TOPIC 3: SERIES AND SEQUENCES

3.2 Series

Learning Objectives

After completing this section you should be able to:

- use standard results for $\sum r$, $\sum r^2$ and $\sum r^3$ to find related sums,
- use the method of differences to obtain the sum of finite series,
- use Taylor's and Maclaurin's series for approximation,
- solve problems involving method of differences as well as Taylor's and Maclaurin's series.

3.2.1 Series Standard Results

1. $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$

2. $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$

3. $\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$

4. $\sum_{r=1}^{n} a = an$ where a is a constant.

Example 1

Show that $\sum_{r=1}^{n} r^2(3-4r) = \frac{1}{2}n(n+1)(1-2n^2)$

Solution

$$\sum_{r=1}^{n} r^2(3-4r) = \sum_{r=1}^{n}(3r^2-4r^3)$$

$$= 3\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r^3$$

$$= 3\left(\frac{n}{6}(n+1)(2n+1)\right) - 4\left(\frac{n^2}{4}(n+1)^2\right)$$

$$= \frac{n}{2}(n+1)(2n+1) - n^2(n+1)^2$$

$$= \frac{1}{2}n(n+1)[2n+1-2n(n+1)]$$

$$= \frac{1}{2}n(n+1)(2n+1-2n^2-2n)$$

$$= \frac{1}{2}n(n+1)(1-2n^2)$$

Example 2

Evaluate $\sum_{r=1}^{2n}\left(3r^2-\frac{1}{2}\right)$ giving your answer in a fully factorized form.

Hence find $\sum_{r=31}^{100}\left(3r^2-\frac{1}{2}\right)$

Solution

$$\sum_{r=1}^{2n}\left(3r^2-\frac{1}{2}\right) = 3\sum_{r=1}^{2n} r^2 - \sum_{r=1}^{2n}\frac{1}{2}$$

$$= 3\left[\frac{2n}{6}(2n+1)(4n+1)\right] - n$$

$$= n(2n+1)(4n+1) - n$$

$$= n[(2n+1)(4n+1) - 1]$$

$= n(8n^2 + 6n)$

$= 2n^2(4n + 3)$

$$\sum_{r=31}^{100}\left(3r^2 - \frac{1}{2}\right) = \sum_{r=1}^{100}\left(3r^2 - \frac{1}{2}\right) - \sum_{r=1}^{30}\left(3r^2 - \frac{1}{2}\right)$$

$= 2(50)^2(4(50) + 3) - 2(15)^2(4(15) + 3)$

$= 1\,015\,000 - 28350$

$= 986\,650$

Exercise 3.2.1

1 (a) Express $4x^3 - 6x^2 + 2x$ in the form $Ax(x+1)(x+2) = Bx(x+1) + Cx$

where A, B and C are constants.

(b) Given that $\sum_{x=1}^{x=n} x(x+1)(x+2) = \frac{n}{4}(n+1)(n+2)(n+3)$ and

$\sum_{x=1}^{x=n} x(x+1) = \frac{n}{3}(n+1)(n+2)$, show that

$$\sum_{x=1}^{x=n} 4x^3 - 6x^2 + 2x = n^2\left(n^2 - 1\right)$$

(Zimsec J2013 p1)

2 Show that $\sum_{r=1}^{n} r^2\left(2r - p\right) = \frac{1}{6}n\left(n+1\right)\left(3n^2 + \left(3 - 2p\right)n - p\right)$ where p is a constant.

Given that the coefficients of n^3 and n^4 in the expression for $\sum_{r=1}^{n} r^2\left(2r - p\right)$ are equal,

find the value of p.

3.2.2 The method of Differences

The method of differences can be used to sum simple finite series whose general term, u_r can be expressed in the form $f(r) - f(r+1)$, or possibly $f(r+1) - f(r)$.

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \left[f(r) - f(r+1) \right]$$

$$= [f(1) - \cancel{f(2)}] + [\cancel{f(2)} - \cancel{f(3)}] + [\cancel{f(3)} - \cancel{f(4)}] + \cdots + [\cancel{f(n-3)} - \cancel{f(n-2)}] +$$

$$[\cancel{f(n-2)} - \cancel{f(n-1)}] + [\cancel{f(n-1)} - \cancel{f(n)}] + [\cancel{f(n)} - f(n+1)]$$

Thus $\sum_{r=1}^{n} \left[f(r) - f(r-1) \right] = f(1) - f(n+1)$

Example 1

(a) Express $\frac{1}{r(r+1)}$ in partial fractions.

