Sequences and Series

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SYLLABUS (6042) REQUIREMENTS

- define a sequence and a series
- use sequence definitions such as $U_n = n^2$ and $U_{n+1} = 2U_n$ to calculate successive terms
- distinguish among periodic, oscillating, converging, and diverging sequences
- define arithmetic and geometric progressions
- find the general terms and other terms of arithmetic and geometric progressions
- solve problems involving sequences
- find the sum of given arithmetic and geometric terms
- use $\sum$ notation
- solve problems involving series

SEQUENCES

Definition

1. A sequence is a set of numbers occurring in a certain order, or
2. A sequence may be defined by a specific formula or an algorithm for determining the members of the sequence successively, or *recursively*, or
3. A set of numbers arranged in order by some fixed rule is called as sequences, or
4. A sequence is simply an ordered list $U_1, U_2, U_3, \ldots, U_n$, of numbers (or terms).

Notes

- $U_1, U_2, U_3, \ldots, U_n$ is normally abbreviated to $\{U_n\}$.
- The "outputs" of a sequence are the **terms** of the sequence
- U_n is normally given/described/defined in one of the two ways:
  (i) as a function of the preceding term(s) [Recurrence relation/ Recursive definition] e.g. $U_{n-1} = 3U_n + 1$. A sequence can be generated by the repeated use of an instruction. The later outputs in terms of previous outputs. We start by defining the first few terms of the sequence, and then describe how later terms are computed in terms of previous terms. Term n is represented by U_n; the next term after this one is represented by U_{n+1}, while the term before U_n is U_{n-1}. For these sequences, the first term must be stated.
  (ii) as a function of its position in the sequence [Functional definition] e.g. $U_n = 2^n + 3$. Using this definition the n^{th} term can be readily calculated. This is also called a formula for the **general term**
- Sequences can be either **finite** or **infinite**
  (i) Finite sequence: It's a sequence with finite terms e.g. $2, 4, 6$ and 8,
  (ii) Infinite sequence: It's a sequence with infinite (unending list) terms e.g. $2, 4, 6, 8, \ldots.$ We usually use $\ldots$ to denote that the sequence continues without bound. The dots at the end indicate that the sequence goes on forever.

Behaviour of a Sequence as $n \to \infty$

Like human beings, sequences portray certain behaviour as n becomes very large.

(a) Convergent Sequence

A sequence which tends to a finite limit, say 'l' is called a *Convergent Sequence*. We say that the sequence converges to 'l' e.g. $1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, 1\frac{1}{5}, \ldots, 1$.

NB: (i) This sequence is generated using the formula $U_n = 1\frac{1}{n}, n \geq 2$.

(ii) Always remember that $\frac{1}{n} \to 0$ when $n \to \infty$.

(b) Divergent Sequence

A sequence which tends to $\pm\infty$ is said to be *Divergent* (or is said to diverge). e.g. $2, 4, 6, 8, \ldots.$

NB: This sequence is generated using the formula $U_n = 2n$

(c) Oscillatory Sequence

A sequence which neither converges nor diverges is called an *Oscillatory/ Oscillating Sequence*, or it is simply a sequence which alternates from negative to positive indefinitely. e.g. $-1, 3, -5, 7, \ldots.$

NB: This sequence is generated using the formula $U_n = (-1)^n(2n-1)$

(d) Periodic Sequence

A sequence which repeats itself after a certain period .e.g. $0, 1, 0, -1, 0, 1, 0, -1$ ....

NB: This sequence is generated using the formula $U_n = Sin(n), n = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi etc$

Examples

Recurrence Relation

Question 1

Consider the recurrence $a_1 = 2$ and $a_n = (a_{n-1}) + 1$ for $n \geq 2$. Write down the first five terms of this sequence.

Suggested Solution

$a_1 = 2$

$a_2 = (a_{2-1}) + 1 = a_1 + 1 = 2 + 1 = 3$

$a_3 = (a_{3-1}) + 1 = a_2 + 1 = 3 + 1 = 4$

$a_4 = (a_{4-1}) + 1 = a_3 + 1 = 4 + 1 = 5$

$a_5 = (a_{5-1}) + 1 = a_4 + 1 = 5 + 1 = 6$

$a_6 = (a_{6-1}) + 1 = a_5 + 1 = 6 + 1 = 7$

Question 2

The sequence $T_1, T_2, T_3, \ldots, T_n$ is such that $T_{r+1} = 2T_r - T_{r-1}$ for $r \geq 1$. Write down the first five terms of this sequence and state the behaviour of this sequence as n tends to infinity. $T_1 = 2$ and $T_2 = 3$.

Suggested Solution

$T_1 = 2$ and $T_2 = 3$.

$T_3 = T_{2+1} = 2T_2 - T_{2-1} = 2T_2 - T_1 = 2(3) - 2 = 6 - 2 = 4$

$T_4 = T_{3+1} = 2T_3 - T_{3-1} = 2T_3 - T_2 = 2(4) - 3 = 8 - 3 = 5$

$T_5 = T_{4+1} = \ 2T_4 - T_{4-1} \ = 2T_4 - T_3 = 2(5) - 4 = 10 - 4 = 6$

$T_6 = T_{5+1} = \ 2T_5 - T_{5-1} \ = 2T_5 - T_4 = 2(6) - 5 = 12 - 5 = 7$

$T_7 = T_{6+1} = \ 2T_6 - T_{6-1} \ = 2T_6 - T_5 = 2(7) - 6 = 14 - 6 = 8$

∴ The sequence is diverging.

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The sequence $U_1, U_2, U_3, \ldots, U_n$ is such that $U_{r+1} = \frac{27}{U_r}$ for $r \geq 1$.

Given also that $U_1 + U_2 = 12$, find possible values of U_1. **[3]**

For each value of U_1, describe the behaviour of the sequence as n tends to infinity. **[2]**

Suggested Solution

$U_1 + U_2 = 12$ (i)

Using $U_{r+1} = \frac{27}{U_r}$

$U_2 = U_{1+1} = \frac{27}{U_1}$

$\therefore U_2 = \frac{27}{U_1}$ (ii)

Substituting (ii) into (i)

$U_1 + U_2 = 12$

$U_1 + \frac{27}{U_1} = 12$

${U_1}^2 + 27 = 12U_1$

${U_1}^2 - 12U_1 = -27$

${U_1}^2 - 12U_1 + (-6)^2 = -27 + (-6)^2$

$(U_1 - 6)^2 = 9$

$U_1 = 6 \pm \sqrt{9}$

$U_1 = 6 \pm 3$

$\therefore U_1 = 3$ or 9

Finding the behaviour

When $U_1 = 9$

$U_2 = U_{1+1} = \frac{27}{U_1} = \frac{27}{9} = 3$

$U_3 = U_{2+1} = \frac{27}{U_2} = \frac{27}{3} = 9$

$U_4 = U_{3+1} = \frac{27}{U_3} = \frac{27}{9} = 3$

$\therefore$ It is a Periodic sequence.

When $U_1 = 3$

$U_2 = U_{1+1} = \frac{27}{U_1} = \frac{27}{3} = 9$

$U_3 = U_{2+1} = \frac{27}{U_2} = \frac{27}{9} = 3$

$U_4 = U_{3+1} = \frac{27}{U_3} = \frac{27}{3} = 9$

$\therefore$ It is a Periodic sequence.

General/Nth Terms

Question 1

Find the general terms of the sequence $2,4,6,8,\ldots$ (even numbers)

Suggested Solution

$T_1 = 2 \times 1 = 2$

$T_2 = 2 \times 2 = 4$

$T_3 = 2 \times 3 = 6$

$T_4 = 2 \times 4 = 8$

NB: The numbers in purple 1, 2, 3, 4 etc indicates the positions.

$\therefore T_n = 2 \times n = 2n.$

Question 2

Find the general terms of the sequence $1,4,9,16,\ldots$ (squares)

Suggested Solution

$T_1 = (1)^2 = 1$

$T_2 = (2)^2 = 4$

$T_3 = (3)^2 = 9$

$T_4 = (4)^2 = 16$

NB: The numbers in purple 1, 2, 3, 4 etc indicates the positions.

