Small Changes and Approximations

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SYLLABUS (6042) REQUIREMENTS

- Define error
- Distinguish between absolute error and relative error
- Estimate errors in calculation including the use of

$$\delta y \approx \frac{dy}{dx}\delta x$$

ERROR

Definition: It is the difference between the measured/calculated value and the true/exact value.

Types Errors

(i) **Absolute Error:** It is the difference between the actual and measured value

$$Absolute\ error = |Actual - Estimate|$$

The absolute error in y is given by δy.

(ii) **Relative Error:** It is the absolute error divided by the actual measurement

$$Relative\ error = \frac{|Actual - Estimate|}{Actual}$$

The relative error in y is given by $\frac{\delta y}{y}$.

(iii) **Percentage Error:** It is the relative error shown as a percentage

$$Percentage\ error = \frac{|Actual - Estimate|}{Actual} \times 100\%$$

The absolute error in y is given by $\frac{\delta y}{y} \times 100\%$.

The percentage change is an increase when it is positive or a decrease when it is negative.

SMALL ERRORS

NOTES

- The small change in y is denoted by δy and the small change in x is denoted by δx.
- δx is pronounced as 'delta x'
- If δx is so small then $\frac{\delta y}{\delta x}$ will be a good approximation for $\frac{dy}{dx}$ i.e $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$
- $\Longrightarrow \delta y \approx \frac{dy}{dx}\delta x$

- The quantity $\frac{dy}{dx}\delta x$ is known as the 'total differential of y' (or simply 'the differential of y'). It provides an approximation (including the appropriate sign) for the increment, δy, in y subject to an increment of δx in x.
- It is important not to use the word 'differential' when referring to 'derivative'. Rather, the correct alternative to 'derivative' is 'differential coefficient'
- $\delta y \approx \frac{dy}{dx}\delta x$is useful in determining the change in one variable given the small change in the second variable.

NB: *It is usually more meaningful to discuss increments in the form of a percentage, since this gives a better idea of how much a variable has a changed in proportion to its original value.*

Derivation of $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ using a simple example

Given that $y = x^2 - 2x + 1$. Lets consider the point $(0; 1)$. Suppose x increased to 0.04, y decreased to 0.9214. Then:

$$\delta x = 0.04 - 0 = 0.04$$

and

$$\delta y = 0.9214 - 1 = -0.0786$$

Now:

$$\frac{\delta y}{\delta x} = \frac{-0.0786}{0.04} = -1.965$$

Also
$$y = x^2 - 2x + 1$$
$$\frac{dy}{dx} = 2x - 2 = 2(0) - 2 = -2$$

Thus

$$-1.965 \approx -2$$

$$\left(\frac{\delta y}{\delta x} = -1.965\right) \approx \left(\frac{dy}{dx} = -2\right)$$

$$\therefore \frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\Rightarrow \delta y \approx \frac{dy}{dx}\delta x$$

WORKED EXAMPLES

Example

An error of 3% is made in measuring the radius of a sphere. Find the percentage error in volume.

$V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2$$

$\delta x = \frac{3}{100} r.$

Now

$$\delta V \approx \frac{dV}{dr}\delta r$$

$$\delta V = (4\pi r^2)\frac{3}{100} r$$

$$\therefore \delta V = \frac{12\pi r^3}{100}$$

Percentage error of y is given by:

$$\frac{\delta V}{V} \times 100\%$$

$$= \frac{\frac{12\pi r^3}{100}}{\frac{4}{3}\pi r^3} \times 100\%$$

$$= \frac{\frac{12}{100}}{\frac{4}{3}} \times 100\%$$

$$= \frac{9}{100} \times 100\%$$

$$= 9\%$$

$\therefore$ The percentage increase is 9%.

November 2000 ZIMSEC Paper 1

The area A of a circle is calculated from a measured value of the radius r. There is a small error δr in measuring the radius.

(i) Find an approximate expression for the measuring error δA in the calculated value of the area.

(ii) Deduce that relative error in the area is approximately twice the relative error in the radius.

Suggested solution

(i) $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

Now

$$\delta A \approx \frac{dA}{dr}\delta r$$
$$\therefore \delta A = 2\pi r \delta r$$

(ii) Relative error of A is given by:

$$\frac{\delta A}{A} = \frac{2\pi r \delta r}{\pi r^2}$$
$$= \frac{2\delta r}{r}$$
$$2\frac{\delta r}{r}$$

∴ The relative error in the area is twice the relative error in the radius.