(b) Hence (i) use the method of differences to evaluate $\sum_{r=1}^{n} \frac{1}{r(r+1)}$

(ii) deduce the value of $\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1} \right)$

Solution

(a) $\frac{1}{r(r+1)} \equiv \frac{A}{r} + \frac{B}{r+1}$

$1 \equiv A(r+1) + Br$

Letting $r = 0 \Longrightarrow A = 1$

Letting $r = -1 \Longrightarrow B = -1$

"Sometimes the only solution is finding a new problem"

$\therefore \frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$

(b) (i) $$\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\ldots+\left(\cancel{\frac{1}{n-2}}-\cancel{\frac{1}{n-1}}\right)\left(\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}\right)\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

(ii) $$\sum_{r=1}^{\infty}\frac{1}{r(r+1)}=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

Example 2

(a) Express $\frac{2}{(r+1)(r+3)}$ in partial fractions. Hence, or otherwise, using the method of differences find the sum of the series $\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)}$

(b) Evaluate $\sum_{r=11}^{39}\frac{2}{(r+1)(r+3)}$

Solution

(a) $\frac{2}{(r+1)(r+3)} \equiv \frac{A}{r+1} + \frac{B}{r+3}$

$2 = A(r+3) + B(r+1)$

If $r = -1 \Rightarrow A = 1$ and if $r = -3 \Rightarrow B = -1$

$\therefore \frac{2}{(r+1)(r+3)} \equiv \frac{1}{r+1} - \frac{1}{r+3}$

$$\sum_{r=1}^{n}\frac{1}{r+1}-\frac{1}{r+3} \quad = \left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)+\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+3}\right)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}$$

$$= \frac{5n^2 + 13n}{6(n+2)(n+3)}$$

$$= \frac{n(5n+13)}{6(n+2)(n+3)}.$$

(b) $$\sum_{r=11}^{39} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{39} \frac{2}{(r+1)(r+3)} - \sum_{r=1}^{10} \frac{2}{(r+1)(r+3)}$$

$$= \frac{39(5(39)+13)}{6(39+2)(39+3)} - \frac{10(5(10)+13)}{6(10+2)(10+3)}$$

$$= \frac{5017}{44772}$$

$$= 0.112$$

Example 3

Show that $(r+1)^3 - (r-1)^3 = 6r^2 + 2$. Hence deduce $\sum_{r=1}^{n} r^2$

Solution

$$(r+1)^3 - (r-1)^3 = (r^3 + 3r^2 + 3r + 1) - (r^3 - 3r^2 + 3r - 1)$$

$$= r^3 - r^3 + 3r^2 + 3r^2 + 3r - 3r + 1 + 1$$

$= 6r^2 + 2$. Shown.

Taking $\sum_{r=1}^{n}\left(6r^2+2\right)=\sum_{r=1}^{n}\left[\left(r+1\right)^3-\left(r-1\right)^3\right]$

$$= (\cancel{2^3} - 0^3) + (\cancel{3^3} - 1^3) + (\cancel{4^3} - \cancel{2^3}) + (\cancel{5^3} - \cancel{3^3}) + (\cancel{6^3} - \cancel{4^3}) \ldots + [\cancel{(n-2)^3} - \cancel{(n-4)^3}] + [\cancel{(n-1)^3} - \cancel{(n-3)^3}] + [n^3 - \cancel{(n-2)^3}] + [(n+1)^3 - \cancel{(n-1)^3}]$$

Look up carefully here the manner in which terms cancel each other

$$= -1^3 + n^3 + (n+1)^3$$

$$\sum_{r=1}^{n}\left(6r^2+2\right) = -1^3 + n^3 + \left(n+1\right)^3 = 2n^3 + 3n^2 + 3n$$

But $\sum_{r=1}^{n}\left(6r^2+2\right) = 6\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2 = 6\sum_{r=1}^{n} r^2 + 2n$

Thus $6\sum_{r=1}^{n} r^2 + 2n = 2n^3 + 3n^2 + 3n$

$$\Rightarrow 6\sum_{r=1}^{n} r^2 = 2n^3 + 3n^2 + 3n - 2n$$

$$\Rightarrow \sum_{r=1}^{n} r^2 = \frac{1}{6}\left(2n^3 + 3n^2 + n\right)$$

$$= \frac{1}{6}n\left(2n^2 + 3n + 1\right)$$

$$= \frac{1}{6}n\left(n+1\right)\left(2n+1\right)$$

Example 4

Express $\frac{5r+4}{r(r+1)(r+2)}$ in partial fractions. Hence, by using the method of differences evaluate

$$\sum_{r=1}^{n}\frac{5r+4}{r(r+1)(r+2)}.$$

Solution

$$\frac{5r+4}{r(r+1)(r+2)} \equiv \frac{A}{r}+\frac{B}{r+1}+\frac{C}{r+2}$$

Using cover up method

$A=2, B=1, C=-3$

$$\therefore \frac{5r+4}{r(r+1)(r+2)} \equiv \frac{2}{r}+\frac{1}{r+1}-\frac{3}{r+2}$$

