Taylor Series & Maclaurin's Series

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SYLLABUS (6042) REQUIREMENTS

- use Taylor's and Maclaurin's series for approximation
- solve problems involving method of differences as well as Taylor's and Maclaurin's series

Taylor's and Maclaurin's Series

Taylor Series

- A Taylor series is a power series which is mainly used to represent any function in terms of a polynomial of degree n near the point $x = a$.
- These functions can be any type like exponential functions, trigonometric functions etc
- It is also called Taylors theorem or Taylors expansion
- The Taylor series expansion of a function $f(x)$ in ascending powers of $(x - a)$ or about the point $x = a$ or centred at a is given by:

$$P_n(x) = f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \infty$$

- This can be written in sigma form as:

$$P_n(x) = f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

NOTE: $f'(x),\ f''(x),\ \ldots,\ f^n(x)$ *are derivatives*

Conditions

- The function $f(x)$ has to be differentiable infinitely many times i.e. we can find each of the first derivative, second derivative, third derivative, and so on forever.
- The function $f(x)$ has to be defined near the value $x = a$.

Steps

To obtain the Taylor series expansion of a function $f(x)$ in ascending powers of $(x - a)$ or about the point $x = a$ or centred at a:

- Decide how many terms of the series you are going to generate *(Normally it is specified in the question)*. Call this number n.
- Calculate the derivatives of the function up to n (the required number of derivatives):

$$f'(x),\ f''(x),\quad \ldots,\quad f^n(x)$$

- Evaluate the function at $x = a$ i.e. $f(a)$
- Evaluate the derivatives at $x = a$ to obtain:

$$f'(a),\ f''(a),\quad \ldots,\quad f^n(a)$$

Maclaurin's Series

- Is the special case of the Taylor expansion where $a = 0$.
- If $f(x)$ can be expanded as an infinite series, then:

$$P_n(x) = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \infty$$

- This can be written in sigma form as:

$$P_n(x) = f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Steps

To obtain the Maclaurin's series expansion of a function $f(x)$ in ascending powers of x:

- Decide how many terms of the series you are going to generate *(Normally it is specified in the question)*. Call this number n.
- Calculate the derivatives of the function up to n (the required number of derivatives):

$$f'(x),\ \ f''(x),\qquad \ldots ,\qquad f^n(x)$$

- Evaluate the function at $x = 0$ i.e. $f(0)$
- Evaluate the derivatives at $x = 0$ to obtain:

$$f'(0),\ \ f''(0),\qquad \ldots,\quad f^n(0)$$

SOLVED PROBLEMS

Question 1

Find the first four terms in the Taylor series for the function $f(x) = (x-1)e^x$ near $x = 1$.

Suggested Solution

Method 1

Find the Taylor series for e^x and then multiply by $(x-1)$.

Let $g(x) = e^x$

$g(x) = e^x \quad \Rightarrow \quad g(1) = e^1 = e$

$g'(x) = e^x \quad \Rightarrow \quad g'(1) = e^1 = e$

$g''(x) = e^x \quad \Rightarrow \quad g''(1) = e^1 = e$

$g'''(x) = e^x \quad \Rightarrow \quad g'''(1) = e^1 = e$

Now:

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \frac{gf'''(a)}{3!}(x-a)^3$$

$$\Rightarrow \; e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3$$

Now:

$f(x) = (x-1)e^x$

$$\Rightarrow \; f(x) = (x-1)\left[e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3\right]$$

$$\therefore \; f(x) = e(x-1) + e(x-1)^2 + \frac{e}{2}(x-1)^3 + \frac{e}{6}(x-1)^4$$

Alternatively:

Method 2

Let $f(x) = (x-1)e^x$.

