x + 1)(ax^2 + bx + c)$$

find all values of x. **[3]**

ZIMSEC JUNE 2008 PAPER 1

Find a quadratic expression $ax^2 + bx + c$ such that $1 - x^3 \equiv (1 - x)(ax^2 + bx + c)$. **[2]**

Hence obtain an expression for $In(ax^2 + bx + c)$ as a difference two logarithms and use it to find an expansion of $In(ax^2 + bx + c)$ in ascending powers of x up to and including the term in x^5. **[5]**

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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