$\therefore T_n = (n)^2 = n^2.$

Question 3

Find the general terms of the sequence $1,3,5,7,\ldots$ (odd numbers)

Suggested Solution

$T_1 = (2 \times 1) - 1 = 1$

$T_2 = (2 \times 2) - 1 = 3$

$T_3 = (2 \times 3) - 1 = 5$

$T_4 = (2 \times 4) - 1 = 7$

NB: The numbers in purple 1, 2, 3, 4 etc indicates the positions.

$\therefore T_n = (2 \times n) - 1 = 2n - 1.$

Question 4

Find the general terms of the sequence $-1,3,-5,7,-9,\ldots$ (odd numbers)

Suggested Solution

NB: $(-1)^1 = -1$ $\quad (-1)^2 = +1$ $\quad (-1)^3 = -1$ $\quad (-1)^4 = +1$

Thus $(-1)^n = -1$ when n is odd and $(-1)^n = +1$ when n is even.

$T_1 = (-1)^{1}[(2 \times 1) - 1] = -1$

$T_2 = (-1)^{2}[(2 \times 2) - 1] = 3$

$T_3 = (-1)^{3}[(2 \times 3) - 1] = -5$

$T_4 = (-1)^{4}[(2 \times 4) - 1] = 7$

NB: The numbers in purple $1, 2, 3, 4$ etc indicates the positions.

$\therefore T_n = (-1)^{n}[(2 \times n) - 1] = (-1)^{n}[2n - 1]$.

Question 5

Find the general terms of the sequence $1, -3, 5, -7, 9, \ldots$ (odd numbers)

Suggested Solution

NB: $(-1)^1 = -1$ $\quad (-1)^2 = +1$ $\quad (-1)^3 = -1$ $\quad (-1)^4 = +1$ **and**

- **(a)** $(-1)^1 \times (-1) = +1$
- **(b)** $(-1)^2 \times (-1) = -1$
- **(c)** $(-1)^3 \times (-1) = +1$
- **(d)** $(-1)^4 \times (-1) = -1$

Thus $(-1)^n \times (-1) = +1$ when n is odd and $(-1)^n \times (-1) = -1$ when n is even.

also $(-1)^n \times (-1) = (-1)^n \times (-1)^1 = (-1)^{n+1}$.

$T_1 = (-1)^{1+1}[(2 \times 1) - 1] = +1$

$T_2 = (-1)^{2+1}[(2 \times 2) - 1] = -3$

$T_3 = (-1)^{3+1}[(2 \times 3) - 1] = +5$

$T_4 = (-1)^{4+1}[(2 \times 4) - 1] = -7$

NB: The numbers in purple $1, 2, 3, 4$ etc indicates the positions.

$\therefore T_n = (-1)^{n+1}[(2 \times n) - 1] = (-1)^{n+1}[2n - 1]$.

Series

Definition

1. Addition/Sum of the terms/elements of a sequence is called a series

Notes

- $1 + 3 + 5 + 7$ is an example of a series.
- Series can be either **finite** or **infinite**
  (i) Finite series: It's a series with finite terms e.g. $2 + 4 + 6 + 8$,
  (ii) Infinite sequence: It's a series with infinite (unending list) terms e.g. $2 + 4 + 6 + 8 +$ .... The dots at the end indicate that the sum goes on forever.
- The sum of terms is often represented by Greek letter sigma ($\sum$)

The $\sum$ notation

- $\sum$ simply means add up terms
- $\sum$ has the minimum(lower limit) and the maximum(upper limit).

$$\sum_{Lower\ limit}^{Upper\ limit} function$$

For example:

(a) The Finite series $2 + 4 + 6 + 8$ can be written in sigma notation as follows:

$$\sum_{r=1}^{4} 2r$$

In this case the lower limit is 1 since $2 \times 1 = 2$ and the maximum or upper limit is 4 since $2 \times 4 = 8$

(b) The Infinite series $-1 + 3 - 5 + 7 + \cdots$ can be written in sigma notation as follows:

$$\sum_{r=1}^{\infty}(-1)^n(2n-1)$$

In this case the lower limit is 1 since $(-1)^1(2\times 1-1)=-1$ and since the series is never ending the maximum value is represented by infinity (∞).

Rules for operating Summations

1) $\sum_{r=1}^{n} a = a \times n = an$

2) $\sum_{r=1}^{n} ar = a\sum_{r=1}^{n} r$

3) $\sum_{r=1}^{n}(ar \pm br) = a\sum_{r=1}^{n} r \pm b\sum_{r=1}^{n} r$

Standard results of Summations

1) $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

2) $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

3) $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

An important property of Summations

$$\sum_{r=N+1}^{M} f(r) = \sum_{r=1}^{M} f(r) - \sum_{r=1}^{N} f(r)$$

I am going to prove the above result using a simple example: Suppose

$$(1)\quad \sum_{r=1}^{5} r = 1 + 2 + 3 + 4 + 5 = 15$$

$$(2)\quad \sum_{r=1}^{2} r = 1 + 2 = 3$$

$$(3)\quad \sum_{r=3}^{5} r = 3 + 4 + 5 = 12$$

Now:

$$\sum_{r=3}^{5} r = \left(\sum_{r=1}^{5} r = 1 + 2 + 3 + 4 + 5 = 15\right) - \left(\sum_{r=1}^{2} r = 1 + 2 = 3\right) = 12$$

$$\sum_{r=3}^{5} r = \left(\sum_{r=1}^{5} r\right) - \left(\sum_{r=1}^{2} r\right)$$

Let $M = 5\ and\ (N + 1) = 3 \Longrightarrow N = 2$ since $N + 1$ and N are consecutive terms, so as $3\ and\ 2$.

$$\Longrightarrow \sum_{r=N+1}^{M} r = \left(\sum_{r=1}^{M} r\right) - \left(\sum_{r=1}^{N} r\right)$$

Therefore:

$$\sum_{r=N+1}^{M} f(r) = \sum_{r=1}^{M} f(r) - \sum_{r=1}^{N} f(r)$$

Examples

SUMMATION SERIES

Question 1

Find:

$$\sum_{r=1}^{20} 3r - 2.$$

Suggested Solution

$$\sum_{r=1}^{20} 3r - 2 = 3\sum_{r=1}^{20} r - \sum_{r=1}^{20} 2$$

We use $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$= 3\left[\frac{1}{2}(20)(20+1)\right] - 2(20)$

$= 630 - 40$

$= 590$

Question 2

Find the value of a and c if:

$$\sum_{r=1}^{n} ar - 3c \equiv n^2 + 2n$$

Suggested Solution

$$\sum_{r=1}^{n} ar - 3c \equiv n^2 + 2n$$

LHS

$$\sum_{r=1}^{n} ar - 3c = \sum_{r=1}^{n} ar - \sum_{r=1}^{n} 3c$$

$$= a\sum_{r=1}^{n} r - 3\sum_{r=1}^{n} c$$

$$= a\left[\frac{1}{2}n(n+1)\right] - 3c(n)$$

$$= \left[\frac{a}{2}n(n+1)\right] - 3cn$$

$$= \frac{a}{2}n^2 + \frac{a}{2}n - 3cn$$

$$= \frac{a}{2}n^2 + \left(\frac{a}{2} - 3c\right)n$$

Comparing the LHS and the RHS

Coefficients of n^2

$$\frac{a}{2} = 1 \Longrightarrow a = 2$$

Coefficients of n

$$\frac{a}{2} - 3c = 2 \Longrightarrow \frac{2}{2} - 3c = 2 \Longrightarrow 1 - 3c = 2 \Longrightarrow 1 - 2 = 3c \Longrightarrow -1 = 3c \therefore c = -\frac{1}{3}.$$