$f(x) = (x-1)e^x \qquad \Rightarrow \qquad f(1) = (1-1)e^1 = 0$

$f'(x) = (x-1)e^x + e^x \qquad \Rightarrow \qquad f'(1) = (1-1)e^1 + e^1 = e$

$f''(x) = (x-1)e^x + 2e^x \qquad \Rightarrow \qquad f''(1) = (1-1)e^1 + 2e^1 = 2e$

$f'''(x) = (x-1)e^x + 3e^x \qquad \Rightarrow \qquad f'''(1) = (1-1)e^1 + 3e^1 = 3e$

$f^{iv}(x) = (x-1)e^x + 4e^x \qquad \Rightarrow \qquad f^{iv}(1) = (1-1)e^1 + 4e^1 = 4e$

Now

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\Rightarrow\ f(x) = (x-1)e^x = 0 + e(x-1) + \frac{2e}{2!}(x-1)^2 + \frac{3e}{3!}(x-1)^3 + \frac{4e}{4!}(x-1)^4$$

$$\therefore\ f(x) = (x-1)e^x = e(x-1) + e(x-1)^2 + \frac{e}{2}(x-1)^3 + \frac{e}{6}(x-1)^4$$

Question 2

Find the first four terms in the Maclaurin's series for the function $f(x) = (x-1)e^x$.

Suggested Solution

Method 1

$f(x) = (x-1)e^x \qquad \Rightarrow \qquad f(0) = (0-1)e^0 = -1$

$f'(x) = (x-1)e^x + e^x \qquad \Rightarrow \qquad f'(0) = (0-1)e^0 + e^0 = 0$

$f''(x) = (x-1)e^x + 2e^x \qquad \Rightarrow \qquad f''(0) = (0-1)e^0 + 2e^0 = 1$

$f'''(x) = (x-1)e^x + 3e^x \qquad \Rightarrow \qquad f'''(0) = (0-1)e^0 + 3e^0 = 2$

$f^{iv}(x) = (x-1)e^x + 4e^x \qquad \Rightarrow \qquad f^{iv}(0) = (0-1)e^0 + 4e^0 = 3$

Now

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \frac{f^{iv}(0)}{4!}(x)^4$$

$$\Rightarrow\ (x-1)e^x = -1 + 0(x) + \frac{1}{2!}(x)^2 + \frac{2}{3!}(x)^3 + \frac{3}{4!}(x)^4$$

$$\therefore\ f(x) = (x-1)e^x = -1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4$$

Method 2

From tables:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

Now:

$$f(x) = (x-1)e^x$$

$$= (x-1)\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4\right)$$

$$= \left(x + x^2 + \frac{1}{2!}x^3 + \frac{1}{3!}x^4 + \cdots - 1 - x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4\right)$$

$$= -1 + x - x + x^2 - \frac{1}{2!}x^2 + \frac{1}{2!}x^3 - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 - \frac{1}{4!}x^4$$

$$= -1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4$$

Question 3

Find the first four terms in the Taylor series for the function $f(x) = Inx$ centred at $a = 1$.

Suggested Solution

$f(x) = Inx \qquad \Rightarrow \quad f(1) = In1 = 0$

$$f'(x) = \frac{1}{x} \qquad \Rightarrow \quad f'(1) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad \Rightarrow \quad f''(1) = -\frac{1}{1} = -1$$

$$f'''(x) = \frac{2}{x^3} \qquad \Rightarrow \quad f'''(1) = \frac{2}{1} = 2$$

$$f'^{v}(x) = -\frac{6}{x^4} \qquad \Rightarrow \quad f'^{v}(1) = -\frac{6}{1} = -6$$

Now

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f'^{v}(a)}{4!}(x-a)^4$$

$$\Rightarrow f(x) = \ Inx = 0 + 1(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4$$

$$\therefore \ f(x) = Inx = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

Question 4

Find the first four terms in the Taylor series for the function $f(x) = Sinx$ centred at $x = \frac{\pi}{4}$.