June 2012 ZIMSEC Paper 1

If $x^3y^2 = 1$, find the approximate percentage decrease in y given that x increases by 0.5%.

Suggested solution

$x^3y^2 = 1$

$$2x^3y\frac{dy}{dx} + 3x^2y^2 = 0$$
$$\frac{dy}{dx} = \frac{-3x^2y^2}{2x^3y}$$

$$\therefore \frac{dy}{dx} = \frac{-3y}{2x}$$

$\delta x = \frac{0.5}{100}x.$

Now

$$\delta y \approx \frac{dy}{dx}\delta x$$
$$\delta y = \frac{-3y}{2x}\left(\frac{0.5}{100}x\right)$$
$$\therefore \delta y = \frac{-1.5y}{200}$$

Percentage error of y is given by:

$$\frac{\delta y}{y} \times 100\%$$

$$= \frac{\frac{-1.5y}{200}}{y} \times 100\%$$
$$= -0.75\%$$

$\therefore$ The percentage decrease is 0.75%.

November 2015 ZIMSEC Paper 1

Given that $y = 2x^3 + \frac{8}{x}$, find the percentage increases in y as x increases from 2 to 2.003.

Suggested solution

$y = 2x^3 + \frac{8}{x} = 2x^3 + 8x^{-1}$

$$\frac{dy}{dx} = 6x^2 - \frac{8}{x^2}$$

$$\frac{dy}{dx} = 6(2)^2 - \frac{8}{(2)^2}$$

$$\therefore \frac{dy}{dx} = 22.$$

$\delta x = 2.003 - 2 = 0.003.$
Now

$$\delta y \approx \frac{dy}{dx}\delta x$$
$$\delta y = 22(0.003)$$
$$\therefore \delta y = 0.066$$

Percentage error of y is given by:

$$\frac{\delta y}{y} \times 100\%$$
$$= \frac{0.066}{2(2)^3 + \frac{8}{(2)}} \times 100\%$$
$$= \frac{0.066}{20} \times 100\%$$

$$= 0.33\%$$

$\therefore$ The percentage increase is 0.33%.

PRACTICE QUESTIONS

[1] If the radius of the sphere increased from $10cm$ to $10.1cm$, what is the approximate increase in surface area? **[ANSWER: $25.13cm^2$]**

[2] The height of a cylinder is $10cm$ and its radius is $4cm$. Find the approximate increase in volume when the radius increases to $4.02cm$. **[ANSWER: $5.03cm^3$]**

[3] An error of 3% is made in measuring the radius of a sphere. Find the percentage error in volume. **[ANSWER: 9%]**

[4] An error of $2\frac{1}{2}\%$ is made in the measurement of the area of a circle. What is the percentage error in

(a) the radius

(b) the circumference. **[ANSWER: a) 1.25% b) 1.25%]**

[5] One side of a rectangle is three times the other. If the perimeter increases by 2%. What is the percentage increase in the area? **[ANSWER: 4%]**

[6] The kinetic energy K of a body of mass m moving with speed V is given by $K = \frac{1}{2}mV^2$. If a body's speed is increased by 1.5%, what is the approximate percentage change in the kinetic energy? **[ANSWER: 3%]**

PAST EXAMINATION QUESTION

November 1996 ZIMSEC Paper

The length u, v and f are related by the equation

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$

Given that f is a constant, but that u and v can vary, show that

$$\frac{dv}{du}=-\frac{v^2}{u^2}$$

Hence find the approximate relative error in v arising from a relative error of 1% in u, in the case when $u = 2v$.

November 1998 ZIMSEC Paper 1

Expand $(1-2x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^2.
State the set of values of x for which the expansion is valid.
Calculate the relative error in using these terms of the expansion as an approximation for $(1-2x)^{-\frac{1}{2}}$ when $x = 0.1$.

June 2007 ZIMSEC Paper 1

Question 5

A gas law is given by the equation $Inp + 1.4Inv = Inc$, where c is a constant.
Find

(i) $\frac{dy}{dx}$ in terms of v and p,
(ii) the percentage change in v given that p increases by 1% and state whether it is an increase or a decrease.

Question 9

The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius.

Show that $V = \frac{1}{6}\pi D^3$, where D is the diameter of the sphere.

Write down an expression for $\frac{dV}{dD}$.

Hence estimate the maximum error in computing the volume of a sphere whose diameter is measured as $16cm$ with error $0.01cm$.