Question 3

Given that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) \quad \text{and} \quad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1),$$

a) Find in terms of n,

$$\sum_{r=1}^{n} (r^2 - 2r),$$

b) Hence evaluate

$$\sum_{r=5}^{20}(r^2 - 2r).$$

Suggested Solution

(a) $\displaystyle\sum_{r=1}^{n}(r^2 - 2r) = \sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r$

$= \dfrac{1}{6}n(n+1)(2n+1) - 2\left[\dfrac{1}{2}n(n+1)\right]$

$= \dfrac{1}{6}n(n+1)(2n+1) - n(n+1)$

$= n(n+1)\left[\dfrac{1}{6}(2n+1) - 1\right]$

$= \dfrac{n(n+1)(2n-5)}{6}$

$\therefore \dfrac{1}{6}\, n(n+1)(2n-5)$

(b) $\displaystyle\sum_{r=5}^{20}(r^2 - 2r) = \sum_{r=1}^{20}(r^2 - 2r) - \sum_{r=1}^{4}(r^2 - 2r)$

Aside

$\displaystyle\sum_{r=1}^{20}(r^2 - 2r) = \frac{1}{6}\,(20)(21)(40-5) = \frac{1}{6}\,(20)(21)(35) = 2\,450$ and

$\displaystyle\sum_{r=1}^{4}(r^2 - 2r) = \frac{1}{6}\,(4)(5)(8-5) = \frac{1}{6}\,(4)(5)(3) = 10.$

Now

$$\sum_{r=5}^{20}(r^2 - 2r) = \sum_{r=1}^{20}(r^2 - 2r) - \sum_{r=1}^{4}(r^2 - 2r) = 2\,450 - 10 = 2\,440.$$

Question 4

Given that

$$\sum_{r=1}^{n}(2r-5)=285,$$

find the value of n.

Suggested Solution

$$\sum_{r=1}^{20} 2r-5=285$$

$$2\sum_{r=1}^{n} r-\sum_{r=1}^{n} 5=285$$

$$\Rightarrow 2\left[\frac{1}{2}(n)(n+1)\right]-5(n)=285$$

$$\Rightarrow n(n+1)-5(n)=285$$

$$\Rightarrow n^2+n-5n=285$$

$$\Rightarrow n^2-4n=285$$

$$\Rightarrow n^2-4n+(-2)^2=285+(-2)^2$$

$$\Rightarrow (n-2)^2=285+4$$

$$\Rightarrow (n-2)^2=289$$

$$\Rightarrow n-2=\pm\sqrt{289}$$

$$\Rightarrow n=2\pm 17$$

$$\Rightarrow n=2+17 \text{ or } 2-17$$

$$\Rightarrow n=19 \text{ or } -15$$

$$\therefore n=19.$$

Question 4

Write the following series in sigma notation:

(a) $1+4+9+16+\cdots$
(b) $-1+3-5+7-9$
(c) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$

Suggested Solution

(i) $1+4+9+16+\cdots=(1)^2+(2)^2+(3)^2+(4)^2+\cdots+(n)^2$

$\therefore$ The answer is $\sum_{r=1}^{\infty} r^2$

(ii) $-1+3-5+7-9$

$\therefore$ The answer is $\sum_{r=1}^{5}(-1)^{n}[2n-1]$

(iii) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots=\frac{1}{(2)^0}-\frac{1}{(2)^1}+\frac{1}{(2)^2}-\frac{1}{(2)^2}+\frac{1}{(2)^4}-\cdots+\frac{1}{(2)^n}+\cdots$

$\therefore$ The answer is $\sum_{r=1}^{\infty}(-1)^{n+1}\left[\frac{1}{2^n}\right]$.

Practice Question

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a) Show that

$$\sum_{r=1}^{n}(r^3-r)=\frac{1}{4}n(n-1)(n+1)(n+2)$$

b) Hence Evaluate

$$\sum_{r=31}^{50}(r^3-r)$$

ARITHMETIC & GEOMETRIC PROGRESSION

Progression

If a sequence of number is such that each term can be obtained from the preceding one by the operation of some law, the sequence is called a progression or simply sequences following certain patterns are more often called **progressions**.

In progressions, we note that each term except the first progresses in a definite manner.

NB: - Each progression is a sequence but each sequence may or may not be a progression

Arithmetic Progression

Definition

(i) An **arithmetic progression** (sometimes called an arithmetic sequence) is a sequence where each term differs from the next by the same, fixed quantity, or

(ii) An arithmetic sequence is a sequence where there is a common difference between any two successive terms, or

(iii) An arithmetic progression, or AP, is a sequence where each new term after the first is obtained by adding a constant d, called the common difference, to the preceding term.

(iv) A sequence in which each term after the first term is obtained from the preceding term by adding a fixed number, is called as an arithmetic sequence or Arithmetic Progression, it is denoted by A.P.

(v) Arithmetic progression (A.P.) is a sequence in which each term except the first is obtained by adding a fixed number (positive or negative) to the preceding term.

Common Difference

The fixed number which is added to the preceding term of an arithmetic sequence is called a common difference.

It is denoted by d.

It is obtained by subtracting the preceding terms from the next term i.e. $T_n - T_{n-1}$.

For example:

Consider the following sequence:

$$5, 10, 15, 20, 25, - - - - -$$

d = Common difference $= T_n - T_{n-1} = T_5 - T_4 = 25 - 20 = 5$, or

d = Common difference $= T_n - T_{n-1} = T_2 - T_1 = 10 - 5 = 5$.

The General Form of an Arithmetic Progression

If the first term of the sequence is 'a' then the arithmetic progression is

$$a, \quad a + d, \quad a + 2d, \quad - - -$$

N^th term or General term (or, last term) of an Arithmetic Progression

If 'a' be the first term and 'd' be the common difference then:

$$T_1 = first\ term = a \qquad = a + (1 - 1)d$$

$$T_2 = 2nd\ term = a + d = a + (2 - 1)d$$

$$T_3 = 3rd\ term = a + 2d = a + (3 - 1)d$$

$$T_4 = 4th\ term = a + 3d = a + (4 - 1)d$$

$$- - - - - - - - - - - - - - - -$$

$$- - - - - - - - - - - - - - - -$$

$$- - - - - - - - - - - - - - - -$$

$$T_n = nth\ term = a + (n - 1)d$$

Therefore, the n^th term of an AP is given by:

$$T_n = a + (n - 1)d$$

in which a = 1st term $d = common\ difference$ and $n = number\ of\ terms$.

Arithmetic Means (A.Ms)

The arithmetic mean of two numbers is equal to one half the sum of the two numbers.

If a, b, c are three consecutive terms in an Arithmetic Progression, then b is called the Arithmetic Mean (A.M) of a and c.

NB: $Common\ difference = T_n - T_{n-1}$

$\Longrightarrow$. If a, b, c are in A.P. then

$b - a = c - b$

$b + b = a + c$

$2b = a + c$

$\therefore b = \frac{a + c}{2}$

Example

Suppose $2, 5$ and 8 are three consecutive terms of an A.P. Show that 5 is the arithmetic mean of 2 and 8.

Solution

Let $a = 2, b = 5$ and $c = 8$.

$b = \frac{a + b}{2}$

$b = \frac{2 + 8}{2}$

$b = \frac{10}{2}$

$\therefore b = 5$ [as required].

Arithmetic Series

The sum of the terms of an Arithmetic sequence is called as Arithmetic series.

For example:

$5, 10, 15, 20, 25, - - - - - - - - -$ is an A.P.

$5 + 10 + 15 + 20 + 25 + - - - -$ is Arithmetic series.