Now,

$$\sum_{r=1}^{n}\frac{5r+4}{r(r+1)(r+2)}$$

$$=\sum_{r=1}^{n}\left(\frac{2}{r}+\frac{1}{r+1}-\frac{3}{r+2}\right)$$

$$=\left(2+\frac{1}{2}-\cancel{1}\right)+\left(1+\cancel{\frac{1}{3}}-\cancel{\frac{3}{4}}\right)\left(\cancel{\frac{2}{3}}+\cancel{\frac{1}{4}}-\cancel{\frac{3}{5}}\right)+\left(\cancel{\frac{1}{2}}+\cancel{\frac{1}{5}}-\cancel{\frac{1}{2}}\right)+\cdots+\left(\cancel{\frac{2}{n-2}}+\cancel{\frac{1}{n-1}}-\cancel{\frac{3}{n}}\right)+$$
$$\left(\cancel{\frac{2}{n-1}}+\cancel{\frac{1}{n}}-\frac{3}{n+1}\right)+\left(\cancel{\frac{2}{n}}+\frac{1}{n+1}-\frac{3}{n+2}\right).$$

Terms marked by
the same colour
of line cancel
each other.

$$=2+\frac{1}{2}+1-\frac{3}{n+1}+\frac{1}{n+1}-\frac{3}{n+2}$$

$$=\frac{7}{2}+\frac{-2}{(n+1)}-\frac{3}{n+2}$$

$$= \frac{7(n+1)(n+2)-4(n+2)-6(n+1)}{2(n+1)(n+2)}$$

$$= \frac{7n^2+21n+14-4n-8-6n-6}{2(n+1)(n+2)}$$

$$= \frac{7n^2+11n}{2(n+1)(n+2)}$$

Example 5

You are given that $\frac{3}{(5+3x)(2+3x)} \equiv \frac{1}{2+3x} - \frac{1}{5+3x}$.

(i) Use this result to find $\sum_{r=1}^{100} \frac{1}{(5+3r)(2+3r)}$, giving your answer as an exact fraction.

(ii) Write down the limit to which $\sum_{r=1}^{n} \frac{1}{(5+3r)(2+3r)}$ converges as n tends to infinity.

Solution

(i) $\sum_{r=1}^{100} \frac{1}{(5+3r)(2+3r)}$

$$= \sum_{r=1}^{n} \frac{1}{2+3r} - \frac{1}{5+3r}$$

$$= \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{8}-\frac{1}{11}\right) + \left(\frac{1}{11}-\frac{1}{14}\right) + \left(\frac{1}{14}-\frac{1}{17}\right) + \cdots + \left(\frac{1}{3n-4}-\frac{1}{3n-1}\right) + \left(\frac{1}{3n-1}-\frac{1}{2+3n}\right) + \left(\frac{1}{2+3n}-\frac{1}{5+3n}\right).$$

$$= \frac{1}{5} - \frac{1}{5+3n}$$

$$= \frac{3n}{5(5+3n)}$$

$$\sum_{r=1}^{n} \frac{1}{(5+3r)(2+3r)} = \frac{1}{3}\left[\frac{3n}{5(5+3n)}\right] = \frac{n}{5(5+3n)}$$

$$\therefore \sum_{r=1}^{100} \frac{1}{(5+3r)(2+3r)} = \frac{100}{5(5+300)} = \frac{4}{61}$$

(ii) $\lim\limits_{n\to\infty} \frac{1}{3}\left(\frac{1}{5} - \frac{1}{5+3n}\right) = \frac{1}{3}\lim\limits_{n\to\infty}\left(\frac{1}{5} - \frac{1}{5+3n}\right) = \frac{1}{3}\left(\frac{1}{5}\right) = \frac{1}{15}$.

Example 6

You are given that $\frac{3}{4(2r-1)} - \frac{1}{2r+1} + \frac{1}{4(2r+3)} = \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$.

(i) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} = \frac{2}{3} - \frac{3}{4(2n+1)} + \frac{1}{4(2n+3)}.$$

(ii) Write down the limit to which $\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$ converges as n tends to infinity.

(iii) Find the sum of the finite series

$$\frac{45}{39x41x43} + \frac{47}{41x43x45} + \frac{49}{43x45x47} + \cdots + \frac{105}{99x101x103},$$ giving your answer to 3 s.f.