Suggested Solution

$$f(x) = Sinx \qquad \Rightarrow \quad f(a) = Sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = Cosx \qquad \Rightarrow \quad f'(a) = Cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -Sinx \qquad \Rightarrow \quad f''(a) = -Sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -Cosx \qquad \Rightarrow \quad f'''(a) = -Cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

NB: *Since the fourth derivative of Sin x is just Sin x, then the sequence repeats forever.*

Now

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{iv}(a)}{4!}(x-a)^4$$

$$\Rightarrow f(x) = \ Sinx = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\frac{\sqrt{2}}{2}}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{\frac{\sqrt{2}}{2}}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{\frac{\sqrt{2}}{2}}{4!}\left(x - \frac{\pi}{4}\right)^4$$

$$\therefore\ f(x) = Sinx = \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} - \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(x - \frac{\pi}{4}\right)^4}{4!}\right]$$

Question 5

Find the Taylor series expansion about $x = 1$ for the function $f(x) = e^{x^2-2x}$ in ascending powers of $(x-1)$ up to and including the term in $(x-1)^2$.

Suggested Solution

$f(x) = e^{x^2-2x}$ $\Rightarrow$ $f(1) = e^{1-2} = e^{-1} = \frac{1}{e}$

$f'(x) = (2x-2)e^{x^2-2x}$ $\Rightarrow$ $f'(1) = (2 \times 1 - 2)e^{1-2} = 0$

$f''(x) = (2x-2)^2 e^{x^2-2x} + 2e^{x^2-2x}$ $\Rightarrow$ $f''(1) = (2-2)^2 e^{1-2} + 2e^{1-2} = 2e^{-1} = \frac{2}{e}$

Now

$$f(x) = f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a)^2$$

$$\Rightarrow f(x) = \ e^{x^2-2x} = \frac{1}{e} + 0(x-1) + \frac{\frac{2}{e}}{2!}(x-1)^2$$

$$\therefore\ f(x) = e^{x^2-2x} = \frac{1}{e} + \frac{1}{e}(x-1)^2$$

Question 6

Given that

$$\frac{dy}{dx} - 2y = 3e^x.$$

If the solution of this linear differential equation passes through the origin,

(i) Write down the first four terms of the Taylor series expansion of y near the origin

(ii) Find the value of y when $x = 0.1$.

Suggested Solution

(i) Taylor series expansion of y near the origin i.e. $y = 0$ when $x = 0$.

$$\frac{dy}{dx} - 2y = 3e^x \qquad \text{(i)}$$

NB: *Differentiate equation* (i)*to get* $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3e^x \qquad \text{(ii)}$$

NB: *Differentiate equation* (ii)*to get* $\frac{d^3y}{dx^3}$

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 3e^x \qquad \text{(iii)}$$

Substituting $y = 0$ and $x = 0$ into the equation $\frac{dy}{dx} - 2y = e^x$ (i):

$$\frac{dy}{dx} - 2(0) = 3e^0 \qquad \therefore \frac{dy}{dx} = 3 \qquad \Rightarrow f'(0) = 3$$

Substituting $x = 0$ and $\frac{dy}{dx} = 3$ into the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x$ (ii):

$$\frac{d^2y}{dx^2} - 2(3) = 3e^0 \qquad \therefore \frac{d^2y}{dx^2} = 9 \qquad \Rightarrow f''(0) = 9$$

Substituting $x = 0$ and $\frac{d^2y}{dx^2} = 9$ into the equation $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = e^x$ (iii):

$$\frac{d^3y}{dx^3} - 2(9) = 3e^0 \qquad \therefore \frac{d^3y}{dx^3} = 21 \qquad \Rightarrow f'''(0) = 21$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\Rightarrow f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \quad \text{since } x = 0.$$

$$\Rightarrow f(x) = 0 + 3x + \frac{9}{2\times 1}x^2 + \frac{21}{3\times 2\times 1}x^3$$

$$\Rightarrow f(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3$$

(ii) Finding y when $x = 0.1$:

$$\Rightarrow f(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3$$

$$= 0.3485$$

Question 7

Given that $y = Sec^2x$.

a) Show that $\frac{d^2y}{dx^2} = 6Sec^4x - 4Sec^2x$.

b) Find a Taylor series expansion of Sec^2x in ascending powers of $\left(x - \frac{\pi}{4}\right)$, up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$.