The sum of n terms of an Arithmetic Sequence

The general form of an arithmetic sequence is: $a, a + d, a + 2d, - - - a + (n - 1)d$.
Let S_n denoted the sum of n terms of an Arithmetic sequence.
Then:

$$S_n = a + (a + d) + (a + 2d) + - - - - - + [a + (n - 1)d]$$

Derivation of the formula

Let n^{th} term $[a + (n - 1)d] = l$
The above series can be written

$$S_n = a + (a + d) + (a + 2d) + - - - - - + l$$

Now, writing the series forwards and backwards, we have

$$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l.$$
$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a.$$

Adding in pairs gives

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l),$$
$$\Longrightarrow 2S_n = n(a + l)$$

Hence

$$S_n = \frac{n}{2}(a + l)$$

Since $l = a + (n - 1)d$
Then

$$S_n = \frac{n}{2}(a + l) = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$$

$$\therefore S_n = \frac{n}{2}[2a + (n - 1)d]$$

Geometric Progression

Definition

(i) A geometric progression is a sequence of numbers each term of which after the first is obtained by multiplying the preceding term by a constant number called the common ratio, or

(ii) A geometric progression (sometimes called a geometric sequence) is a sequence where the ratio between subsequent terms is the same, fixed quantity, or

(iii) A geometric sequence is a sequence where each term is obtained by multiplying the preceding term by a certain constant factor, or

(iv) A Geometric progression (G.P.) is a sequence in which each term except the first is obtained by multiplying the previous term by a non-zero constant called the common ratio, or

(v) A geometric progression, or GP, is a sequence where each new term after the first is obtained by multiplying the preceding term by a constant r, called the common ratio.

Common Ratio

In geometric progression, the ratio between any two consecutive terms remains constant

It is denoted by r.

It is obtained by dividing the next term with the preceding term i.e. $T_n \div T_{n-1}$.

For example:

Consider the following sequence:

$$2, 4, 8, 16, 32, - - - - -$$

$d =$ Common ratio $= T_n \div T_{n-1} = T_5 \div T_4 = 32 \div 16 = 2$, or

$d =$ Common ratio $= T_n \div T_{n-1} = T_2 \div T_1 = 4 \div 2 = 2$.

The General Form of a Geometric Progression

If the first term of the sequence is 'a' then the arithmetic progression is

$$a, \quad ar, \quad ar^2, \quad - - -$$

N[th] term or General term(or last term) of a Geometric Progression (G.P)

If 'a' is the first term and 'r' is the common ratio then the general form of G.P is

$$a, ar, ar^2, ar^3, \ldots$$

$$T_1 = first\ term = a = ar^{(1-1)}$$
$$T_2 = 2nd\ term = ar = ar^{(2-1)}$$
$$T_3 = 3rd\ term = ar^2 = ar^{(3-1)}$$
$$T_4 = 4th\ term = ar^3 = ar^{(4-1)}$$
$$- - - - - - - - - - - - - - -$$
$$- - - - - - - - - - - - - - -$$
$$- - - - - - - - - - - - - - -$$
$$T_n = nth\ term = ar^{n-1}$$

Therefore, the n^{th} term of an GP is given by:

$$T_n = a\,r^{n-1}$$

Geometric Mean (G.M.)

The Geometric Mean between two quantities is equal to the square root of their product

When three quantities are in G.P., the middle one is called the Geometric Mean (G.M.) between the other two. If a, b, c are three consecutive terms in an Geometric Progression, then b is called the Geometric Mean (G.M) of a and c.

NB: $Common\ ration = T_n \div T_{n-1}$

$\Longrightarrow$. If a, b, c are in G.P. then

$$\frac{b}{a} = \frac{c}{b}$$

$$b^2 = ac$$

$$\therefore b = \sqrt{a\,c}$$

Example

Suppose 2, 4 and 8 are three consecutive terms of a G.P. Show that 4 is the geometric mean of 2 and 8.

Solution

Let $a = 2, b = 4$ and $c = 8$.

$b = \sqrt{a\,c}$

$b = \sqrt{2 \times 8}$

$b = \sqrt{16}$

$\therefore b = 4$ [as required].

Geometric Series

A geometric series is the sum of the terms of a geometric sequence.

If $a, ar, ar^2, \dots, ar^{n-1}$ is a geometric sequence,

Then geometric series is $a + ar + ar^2 + \dots\dots + ar^{n-1}$.

Sum of n Terms of a Geometric Series

Deriving the formula

Let S_n be the sum of geometric series

i.e.

$$S_n = a + ar + ar^2 + \dots\dots + ar^{n-1} \qquad (1)$$

Multiplying by r on both sides

$$rS_n = ar + ar^2 + ar^3 + \cdots\dots + ar^{n-1} + ar^n \qquad (2)$$

Now, subtracting (2) from (1), we get

$$S_n - rS_n = a - ar^n$$

so that

$$S_n(1 - r) = a(1 - r^n)$$

Now divide by $1 - r$ (as long as $r \neq 1$) to give

$$S_n = \frac{a(1 - r^n)}{(1 - r)}, \quad r \neq 1$$

The above formula is used when $|r| < 1$.

If $|r| > 1$ then we use the following formula:

$$S_n = \frac{a(r^n - 1)}{(r - 1)}, \quad r \neq 1$$

Sum to infinity of a Geometric Series

We want to examine this formula

$$\frac{a(1 - r^n)}{(1 - r)}$$

Let's consider $r = \frac{1}{4}$ since $|r| < 1$.

Now the formula contains the term $r^n$and, as $|r| < 1$ i.e. $-1 < r < 1$, this term will get closer and closer to zero as n gets larger and larger.

NB: $\frac{1}{\infty} \approx 0$. Thus, r^n i.e $\left(\frac{1}{4}\right)^n \Longrightarrow 0$ as $n \to \infty$.

$$\Longrightarrow Lim_{n\to\infty}\left[\frac{a(1 - r^n)}{(1 - r)}\right] = \left[\frac{a\left(1 - \frac{1}{\infty}\right)}{(1 - r)}\right] = \left[\frac{a(1 - 0)}{(1 - r)}\right] = \frac{a}{1 - r}$$

Now, if $-1 < r < 1$, we can say that the 'sum to infinity' of a geometric series is

$$S_\infty = \frac{a}{1 - r}$$

Recurring Decimals

Definition

It is non-terminating decimal fraction in which some digits are repeated again and again in the same order in its decimal parts

Notes

When we attempt to express a common fraction such as $\frac{3}{4}$ or $\frac{4}{11}$ as a decimal fraction, the decimal always either terminates or ultimately repeats. Thus:

$\frac{3}{4} = 0.75$ (Decimal terminate) and $\frac{4}{11} = 0.363636$ (Decimal repeats)

We can express the recurring decimal fraction $0.\dot{3}\dot{6}$ (or $0.\overline{36}$) as a common fraction.
The bar ($0.\overline{36}$) means that the numbers appearing under it are repeated endlessly. i.e.
$0.\overline{36}$ means $0.363636 - - - -$
Thus a non-terminating decimal fraction in which some digits are repeated again and again in the same order in its decimal parts is called a recurring decimal fraction.

Questions

Recurring Decimals

Example 1

Find the fraction equivalent to the recurring decimals $0.\dot{3}\dot{4}\dot{5}$.

Solution

Let $S = 0.\dot{3}\dot{4}\dot{5}$

$S = 0.345\,345\,345 - - - - - - - - - - - - \infty$

$S = 0.345 + 0.000345 + 0.000000345 - - -$

$$S = \frac{345}{1000} + \frac{345}{1\,000\,000} + \frac{345}{1\,000\,000\,000} + - - -$$

$$S = \frac{345}{1000} + \frac{345}{1\,000}\left(\frac{1}{1\,000}\right) + \frac{3453}{1\,000}\left(\frac{1}{1\,000}\right)\left(\frac{1}{1\,000}\right) + - - -$$

$$S = \frac{345}{1000} + \frac{345}{1\,000}\left(\frac{1}{1\,000}\right) + \frac{345}{1\,000}\left(\frac{1}{1\,000}\right)^2 + - - -$$

Now its a GP with a $= \frac{345}{1000}$ and $r = \frac{1}{1\,000}$.