Solution

$$\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$$

$$= \sum_{r=1}^{n} \frac{3}{4(2r-1)} - \frac{1}{2r+1} + \frac{1}{4(2r+3)}$$

$$= \left(\frac{3}{4} - \frac{1}{3} + \cancel{\frac{1}{20}}\right) + \left(\frac{1}{4} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{28}}\right) + \left(\cancel{\frac{3}{20}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{36}}\right) + \left(\cancel{\frac{3}{28}} - \cancel{\frac{1}{9}} + \frac{1}{44}\right) + \cdots + \left(\frac{3}{8n-20} - \cancel{\frac{1}{2n-3}} + \cancel{\frac{1}{8n-4}}\right) + \left(\cancel{\frac{3}{8n-12}} - \cancel{\frac{1}{2n-1}} + \frac{1}{8n+4}\right) + \left(\cancel{\frac{3}{8n-4}} - \frac{1}{2n+1} + \frac{1}{8n+12}\right)$$

$$= \frac{3}{4} - \frac{1}{3} + \frac{1}{4} + \frac{1}{8n+4} - \frac{1}{2n+1} + \frac{1}{8n+12}$$

$$= \frac{1}{4} + \frac{1}{4(2n+1)} - \frac{1}{2n+1} + \frac{1}{4(2n+3)}$$

$$= \frac{2}{3} - \frac{3}{4(2n+1)} + \frac{1}{4(2n+3)}$$

(ii) $\frac{2}{3}$

(iii) $$\sum_{r=20}^{50} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} = \sum_{r=1}^{50} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} - \sum_{r=1}^{19} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$$

$$= \left(\frac{2}{3} - \frac{3}{4(2(50)+1)} + \frac{1}{4(2(50)+3)}\right) - \left(\frac{2}{3} - \frac{3}{4(2(19)+1)} + \frac{1}{4(2(19)+3)}\right)$$

$$= 0.661668108 - 0.653533458$$

$$= 0.008134649$$

$$= 0.00813 \text{ to 3s.f.}$$

Exercise 3.2.2

1 Express $\frac{1}{r(r+2)}$ in partial fractions. Hence find the sum of the series $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ using the method of differences.

Evaluate, correct to 4 decimal places the value of $\sum_{r=50}^{100} \frac{4}{r(r+2)}$

2 (a) Given that $f(r) = \frac{1}{4r-1}$, show that $f(r) - f(r+1) = \frac{A}{(4r-1)(4r+3)}$ where A is an integer.

Find the value of $\sum_{r=1}^{40} \frac{1}{(4r-1)(4r+3)}$ using the method of differences.

3 Show that $4r^3 = r^2(r+1)^2 - (r-1)^2r^2$. Hence prove, by the method of differences that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

4 Prove the identity $\frac{2r-1}{r(r-1)} - \frac{2r+1}{r(r+1)} \equiv \frac{2}{(r-1)(r+1)}$.

Hence, using the method of differences, prove that $\sum_{r=1}^{n} \frac{2}{(r-1)(r+1)} = \frac{3}{2} - \frac{2n+1}{n(n+1)}$

Evaluate $\sum_{r=1}^{\infty} \frac{2}{(r-1)(r+1)}$.

5 (a) Show that $(r+1)! - (r-1)! = (r^2+r-1)(r-1)!$

(b) Hence show that $\sum_{r=1}^{n} (r^2+r-1)(r-1)! = (n+2)n! - 2$

6 Find $\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}$. Hence or otherwise deduce the value of $\sum_{r=1}^{\infty} \frac{2}{r(r+1)(r+2)}$.

7 (a) Express $\frac{1}{(x+1)(x+2)}$ in partial fractions.

(b) Hence, or otherwise, show that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{n+2}$

"Sometimes the only solution is finding a new problem"

8 $f(r) = \frac{1}{r(r+1)}, r \in \mathbb{Z}^+$

(a) Show that $f(r) - f(r+1) = \frac{k}{r(r+1)(r+2)}$, stating the value of k.

(b) Hence show, by the method of differenes, that

$$\sum_{r=1}^{2n} \frac{1}{r(r+1)(r+2)} = \frac{n(2n+3)}{4(n+1)(2n+1)}$$

9 (a) Given that $f(r) = r^2(2r^2 - 1)$, show that $f(r) - f(r-1) = (2r-1)^3$

(b) Use the method of differences to show that $\sum_{r=n+1}^{2n} (2r-1)^3 = 3n^2(10n^2-1)$

10 Show that $\frac{r+1}{r+2} - \frac{r}{r+1} = \frac{1}{(r+1)(r+2)}, r \in \mathbb{Z}^+$.

Hence. or otherwise find $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$.

11 (a) Express $f(x) = \frac{1}{(x+1)(x+2)(x+3)}$ in partial fractions.

(b) Hence find $\sum_{r=1}^{n} f(r)$.

12 (a) Show that $\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r2^r}{(r+1)(r+2)}$.

(b) Hence find $\sum_{r=1}^{30} \frac{r2^r}{(r+1)(r+2)}$ giving your answer in the form $2^n - 1$, where n is an integer.

13 Express as a simplified single fraction $\frac{1}{(r-1)^2} - \frac{1}{r^2}$. Hence prove by the method of differences that $\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}$.

14 Find the sum of the series $\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ln\frac{n}{n+1}$.

"Sometimes the only solution is finding a new problem"

15 Show that $\frac{1}{2r+1}-\frac{1}{2r+3}\equiv\frac{2}{(2r+1)(2r+3)}$.

Use the method of differences to find $\sum_{r=1}^{30}\frac{2}{(2r+1)(2r+3)}$

16 (a) Show that $\frac{1}{5r-2}-\frac{1}{5r+3}=\frac{A}{(5r-2)(5r+3)}$, stating the value of the constant A.