November 2007 ZIMSEC Paper 1

A student in a physics class measured the diameter x, of a cylindrical wire of fixed length l. He used this to calculate the volume V of the wire. If an error δx, in measurement of the diameter, results in an error of δv in the calculated volume, write down the relationship between δx and δv. Hence show that the relative error in the calculated volume is approximately twice the relative error in the diameter.

November 2009 ZIMSEC Paper 1

The two variables x and y are related by the equation $y = 5 - \frac{3}{x}$.Given that the value of y increases from 4 by a small amount $\frac{p}{25}$, determine in terms of p,

(i) the approximate change in x,
(ii) the corresponding percentage change in x.

June 2013 ZIMSEC Paper 1

If 1% error is made in measuring the diameter of a sphere, find approximately, the resulting percentage error in

(i) the volume
(ii) the surface are of the sphere

$$\begin{bmatrix} \boldsymbol{Volume\ of\ a\ sphere = \frac{4}{3}\pi r^3} \\ \boldsymbol{Surface\ area\ of\ a\ sphere = 4\pi r^2} \end{bmatrix}$$

November 2013 ZIMSEC Paper 1

Given that $x = yInx = Iny$,

(i) find $\frac{dy}{dx}$ in terms of x and y,
(ii) find an expression for the approximate value of y when $x = 1 + h$ where h is a small increase in x

June 2014 ZIMSEC Paper 1

Given that $y = \frac{1}{\sqrt{2x-1}}$,find the percentage change in y if x is decreased by 5% when $x = 5$.

November 2014 ZIMSEC Paper 1

Given that $F = kS^2$, where k is a constant, find the percentage change in F if S increases by 5%.

June 2017 ZIMSEC Paper 1

The radius of a spherical globe was increased from $11cm$ to $11.02cm$. Determine the percentage increase in the volume of the globe.

$$\left[\boldsymbol{Volume\ of\ sphere} = \frac{4}{3}\pi r^3\right]$$

November 2017 ZIMSEC Paper 1

Quantities x and y are related by the equation $x^3y^2 = c$ where c is a constant. Find the approximate percentage decrease in y if x increases by 0.5%.

APPROXIMATIONS

NOTES

A more rigorous approach to the calculation of δy is to use the result known as 'Taylor's Theorem', which, in this context would give the formula

$$f(x+\ \delta x) = f(x) + f^1(x)(\delta x) + \frac{f^2(0)}{2!}(\delta x)^2 + \frac{f^3(x)}{3!}(\delta x)^3 + \cdots$$

Hence, if δx is small enough for powers of two and above to be neglected then

$$f(x+\ \delta x) - f(x) \approx f^1(x)(\delta x)$$

$$\Rightarrow \boldsymbol{f(x+\ \delta x) \approx f(x) + \frac{dy}{dx}(\delta x)}$$

WORKED EXAMPLE

November 2018 ZIMSEC Paper 2

The equation $y = \sqrt{x}$ is used to approximate the square root of a number. Find an approximation for $\sqrt{16.01}$

Suggested solution

$f(x) = \sqrt{16.} = 4$ and $f(x + \delta x) = \sqrt{16.01} = \sqrt{16 + 0.01}$

$\therefore \delta x = 0.01$ and $x = 16$

$y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Now

$$\frac{dy}{dx}(\delta x) = \frac{1}{2\sqrt{16}}(0.01)$$

$$\frac{dy}{dx}(\delta x) = \frac{1}{8}(0.01) = 0.00125$$

Now

$$f(x + \delta x) \approx f(x) + \frac{dy}{dx}(\delta x)$$

$$f(x + \delta x) = 4 + 0.00125$$

$$f(x + \delta x) = 4.00125$$

$\therefore$ The an approximation for $\sqrt{16.01}$ is 4.00125.

PRACTICE QUESTIONS

[1] Find an approximation for $\sqrt{4.003}$ without using a calculator.

[2] Find an approximation for $\sqrt{81.7}$ without using a calculator.

[3] Find an approximation for $\sqrt{36.01}$ without using a calculator.

[4] Find an approximation for $\sqrt{49.13}$ without using a calculator.

[5] Find an approximation for $\sqrt{25.04}$ without using a calculator.

[6] Find an approximation for $\sqrt{16.33}$ without using a calculator.

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONTRIBUTIONS ARE WELCOME; FEEL FREE TO CONTACT ME SO THAT WE CAN IMPROVE THE DOCUMENT TOGETHER.

ENJOY

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