Therefore:

$$S_\infty = \frac{a}{1 - r} = \frac{\frac{345}{1000}}{1 - \frac{1}{1\,000}} = \frac{\frac{345}{1000}}{\frac{999}{1\,000}} = \frac{345}{1000} \times \frac{1000}{999} = \frac{345}{999} = \frac{115}{333}$$

Practice Question

ZIMSEC June 2004 Paper 1 #4

By using an infinite geometric progression show that $0.4\dot{3}2\dot{1}$(i. e $0.4321212121\ldots$ is equal to $\frac{115}{333}$. **[4]**

Question 2

By using an infinite geometric progression, show that the recurring decimal $0.2\dot{5}\dot{3}$ (i.e. 0.253535353 ...) is equal to $\frac{251}{990}$.

Arithmetic Progression

WORKED PROBLEMS

ZIMSEC June 2018 Paper 1

The salary scale of an employee begins at \$315 a month and rises to a maximum of \$765 by equal monthly increments of \$25

Find the number of months it takes for the salary to reach the maximum. **[3]**

Solution

$Year_{\ 1} = \$315$

$Year_{\ 2} = \$315 + \$25$

$Year_{\ 3} = \$315 + \$25 + \$25$

$Year_{\ 4} = \$315 + \$25 + \$25 + \$25$

Thus it’s an AP with $a = \$315$ and $d = \$25$

$Year_{\ n} = \$765$

$\Longrightarrow a + (n-1)d = \$765$

$\Longrightarrow 315 + (n-1)25 = 765$

$\Longrightarrow (n-1)25 = 765 - 315$

$\Longrightarrow n - 1 = 18$

$\Longrightarrow n = 18 + 1$

$\therefore n = 19$

ZIMSEC November 2018 Paper 1

The sum of the 5^{th} and 10^{th} terms of an arithmetic progression is 27. The third term is – 9.

Find the

(i) first term and the common difference, **[4]**

(ii) smallest value of n, for which $S_n > 0$, where S_n is the sum of the first n terms. [**3**]

Solution

(i) $T_5 + T_{10} = 27$ (i)

$T_3 = -9$ (ii)

Now $T_n = a + (n-1)d$

$\Longrightarrow a + 4d + a + 9d = 27 \quad \Longrightarrow 2a + 13d = 27$ (iii)

and $a + 2d = -9$ (iv)

(iv) × 2 gives $2a + 4d = -18$ (v)

Subtracting (iv) from (iii):

$9d = 45$

$\therefore d = 5$

$a + 2d = -9$ (iv)

$\Longrightarrow a + 2(5) = -9$

$\Longrightarrow a = -9 - 10 = -19.$

(ii) $S_n > 0$

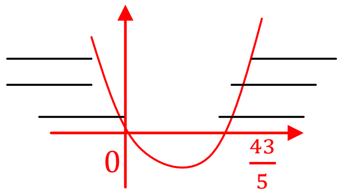
$\frac{n}{2}[2a + (n-1)d] > 0$

$\frac{n}{2}[2(-19) + (n-1)5] > 0$

$\frac{n}{2}[-38 + 5n - 5] > 0$

$n(5n - 43) > 0$

The critical values are $n = 0$ and $n = \frac{43}{5}$



$n \in \Re : n < 0$ or $n > \dfrac{43}{5}$

Thus the smallest number of n is 9 (*since n will never be negative*)

ZIMSEC June 2017 Paper 1

The sum of the first n terms of an arithmetic progression is $\frac{3}{2}n^2 - 2n$.

Find the

(i) first 3 terms of the progression,
(ii) n^{th} term of the progression. **[5]**

Solution

(i) Let the sum of terms be S_n and the terms be T_n.

Note

$S_1 = T_1$
$S_2 = T_1 + T_2$
$S_3 = T_1 + T_2 + T_3$

$S_n = \frac{3}{2}n^2 - 2n$

When $n = 1$, $S_1 = \frac{3}{2}(1)^2 - 2(1) = \frac{3}{2}$

$\Rightarrow T_1 = \dfrac{3}{2} - 2 = -\dfrac{1}{2}$

When $n = 2$, $S_2 = \frac{3}{2}(2)^2 - 2(2) = 2$

$\Longrightarrow T_1 + T_2 = 2 \Longrightarrow T_2 = 2 - (T_1) = 2 - \left(-\frac{1}{2}\right) = 2\frac{1}{2}$

When $n = 3$, $S_3 = \frac{3}{2}(3)^2 - 2(3) = 7\frac{1}{2}$

$\Longrightarrow T_1 + T_2 + T_3 = 7\frac{1}{2} \Longrightarrow T_3 = 7\frac{1}{2} - (T_1 + T_2) = 7\frac{1}{2} - (2) = 5\frac{1}{2}$

$\therefore$ The first three terms are $-\frac{1}{2}$, $2\frac{1}{2}$ and $5\frac{1}{2}$.

(ii) The sequence above is in AP with $a = -\frac{1}{2}$ and $d = 5\frac{1}{2} - 2\frac{1}{2} = 3$ or $2\frac{1}{2} - \left(-\frac{1}{2}\right) = 3$

$\therefore T_n = a + (n-1)d = -\frac{1}{2} + (n-1)3 = -\frac{1}{2} + 3n - 3 = \boldsymbol{3n - \frac{7}{2}}$

ZIMSEC November 2002 Paper 1

A sequence U_r is defined by

$$U_r = (n - 3r).$$

(i) Write down the first 3 terms of the sequence

(ii) Find in terms of n a formula for

$$\sum_{r=n}^{2n} U_r$$

Solution

(i) $U_r = (n - 3r)$

$U_1 = (n - 3 \times 1) = n - 3$

$U_2 = (n - 3 \times 2) = n - 6$

$U_3 = (n - 3 \times 3) = n - 9$

(ii) The sequence above is in AP with $a = n - 3$ and $d = (n-6) - (n-3) = -3$ or $(n-9) - (n-6) = -3$.

Now

$$\sum_{r=n}^{2n} \boldsymbol{U_r} = \sum_{r=1}^{2n} \boldsymbol{U_r} - \sum_{r=1}^{n-1} \boldsymbol{U_r}$$

Also

$$\sum_{r=1}^{2n} U_r = Sum\ of\ the\ 1st\ (2n) terms\ of\ an\ AP,$$

and

$$\sum_{r=1}^{n-1} U_r = Sum\ of\ the\ 1st\ (n-1) terms\ of\ an\ AP.$$

Aside

Now using the sum of 'n' terms formula for an AP.

$$\sum_{r=1}^{n} U_r = \frac{n}{2}[2a + (n-1)d]$$

$$\sum_{r=1}^{2n} U_r = \frac{2n}{2}[2(n-3) + (n-1) \times -3]$$

$$= n[2n - 6 - 3n + 3]$$

$$= n[-n - 3] = -n^2 - 3n.$$

$$\sum_{r=1}^{n-1} U_r = \frac{n-1}{2}[2(n-3) + (\{n-1\} - 1) \times -3]$$

$$= \frac{n-1}{2}[2n - 6 + (n-2) \times -3]$$

$$= \frac{n-1}{2}[2n - 6 - 3n + 6]$$

$$= \frac{n-1}{2}(-n) = \frac{-n^2 + n}{2} = -\frac{n^2}{2} + \frac{n}{2}$$

$$\therefore \sum_{r=n}^{2n} U_r = \sum_{r=1}^{2n} U_r - \sum_{r=1}^{n-1} U_r = (-n^2 - 3n) - \left(-\frac{n^2}{2} + \frac{n}{2}\right) = -\boldsymbol{\frac{n^2}{2}} - \boldsymbol{\frac{7n}{2}}.$$