(b) Hence show that $\sum_{r=1}^{n}\frac{1}{(5r-2)(5r+3)}=\frac{n}{3(5n+3)}$

(c) Find the value of $\sum_{r=1}^{\infty}\frac{1}{(5r-2)(5r+3)}$.

3.2.3 Maclaurin's Series

The Maclaurin's theorem for a function $f(x)$ is given by

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

Example 1

Use the Maclaurin's theorem to find the first three non-zero terms of the expansion of $\sin x$. Hence find sin 0.1 radians correct to seven dp.

Solution

Let $f(x)=\sin x$. $\quad f(0)=\sin 0=0$

$f'(x)=\cos x$. $\quad f'(0)=1$

$f''(x)=-\sin x$. $\quad f''(0)=0$

$f'''(x)=-\cos x$. $\quad f'''(0)=-1$

$f'^{v}(x)=\sin x$. $\quad f'^{v}(0)=0$

"Sometimes the only solution is finding a new problem"

$f^{v}(x) = \cos x.$ $\qquad f^{v}(0) = 1$

By Maclaurin's theorem,

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$$\Longrightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin 0.1 = 0.1 - \frac{0.1^3}{3!} + \frac{0.1^3}{3!}$$

$= 0.0998334$ correct to 7 d.p.

Exercise 3.2.3

1 Find the first three non-zero terms of the Maclaurin's expansions of the following functions.

(a) $e^{-x}\sin x$ (b) $\sec x$
(c) $e^{2x}\cos x$ (d) $e^{\sin x}$

2 Given that $f(x) = xe^{3x}$, $f'(x) = 3xe^{3x} + e^{3x}$ and $f''(x) = 3e^{3x}(3x+2)$.

(i) find $f'''(x)$, $f^{iv}(x)$ and hence write down the Maclaurin series of $f(x)$ up to the term in x^4.

3 Given that $f(x) = e^{3x}\sin x$ and $f'(x) = e^{3x}\cos x + 3e^{3x}\sin x$,

(i) find $f''(x)$ and $f'''(x)$ and hence write down the Maclaurin's series of $f(x)$ up to and including the term in x^3.

(ii) evaluate $f(0.5)$ giving your answer correct to 4 decimal places.

(iii) use the Maclaurin's series obtained in (i) to estimate the value of $f(0.5)$ and state the absolute error in using Maclaurin as an approximation

4 Given that $f(x) = \ln(2-3x)$,

(i) show that $f'(x) = \frac{-9}{(2-3x)^2}$,

(ii) find $f''(x)$.

Hence obtain a Maclaurin's series expansion for $\ln(2-3x)$, neglecting terms in x^4 and higher degree. State clearly what circumstances might justify neglecting these terms.

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5 Given that $f(x) = \tan x$, $x \neq (2n+1)\pi$. By expressing $f(x)$ as $\frac{\sin x}{\cos x}$
$f'(x) = 1 + tan^2 x$.

(a) (i) Obtain expressions for $f''(x)$ and $f'''(x)$.

(ii) Show that the Maclaurin's series for $\tan x$ is $x + \frac{x^3}{3}$ is x is small enough for the terms in x^4 and higher powers to be neglected.

(iii) State one reason why Maclaurin's series for $\cot x$ cannot be obtained.

6 It is given that $e^{-3y}(1-4x) = (7+3x)$.

(i) Show that $y = -\frac{1}{3}[\ln(7+3x) - \ln(1-4x)]$.

(ii) Hence obtain the Maclaurin expansion of y up to and including the term in x^3.

3.2.4 Problems involving implicit functions and differential equations

Example 1

Given that $\ln y = xy$, where $y > 0$,

(i) 1. show that $\frac{dy}{dx} = \frac{y^2}{1-xy}$,

2. find $\frac{d^2y}{dx^2}$.

(ii) Hence or otherwise, find the Maclaurin expansion of y up to and including the term in x^2.

Solution

$\ln y = xy$, where $y > 0$,

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(xy)$

$$\frac{1}{y}\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - x\frac{dy}{dx} = y$$

$$\left(\frac{1}{y} - x\right)\frac{dy}{dx} = y$$

$$\left(\frac{1-xy}{y}\right)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = y\left(\frac{y}{1-xy}\right) = \frac{y^2}{1-xy}$$

$$\frac{d^2y}{dx^2} = \frac{(1-xy)\left(2y\frac{dy}{dx}\right) - y^2\left(0-\left(y+x\frac{dy}{dx}\right)\right)}{(1-xy)^2}$$

$$= \frac{(1-xy)\left(\frac{2y^3}{1-xy}\right) + y^2\left(y+\frac{xy^2}{1-xy}\right)}{(1-xy)^2}$$

$$= \frac{3y^3 + \frac{xy^4}{1-xy}}{(1-xy)^2}$$

$$= \frac{3y^3}{(1-xy)^2} + \frac{xy^4}{(1-xy)^3}$$

$$= \frac{3y^3 - 3xy^4 + xy^4}{(1-xy)^3}$$

$$= \frac{3y^3 - 2xy^4}{(1-xy)^3}$$

$$= \frac{(3-2xy)y^3}{(1-xy)^3}$$

(ii) When $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$ and $\frac{d^2y}{dx^2} = 3$

Hence the Maclaurins series for y is given by

$$y = 1 + x + \frac{3x^2}{2} + \cdots$$

Example 2

(a) If $y = (1 + 3e^{-x})^{\frac{1}{2}}$, find $\frac{dy}{dx}$ at $x = 0$.

(b) Show that $2y\frac{dy}{dx} = 1 - y^2$.

(c) By further differentiating, use the Maclaurin's theorem to find the series expansion for y in ascending powers of x, up to and including the term in x^2.

Solution

$y = (1 + 3e^{-x})^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(1 + 3e^{-x})^{-\frac{1}{2}}(-3e^{-x})$

$= \frac{-3e^{-x}}{2(1+3e^{-x})^{\frac{1}{2}}}$

When $x = 0$,

$\frac{dy}{dx} = \frac{-3}{2(4)^{\frac{1}{2}}} = -\frac{3}{4}$ OR $\frac{3}{4}$

(b) $y = (1 + 3e^{-x})^{\frac{1}{2}} \Longrightarrow 3e^{-x} = y^2 - 1$

$\frac{dy}{dx} = \frac{-3e^{-x}}{2(1+3e^{-x})^{\frac{1}{2}}}$

$\Longrightarrow \frac{dy}{dx} = \frac{1-y^2}{2y}$

$\Longrightarrow 2y\frac{dy}{dx} = 1 - y^2$

(c) $y = (1 + 3e^{-x})^{\frac{1}{2}}$, When $x = 0$, $y = 2$ OR -2

$2y\frac{dy}{dx} = 1 - y^2$

$\Longrightarrow \frac{dy}{dx} = \frac{1-y^2}{2y}$ When $x = 0$, $\frac{dy}{dx} = \frac{1-(\pm 2)}{2(\pm 2)} = -\frac{3}{4}$ or $\frac{3}{4}$

Taking $2y\frac{dy}{dx} = 1 - y^2$

$\Longrightarrow \frac{d}{dx}\left(2y\frac{dy}{dx}\right) = \frac{d}{dx}(1) - \frac{d}{dx}(y^2)$

$$\Longrightarrow 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = -2y\frac{dy}{dx}$$

$$\Longrightarrow y\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx}$$

When $x = 0$, $2\frac{d^2y}{dx^2} = -\left(-\frac{3}{4}\right)^2 - 2\left(-\frac{3}{4}\right)$ OR $-2\frac{d^2y}{dx^2} = -\left(\frac{3}{4}\right)^2 - (-2)\left(\frac{3}{4}\right)$

$$\Longrightarrow \frac{d^2y}{dx^2} = \frac{15}{32} \text{ OR } \frac{-15}{32}$$

Therefore the Maclaurin's series for y is given by

$y = 2 - \frac{3}{4}x + \frac{15}{64}x^2 + \cdots$ OR $y = -2 + \frac{3}{4}x - \frac{15}{64}x^2 + \cdots$

Exercise 3.2.4

1 If $y = \ln\cos x$, prove that $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\frac{dy}{dx} = 0$.

Hence, or otherwise, obtain the Maclaurin expansion of y in terms of x up to and including the term in x^4. Using $x = \frac{\pi}{4}$ show that $\ln 2 \approx \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

2 Given that $\frac{dy}{dx} = x + 2y$ and that $y = 1$ when $x = 0$,

(a) show that $\frac{d^2y}{dx^2} = 5$,

(b) obtain the Maclaurin's series for y in ascending powers of x as far as the term in x^3.

3 Use a Maclaurin's series to find a fifth degree polynomial that is a solution to the differential equation $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0$ given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$

4 Use a Maclaurin's series to find a fourth degree polynomial that is a solution to the differential equation $\frac{d^2y}{dx^2} + y = x$ given that $y = 0$ and $\frac{dy}{dx} = 1 - \frac{\pi}{2}$ when $x = 0$. Find an approximate value of y when $x = \frac{\pi}{4}$.

5 Use a Maclaurin's series to find the series solution (first 6 terms) of the differential equation $\frac{dy}{dx} = y^2 - x$ given that $y = 1$ when $x = 0$.