Geometric Progression

WORKED PROBLEMS

ZIMSEC November 2003 Paper 1

Given that $a_n = 4 + (0.1)^n$, show that

$$\sum_{n=N+1}^{2N} a_n = 4N + \frac{(0.1)^N}{9}[1 - (0.1)^N]$$

Solution

$$\sum_{n=N+1}^{2N} a_n = \sum_{n=1}^{2N} a_n - \sum_{n=1}^{N} a_n$$

Aside

$$\sum_{n=1}^{2N} a_n = \sum_{n=1}^{2N} 4 + (0.1)^n = \sum_{n=1}^{2N} 4 + \sum_{n=1}^{2N} (0.1)^n$$

$$= (4 \times 2N) + (0.1)^1 + (0.1)^2 + (0.1)^3 + \cdots. + (0.1)^{2N}$$

$$= 8N + Sum\ of\ (2N) terms\ of\ a\ GP$$

Using:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= 8N + \frac{0.1(1 - (0.1)^{2N})}{1 - 0.5}$$

$$= 8N + \frac{0.1(1 - (0.1)^{2N})}{0.9}$$

$$= 8N + \frac{(1 - (0.1)^{2N})}{9}$$

$$= 8N + \frac{1}{9} - \frac{(0.1)^{2N}}{9}$$

Also:

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} 4 + (0.1)^n = \sum_{n=1}^{N} 4 + \sum_{n=1}^{N} (0.1)^n$$

$$= (4 \times N) + (0.1)^1 + (0.1)^2 + (0.1)^3 + \cdots . + (0.1)^N$$

$$= 4N + Sum\ of\ (N) terms\ of\ a\ GP$$

Using:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= 4N + \frac{0.1(1 - (0.1)^N)}{1 - 0.5}$$

$$= 4N + \frac{0.1(1 - (0.1)^N)}{0.9}$$

$$= 4N + \frac{(1 - (0.1)^N)}{9}$$

$$= 4N + \frac{1}{9} - \frac{(0.1)^N}{9}$$

Now

$$\sum_{n=N+1}^{2N} a_n = \sum_{n=1}^{2N} a_n - \sum_{n=1}^{N} a_n = \left[8N + \frac{1}{9} - \frac{(0.1)^{2N}}{9}\right] - \left[4N + \frac{1}{9} - \frac{(0.1)^N}{9}\right]$$

$$= 8N + \frac{1}{9} - \frac{(0.1)^{2N}}{9} - 4N - \frac{1}{9} + \frac{(0.1)^N}{9}$$

$$= 4N + \frac{(0.1)^N}{9} - \frac{(0.1)^{2N}}{9}$$

$$\boldsymbol{= 4N + \frac{(0.1)^N}{9}[1 - (0.1)^N]\ \ (as\ required)}$$

ZIMSEC June 2018 Paper 1

Show that

$$\sum_{r=1}^{16}\left(\sqrt{3}\right)^r = 9\,840 + 3\,280\sqrt{3}.$$

Solution

$$\sum_{r=1}^{16}\left(\sqrt{3}\right)^r = \sqrt{3} + \left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^3 + \cdots + \left(\sqrt{3}\right)^r$$

Thus the above sequence is a GP with $a = \sqrt{3}$ and $r = \sqrt{3}$.

Now $\sum_{r=1}^{16}\left(\sqrt{3}\right)^r = Sum\ of\ 16\ terms\ of\ a\ GP.$

Using:

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{since } |r| > 1$$

$$S_{16} = \frac{\sqrt{3}\left[\left(\sqrt{3}\right)^{16} - 1\right]}{\sqrt{3} - 1}$$

$$= \frac{\sqrt{3}[6\,561 - 1]}{\sqrt{3} - 1} = \frac{\sqrt{3}[6\,560]}{\sqrt{3} - 1} = \frac{6\,560\sqrt{3}}{\sqrt{3} - 1}$$

$$= \frac{6\,560\sqrt{3}(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$= \frac{6\,560(3) + 6\,560\sqrt{3}}{\left(\sqrt{3}\right)^2 - (1)^2}$$

$$= \frac{19\,680 + 6560\sqrt{3}}{2}$$

$$= \mathbf{9\,840 + 3\,280\sqrt{3}} \quad (\textbf{\textit{as required}})$$

ZIMSEC November 2018 Paper 1

A geometric series has a common ratio $r < 0$, and its 4th and 6th terms are 2.56 and 1.6348, respectively.

Find the

(i) first terms and the common ratio, **[3]**
(ii) the sum to infinity of the series. **[2]**

Solution

(i) $T_4 = 2.56$ (i)

$T_6 = 1.6348$ (ii)

Now $T_n = ar^{n-1}$

$\Longrightarrow ar^3 = 2.56$ (iii)

$\Longrightarrow ar^5 = 1.6348$ (iv)

(iv) ÷ (iii) gives:

$$r^2 = \frac{4087}{6400}$$

$$\Longrightarrow r = \pm\sqrt{\frac{4087}{6400}}$$

$$\therefore\ r = -\frac{\sqrt{4087}}{80}\quad since\ r < 0$$

$ar^3 = 2.56$ (iii)

$$\Longrightarrow\ a = \frac{2.56}{r^3} = \frac{2.56}{\left(-\frac{\sqrt{4087}}{80}\right)^3}$$

$$\therefore\ a = -5.016524871 = -5.0(\text{to } 2sf).$$

(ii) $S_\infty = \frac{a}{1-r}$

$$= \frac{-5.016524871}{1-\left(-\frac{\sqrt{4087}}{80}\right)}$$

$$= \frac{-5.016524871}{1-\left(-\frac{\sqrt{4087}}{80}\right)}$$

$$= -2.788320495 = -2.8(\text{to } 2sf)$$

ZIMSEC June 2017 Paper 1

(i) Write down the first 3 terms of the sequence U_r, where

$$U_r = 2\left(\frac{1}{3}\right)^r, r = 1; 2; \dots$$

(ii) Find the value of n for which

$$\sum_{r=1}^{n} 2\left(\frac{1}{3}\right)^r = \frac{80}{81}$$

[5]

<u>**Solution**</u>

(i) $U_r = 2\left(\frac{1}{3}\right)^r$

$$U_1 = 2\left(\frac{1}{3}\right)^1 = 2\left(\frac{1}{3}\right) = \frac{2}{3}$$

$$U_r = 2\left(\frac{1}{3}\right)^2 = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}$$

$$U_3 = 2\left(\frac{1}{3}\right)^3 = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{27}$$

(ii) The above sequence is in GP with $a = \frac{2}{3}$ and $r = \frac{1}{3}$

$$\sum_{r=1}^{n} 2\left(\frac{1}{3}\right)^r = \frac{80}{81}$$

Now:

$$\sum_{r=1}^{n} 2\left(\frac{1}{3}\right)^r$$

is the sum of 'n' terms of the GP.
Using

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\sum_{r=1}^{n} 2\left(\frac{1}{3}\right)^r = \frac{80}{81} \Rightarrow \frac{\frac{2}{3}\left[1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}} = \frac{80}{81}$$

$$\Rightarrow \frac{\frac{2}{3}\left[1-\left(\frac{1}{3}\right)^n\right]}{\frac{2}{3}} = \frac{80}{81}$$

$$\Rightarrow \frac{\frac{2}{3}\left[1-\left(\frac{1}{3}\right)^n\right]}{\frac{2}{3}} = \frac{80}{81}$$

$$\Rightarrow 1-\left(\frac{1}{3}\right)^n = \frac{80}{81}$$

$$\Rightarrow 1-\frac{80}{81} = \left(\frac{1}{3}\right)^n$$

$$\Rightarrow \frac{1}{81} = \left(\frac{1}{3}\right)^n$$

$$\Rightarrow In\left(\frac{1}{81}\right) = In\left(\frac{1}{3}\right)^n$$

$$\Rightarrow In\left(\frac{1}{81}\right) = nIn\left(\frac{1}{3}\right)$$

$$\therefore n = \frac{In\left(\frac{1}{81}\right)}{In\left(\frac{1}{3}\right)} = 4.$$

Practise Problems

Question 1

a) Trockers saves \$10 in the first month, \$12 in the second month and increases the monthly savings by \$2 each month. How long will it take Trockers to save \$500? **[4]**

b) The first term of a *GP* exceeds the second term by 4 and the sum of the *2nd* and *3rd* terms is $2\frac{2}{3}$. Find the first three terms given that $r > 0$. **[4]**