6 Given that $y = \ln\left(e^{-x} - \frac{1}{2}\right)$, show that $\frac{dy}{dx} = -\frac{1}{2}(e^{-y} + 2)$.

By finding the second derivative, or otherwise, show that the series expansion of y in ascending powers of x, up to and including the term in x^2 is $-\ln 2 - 2x - x^2$.

7 Given that $\ln y = xy$, where $y > 0$,

(i) 1. show that $\frac{dy}{dx} = \frac{y^2}{1-xy}$,

2. find $\frac{d^2y}{dx^2}$.

(ii) Hence or otherwise, find the Maclaurin expansion of y up to and including the term in x^2.

8 (a) If $y = (1 + 3e^{-x})^{\frac{1}{2}}$, find $\frac{dy}{dx}$ at $x = 0$.

(b) Show that $2y\frac{dy}{dx} = 1 - y^2$.

(c) By further differentiating, use the Maclaurin's theorem to find the series expansion for y in ascending powers of x, up to and including the term in x^2.

3.2.5 Taylor Series

The Taylor Series for the function f about the point a is

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Observe that when $a = 0$, the Taylor series becomes the Maclaurin's series.

Example 1

Find the Taylor Polynomial of degree 3 for $f(x) = e^x$ about $x = 2$.

Solution

$f(x) = e^x$ $\qquad$ $f(2) = e^2$

$f'(x) = e^x$ $\qquad$ $f'(2) = e^2$

$f''(x) = e^x$ $\qquad$ $f''(2) = e^2$

Solution

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

When $a = 2$

$$f(x) = e^2 + e^2(x-2) + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{6!} + \cdots$$

$$\therefore e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \cdots$$

Example 2

Expand $\sin x$ as a series of ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$. Use your series to find an approximate value of $\sin 31°$ correct to 8 decimal places.

Solution

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$a = \frac{\pi}{6}$

$f(x) = \sin x \qquad f\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$

$\therefore \sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\left(x-\frac{\pi}{6}\right)}{1!} - \frac{1}{2}\frac{\left(x-\frac{\pi}{6}\right)^2}{2!} - \frac{\sqrt{3}}{2}\frac{\left(x-\frac{\pi}{6}\right)^3}{3!} + \cdots$

$= \frac{1}{2} + \frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)}{2} - \frac{\left(x-\frac{\pi}{6}\right)^2}{4} - \frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)^3}{12} + \cdots$

If $x = 31°$, we let $a = 30°$, then $x - a = 1°$ which is small.

$1° = \frac{\pi}{180}$ radians

$\therefore \sin 31° = \frac{1}{2} + \frac{\sqrt{3}\left(\frac{\pi}{180}\right)}{2} - \frac{\left(\frac{\pi}{180}\right)^2}{4} - \frac{\sqrt{3}\left(\frac{\pi}{180}\right)^3}{12} + \cdots$

$= 0.51503807$

Exercise 3.2.5

1 Use Taylor series to expand the following about the point where $x = 1$ up to and including the third term.

(a) $e^x \cos x$ (b) $\sin x \cos x$ (b) $\ln(x+1)$ (c) $\frac{1}{x}$

2 Find the first three terms of the Taylor series expansion of $\cos x \ln x$ about the point $x = 1$.

3 Find the Taylor series expansion of $x \ln x$ about $x = 1$ (i.e in ascending powers of $(x-1)$, up to and including the term in $(x-1)^5$. Hence find the value of $xln\ x$ when $x = 1.1$ correct to 5 decimal places.

4 Find the Taylor series expansion of $x^2 \ln x$ in ascending powers of $(x-1)$, up to and including the term in $(x-1)^4$.

5 **F**ind the Taylor Series for each of the following functions about the given points.

(a) $\sin x$ about $x = \frac{3\pi}{2}$

(b) e^{1-8x} about $x = 3$

(c) $\ln(1-x)$ about $x = -2$

3.2.6 Using Taylor series to solve differential equations

Example 1

Use a Taylor series to find the series solution (first 6 terms) of the differential equation $\frac{dy}{dx} = y^2 - x$ given that $y = 1$ when $x = 0$.

Solution

When $x = 0$ and $y = 1$

$y = 1$

$\frac{dy}{dx} = 1^2 - 0 = 1$

$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - 1 = 2(1)(1) - 1 = 1$

$\frac{d^3y}{dx^3} = 2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - 0 = 2[1(1) + 1^2] = 4$

$\frac{d^4y}{dx^4} = 2\left[y\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\frac{dy}{dx} + 2\frac{dy}{dx}\frac{d^2y}{dx^2}\right] = 2[1(4) + 1(1) + 2(1)(1)] = 14$

$\frac{d^5y}{dx^5} = 2\left[y\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3}\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx}\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2\right)\right] = 2[1(14) + 4(1) + 1^2 + 1(4) + 2(1(4) + 1^2)]$

$= 66$

Now using the Taylor series $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$

Where $a = o$ since $x = 0$

$f(x) = 1 + (x-0)(1) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(4) + \frac{(x-0)^4}{4!}(14) + \frac{(x-0)^5}{5!}(66) + \cdots$

$\Longrightarrow y = 1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{7}{12}x^4 + \frac{11}{20}x^5 + \cdots$

Example 2

Find a third degree polynomial which is a solution to the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos x$ given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 1$. Find y when $x = 1.1$.