Question 2

a) A sequence is defined by the recurrence relation $U_{n+1} = |6 - U_n|$. If $U_1 = 2$, describe the behaviour of the sequence. **[2]**

b) The first term of an arithmetic progression is 4 and the sum of its first 11 terms is 209. Find the common difference and the 4th term. **[3]**

Question 3

Given that a sequence $U_1, U_2, U_3, \ldots$ is defined by $U_1 = 3$; $(n \geq 1)$ where $U_{n+1} = 3 - \frac{2}{U_n}$. Find the next three terms of the sequence and describe the behaviour of this sequence as $n \to \infty$. **[4]**

Show that these terms are the same as those we get when using the relation

$$U_n = \frac{2^{n+1} - 1}{2^n - 1}, n \in \mathbb{Z}^+.$$ **[3]**

Question 4

a) If the n[th] term of an arithmetic progression is $4n - 7$, find the sum of the first 40 terms. **[4]**

b) A geometric progression has first term a and common ratio r. Given that the sum of first n terms is 422, show that $ar^{n-1} = \frac{422(r-1)+a}{r}$. **[4]**

If, in addition, the sum of the first and the terms are 32 and 162 respectively, find

r and n. **[4]**

Question 5

a) Write down the first five terms of the sequence U_n, where

$$U_{n+2} = U_{n+1} + 2U_n,\ U_1 = 2 \text{ and } U_2 = 3$$

States whether the sequence oscillates, converges or diverges. **[4]**

b) Given that the sum of the first two terms of a geometric progression is 90 and the sum to infinity is $91\frac{3}{7}$, find two possible values of the common ratio. **[4]**

Question 6

(a) An arithmetic progression has 13 terms whose sum is 143. The third term is 5. Find the first term. **[3]**

(b) The sum of the first 3 terms of a GP is 7 times the first term. Find the possible values of the common ratio r. **[3]**

Hence, using the value of the first term you obtained in 14 (a) above, find the sum to infinity of this progression. **[3]**

Question 7

The 1st term of an AP is 2 and the common difference is 3. Find the sum of n (S_n) of this progression and the least value of n such that $S_n > 10\,000$. Hence find **[4]**

$$S_n = \sum_{r=0}^{n-1} 3^r$$

[3]

Question 8

The population of a colony of insects increases in such a way that if it is N at the beginning of a week, then at the end of the week it is $a + bN$, where a and b are constants and

$a < b < 1$.

Starting from the beginning of week when the population is N write down an expression for the population at the end of two weeks. Show that at the end of c weeks the population is $a\left(\frac{1-b^c}{1-b}\right) + b^c N$. **[4]**

When $a = 2000$ and $b = 0.2$, it is known that the population takes about 4 weeks to increase from N to $2N$. Estimate the value of N from this information. **[2]**

Question 9

(a)The positive numbers $p, 6$ and q are three consecutive terms of a GP. p and q are also the 3rd and 5th terms respectively on an AP whose common difference is $2\frac{1}{2}$. Find the possible values of p and q and the common ratio of the GP. **[6]**

(b)The sum to infinity of a GP is $\frac{2}{3}$ and the first term is $\frac{1}{12}$. Find the common ratio. **[3]**

Question 10

The annual output from an oil well diminishes each year by 5%. $200\,000$ barrel were extracted in the first year. It will become uneconomic to use the well when the output isles that 80 000 barrels per year.

a) For how many years can the well be used? **[4]**
b) How much oil (in barrels correct to 2 $s.f.$) will have been extracted by then? **[4]**

Question 11

(a)The 4th and 8th terms of a GP are 3 and $\frac{1}{27}$ respectively. Find the possible values of a and r.

[6]

(b)If $x+2, x+3$ and $2x^2+1$ are three consecutive terms of an AP, find the possible values of x. **[4]**

Past Examination Questions

ZIMSEC November 2017 Paper 1

In a geometric progression the first term is a the common ratio is r where $0 < r < 1$'

If the sum of the first 4 terms is half the sun to infinity, find

(i) the exact value of r, **[3]**
(ii) the 9^{th} term when $a = 2$. **[2]**

ZIMSEC November 2016 Paper1

(a) The present population of a country is 27 million. The population is projected to be 38 million in 36 years time.
Calculate the annual growth rate correct to two decimal places, if this projection is based on a geometric sequence. **[3]**
(b) Find the sum, S, defined by

$$S = \sum_{n=1}^{15}\left(2n + \frac{1}{2}\right)$$ [3]

November June 2015 Paper 1

(a) The 4^{th} term of an arithmetic progression is 42 and the sum of the first 3 terms of the series is 12.

Find the

(i) first term and the common difference,
(ii) sum of the first twenty terms. **[6]**

(b) The 3rd term of a geometric progression is 36 and the 5th terms of the series is 12.

Find the

(i) first term and the common ratio, r, given $r < 0$,
(ii) the sum to infinity of the series. **[6]**

ZIMSEC November 2007 Paper 1

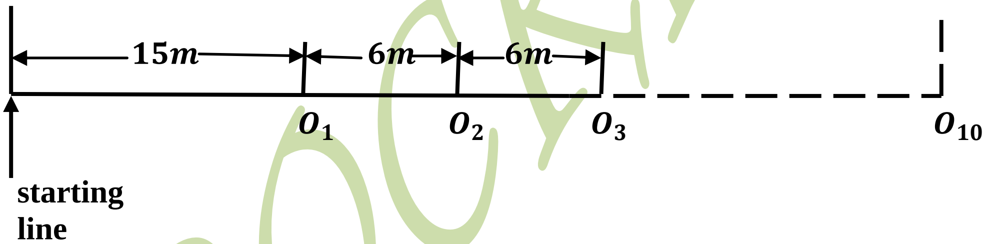
(a) It is given that

$$\sum_{r=1}^{n} U_r = n^2 - 7n$$

Find

(i) $u_1, u_2, u_3,$ **[4]**
(ii) an expression for u_r in terms of r. **[2]**

(b) The diagram shows 10 oranges O_1, O_2, O_3, ... O_{10} placed in a straight line $6m$ apart with the first orange $15m$ from the starting line.



In a race, each competitor runs and collect the oranges one at a time starting with O_1 and returns it to a box at a starting line. Find the total distance covered by a contestant who manages to collect all the oranges. **[4]**

ZIMSEC June 2013 Paper 1

(a) The sum of the first 5 terms of a geometric series is 5 and the sum of the fifth to the ninth is 80,

Show that the common ratio is ± 2 and hence find two possible values of a. **[6]**

(b) (i) Express $4x^3 - 6x^2 + 2x$ in the form $Ax(x+1)(x+2) + Bx(x+1) + Cx$ where A, B and C are constants. **[3]**

(iii) Given that

$$\sum_{x=1}^{x=n} x(x+1)(x+2) = \frac{n}{4}(n+1)(n+2)(n+3) \text{ and } \sum_{x=1}^{x=n} x(x+1) = \frac{n}{3}(n+1)(n+2),$$

show that

$$\sum_{x=1}^{x=n} 4x^3 - 6x^2 + 2x = n^2(n^2-1).$$ **[4]**

ZIMSEC November 2000 Paper 1

(a) The nth term of an arithmetic progression is denoted by u_n. Given that $u_{10} = 82$ and $u_{20} = 172$, find the sum of the first 50 terms of the progression. **[6]**

(b) The nth term of a sequence is given by

$$v_n = 9 - 4\left(\frac{1}{2}\right)^{n-1}, \text{for } n \geq 1$$

(i) Show that

$$\sum_{n=1}^{N} v_n = 9N - 8 + 8\left(\frac{1}{2}\right)^{N}.$$

[4]

(ii) Deduce that, for large N,

$$\sum_{n=1}^{N} v_n \approx u_N,$$

Where u_N is the Nth term of the arithmetic progression in part **(a)**. **[2]**

ZIMSEC November 1998 Paper 1

(i) A sequence of positive integers $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 2 \text{ and } u_{n+1} = 2u_n \text{ for } n \geq 1.$$

(a) Write down the first four terms of this sequence. **[1]**

(b) State what type of sequence this is, and express u_n in terms of n. **[2]**

(ii) A sequence of positive integers $v_1, v_2, v_3, \ldots$ is given by

$$v_1 = 2 \text{ and } v_{n+1} = 2v_n - 1 \text{ for } n \geq 1.$$

(a) Show that the relation between v_{n+1} and v_n may be written in the form

$$v_{n+1} - 1 = 2(v_n - 1)$$ **[1]**

(b) Hence, by using the results in part **(i)**, show that $v_n = 2^n + 1$ for $n \geq 1$. **[2]**

(iii) Express

$$\sum_{n=1}^{N} v_n \text{ in terms of } N. \qquad [4]$$

ZIMSEC June 1998 Paper 1

(a) Show that

$$\sum_{r=1}^{50}(3r-2) = 3725. \qquad [3]$$

Hence write down the value of

$$\sum_{r=1}^{50}(2-3r) \qquad [1]$$

(b) I save $\$c$ every month, at 0.5% per month interest on the total amount saved. It is given that the rate at end of n months is

$$\$(1.005c + 1.005^2c + 1.005^3c + \cdots + 1.005^n c)$$

Find the number of months at the end of which the total exceeds $\$100c$. **[4]**

Ministry of Education and Cambridge November 1997 Paper 1

(i) Evaluate

$$\sum_{r=1}^{500}(3r+2) \qquad [3]$$

(ii) Given that

$$\sum_{r=1}^{n}(ar+b) \equiv n^2,$$

find the constants a and b.

ZIMSEC November 1996 Paper 1

When a new stretch of motorway was opened, the peak rate at which traffic flowed was to be 5000 vehicles per hour.

a) One mathematical model for future rates of flow was that the peak rate would increase by 6% each year. Using this model,
   (i) Express the peak rate of flow after n years in terms of n. **[2]**

(ii) Find the integer value of n for which the peak rate of flow first exceeds 20 000 vehicles per hour. **[2]**

b) An alternative model stated that peak rate of flow after n years, denoted f_n vehicles per hour, would be given by

$$f_n = 5000\{6 - 5 \times (0.96)^n$$

Using this model,

(i) find the predicted percentage increase in the peak rate of flow over the first year, **[2]**

(ii) find the integer value of n for which f_n first exceeds 20 000. **[2]**

Describe how the predictions of the models in **(a)** and **(b)** differs as n becomes large. **[2]**

ZIMSEC June 2011 Paper 1

A teacher earns $\$x$ in his first year of working.

If his annual salary increases by 10% of his first year's salary,

(i) show that his total salary after n years is $\frac{nx(n+19)}{20}$,

(ii) Hence calculate the least value of n when his total salary exceeds 100 times his first salary. **[8]**

ZIMSEC November 2012 Paper 1

The 6th term of a geometric progression is $\frac{1}{81}$ and the 3rd term is $-\frac{1}{3}$.

Find

(a) the value of

(i) common ratio,

(ii) first term **[4]**

(b) the sum of the first n terms of the progression, **[2]**

(c) the value of the sum to infinity of the progression. **[1]**

ZIMSEC November 2013 Paper 1

A circular plank is cut into 12 sectors whose areas are in arithmetic progression. If the area of the largest is twice that of the smallest, find the angle in terms of π between the straight edges of the smallest sector. **[4]**

ZIMSEC November 2005 Paper 1

The sequence U_1, U_2, U_3 is defined by $U_{n+1} = U^2{}_n - 1$.

(a) Describe the behaviour of the sequence for each of the following cases.

(i) $U_1 = 0$ **[2]**

(ii) $U_2 = 2$ **[2]**

(b) Given that $U_2 = U_1$, find two possible values of U_1 in exact form. **[2]**

(c) Given also that $U_3 = U_1$, show that ${U^4}_1 - 2{U^2}_1 - U_1 = 0$. **[3]**

ZIMSEC June 2006 Paper 1

(a) Write down the first five terms of the sequence U_n, where $U_{n+2} = U_{n+1} + 2U_n$, $U_1 = 2$ and $U_2 = 3$. **[3]**

State whether the sequence oscillates, converges or diverges. **[1]**

(b) Given that the sum of the first two terms of a geometric progression is 90 and the sum to infinity is $91\frac{3}{7}$, find the two possible values of the common ratio. **[4]**

ZIMSEC June 2007 Paper 1

Two sequences are defined for $n = 1, 2, 3, \ \dots$, as follows

$$U_n = 10n - 3,$$

$$V_n \left[1 - \left(\frac{1}{n}\right)^n\right].$$

(i) Describe the behaviour of each sequence as $n \to \infty$. **[2]**

(ii) Given that

$$\sum_{n=1}^{N} U_n = 259,$$

find N. **[5]**

ZIMSEC June 2008 Paper 1

In an arithmetic progression the sum of the first term and the fifth term is zero. The thirteenth term is 20.

Find

(i) the first term and the common difference, **[4]**

(ii) the ratio of the seventh term to the fifth term. **[2]**

ZIMSEC June 2014 Paper 1

(a) The numbers $p, 10$ and q are three consecutive terms of an arithmetic progression. The numbers $p, 6$ and q are three consecutive terms of a geometric progression.

(i) By first forming two equations in $p, 10$ and p and $p, 10$ and q show that $p^2 - 20p + 36 = 0$.

(ii) Hence find the values of p and q for which the geometric progression converges.

[5]

(b) A woman measures the height of her child at birth and at monthly intervals afterwards.
The child's height increases by 5% per month. Find the number of measurements she has made before the child's height is twice what it was at birth. **[4]**

ZIMSEC November 2006 Paper 1

Question 1

A sequence is defined by $U_{n+1} = \frac{1}{1-U_n}$, with $U_1 = a$, where $a \neq 1$.

Show that $U_2 = U_5$. **[4]**

Question 2

(a) Given that

$$\sum_{r=1}^{n} (2r - 5) = 255,$$

find the value of n.

(b) After running a $40km$ marathon race, an athlete 'trains down' by running 80% of the distance run the previous day, starting the day after the competition.

Find the

(i) the distance run on the tenth day after the marathon race **[4]**

(ii) the first day on which the athlete will have run a total of more than $155km$ after the marathon race. **[5]**

ZIMSEC November 2012 Paper 1

A thin string of length $220cm$ is divided (without cutting it up), into n parts in he ratio $1: 2: 3: \ldots n$.

(a) Show that the length of the longest segment is $\frac{440}{n+1} cm$. **[3]**

(b) The string is made to form a circle so that the ends just touch each other.

(i) Find the radius of this circle.
(ii) Show that the angle subtended at the centre by the largest segment in part **(a)** is approximately $\frac{12.57}{n+1}$ radians.
(iii) Hence find the smallest value of n for which this angle is less than $\frac{\pi}{12}$ radians.

[6]

ZIMSEC June 2000 Paper 1

(i) The first four terms u_1, u_2, u_3, u_4 of an arithmetic progression are such that

$$u_4 - u_2 = 15 \quad \text{and} \quad 4u_3 = 9u_1.$$

Find the value of u_1. **[3]**

(ii) The four terms v_1, v_2, v_3, v_4 of a geometric progression are such that

$$v_4 - v_2 = 15 \quad \text{and} \quad 4v_3 = 9v_1.$$

Find the value of v_1. **[5]**

Ministry of Education and Cambridge June 1997 Paper 1.

Find the values of θ, such that $-\pi < \theta < \pi$, for which the geometric progression

$$1 + 2cos^2\theta + 4cos^4\theta + 8cos^6\theta + \cdots$$

has a sum to infinity. **[5]**

Show that, for this set of values of θ, the sum to infinity of the progression is $-sec2\theta$. **[2]**

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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