Solution

When $x = 1$, $y = 1$ and $\frac{dy}{dx} = 2$

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos x \Longrightarrow \frac{d^2y}{dx^2} = \cos 1 - 2 - 1 = \cos 1 - 3$

$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} = -\sin x \Longrightarrow \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} = -\sin x \Longrightarrow \frac{d^3y}{dx^3} = -\sin 1 - \cos 1 + 3 - 2 = -(\sin 1 + \cos 1 + 1).$

Using $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$

Where $a = 1$ since $x = 1$

$$y = 1 + (x-1)(2) + \frac{(x-1)^2}{2!}(\cos 1 - 3) - \frac{(x-1)^3}{3!}(\sin 1 + \cos 1 + 1) + \cdots$$

When $x = 1.1$ $\quad y = 1 + 2(0.1) + \frac{0.1^2}{2}(\cos 1 - 3) - \frac{0.1^3}{6}(\sin 1 + \cos 1 + 1)$

$$= 1.187$$

Exercise 3.2.6

Use the Taylor's theorem to find a polynomial (up to the specified degree) which approximates y given that

1 $\frac{dy}{dx} = y\cos x$ given that $y = 1$ when $x = 0$ (2nd degree)

2 $x\frac{dy}{dx} + y = 0$ given that $y = 1$ when $x = 1$ (3rd degree)

Hence find y when $x = 1.1$, 1.5, and 1.9.

3 $\frac{dy}{dx} - x^2 = y^2$ given that $y = 0$ when $x = -1$ (2nd degree)

4 $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$ (3rd degree)

5 $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ given that $y = 1$ and $\frac{dy}{dx} = 4$ when $x = 0$ (3rd degree)

Find y when $x = -0.1$.

6 $\frac{dy}{dx} = \cos x + \cos y$ given that $y = 2$ when $x = \frac{\pi}{2}$ (2nd degree)

Find y when $x = \frac{\pi}{3}$.

7 $\frac{d^2y}{dx^2} - 2xy = 0$ given that $y = 1$ and $\frac{dy}{dx} = -3$ when $x = 0$ (6th degree)

8 $\frac{d^2y}{dx^2} + \frac{dy}{dx} + x^2y = 0$ given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$ (3rd degree)

9 $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y^2 = 0$ given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$ (4th degree)

10 $\frac{d^2y}{dx^2} = y$ given that $y = 1$ and $\frac{dy}{dx} = -1$ when $x = 2$ (3rd degree)

11 $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0$ given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$ (8th degree)

12 $\frac{d^2y}{dx^2} + y = x$ given that $y = 0$ and $\frac{dy}{dx} = 1 - \frac{\pi}{2}$ when $x = 0$. (7th degree)

Find an approximate value of y when $x = \frac{\pi}{4}$.

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Miscellaneous Exercises 3.2

1 (a) Use the series expansion of $\text{Cos}\, x$ to expand $(1+x)^2 \text{Cos}\, x$ up to and including the term in x^3. [4]

(b) Given that $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = 0$, where $y = 1$ at $x = 0$ and $\frac{dy}{dx} = 2$ at $x = 0$.

Use the taylor series to find the first three terms of $(1+x)^2 \text{Cos}\, x$. [4]

2 (a) Show that $\sum_{r=2}^{n} \frac{r+3}{r^3 - r} = 1\frac{1}{2} - \frac{n+2}{n(n+1)}$ [10]

(b) Write down (i) the 5th term, (ii) the sum of the first 5 terms and,

(iii) the limit of this sum as $n \to \infty$ in the series. [5]

3 The function $f(x) = \frac{16x}{(2x-1)(2x+1)(2x+3)}$. Express $f(x)$ in partial fractions.

Hence show that $\sum_{r=1}^{n} f(r) = \frac{8n(n+1)}{(2n+1)(2n+3)}$. [9]

4 Given that $y^2 = \sec x + \tan x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0$.

Show that

(a) $\frac{dy}{dx} = \frac{1}{2} y \sec x$

(b) $\frac{d^2y}{dx^2} = \frac{1}{4} y \sec x\, (\sec x + 2\tan x)$

Given that x is small and that terms in x^3 and higher powers of x may be neglected, use Maclaurin's expansion to express y in the form $A + Bx + Cx^2$, stating the values of A, B and C.

4 If $y = \tan(e^x - 1)$, prove that $\frac{d^2y}{dx^2} = \frac{dy}{dx}(1 + 2e^x y)$.

Obtain the Maclaurin expansion of y as far as the term in x^4 inclusive.

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