Suggested Solution

a) $y = Sec^2x = \frac{1}{Cos^2x}$

$$\frac{dy}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

$$= \frac{Cos^2x(0) - 1(2Cosx \times -Sinx)}{(Cos^2x)^2}$$

$$= \frac{2CosxSinx}{Cos^4x}$$

$$= \frac{2Sinx}{Cos^3x}$$

NB: *Differentiate* $\frac{dy}{dx}$ *to get* $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

$$= \frac{Cos^3x(2Cosx) - 2Sinx(3Cos^2x \times -Sinx)}{(Cos^3x)^2}$$

$$= \frac{2Cos^4x + 6Sin^2xCos^2x}{Cos^6x}$$

NB: $Sin^2x \equiv 1 - Cos^2x$

$$= \frac{2Cos^4x + 6Cos^2x(1 - Cos^2x)}{Cos^6x}$$

$$= \frac{2Cos^4x + 6Cos^2x - 6Cos^4x}{Cos^6x}$$

$$= \frac{6Cos^2x - 4Cos^4x}{Cos^6x}$$

$$= \frac{6Cos^2x}{Cos^6x} - \frac{4Cos^4x}{Cos^6x}$$

$$= \frac{6}{Cos^4x} - \frac{4}{Cos^2x}$$

NB: $Secx \equiv \frac{1}{Cosx}$ *and* $Sec^nx \equiv \frac{1}{Cos^nx}$

$$= 6Sec^4x - 4Sec^2x$$

b) NB: *Differentiate* $\frac{d^2y}{dx^2}$ *to get* $\frac{d^3y}{dx^3}$

$$\frac{d^2y}{dx^2} = \frac{6}{Cos^4x} - \frac{4}{Cos^2x}$$

$$\frac{d^3y}{dx^3} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

$$= \left[\frac{Cos^4x(0) - 6[4Cos^3x \times (-Sinx)]}{(Cos^4x)^2}\right] - \left[\frac{Cos^2x(0) - 4[2Cosx \times (-Sinx)]}{(Cos^2x)^2}\right]$$

$$= \frac{24Cos^3xSinx}{Cos^8x} - \frac{8CosxSinx}{Cos^4x}$$

$$= \frac{24Sinx}{Cos^5x} - \frac{8Sinx}{Cos^3x}$$

Now

$$f(x) = \frac{1}{Cos^2x} \quad \Rightarrow f\left(\frac{\pi}{4}\right) = 2$$

$$f'(x) = \frac{2Sinx}{Cos^3x} \quad \Rightarrow f'\left(\frac{\pi}{4}\right) = 4$$

$$f''(x) = \frac{6}{Cos^4x} - \frac{4}{Cos^2x} \quad \Rightarrow f''\left(\frac{\pi}{4}\right) = 16$$

$$f'''(x) = \frac{24Sinx}{Cos^5x} - \frac{8Sinx}{Cos^3x} \quad \Rightarrow f'''\left(\frac{\pi}{4}\right) = 80$$

Now

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{iv}(a)}{4!}(x-a)^4$$

$$\Rightarrow f(x) = Sec^2x = 2 + 4\left(x - \frac{\pi}{4}\right) + \frac{16}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{80}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$\therefore\ f(x) = Sec^2x = 2 + 4\left(x - \frac{\pi}{4}\right) + 8\left(x - \frac{\pi}{4}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{4}\right)^3$$

Question 8

Given that

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x, \qquad \text{at } x = 1, y = 0 \text{ and } \frac{dy}{dx} = 2$$

Use Taylor's series to find a solution of the differential equation in ascending powers of $(x - 1)$ up to and including the term in $(x - 1)^3$

Suggested Solution

When $x = 1$:

$y = 0 \qquad \Rightarrow f(1) = 0$

$\frac{dy}{dx} = 2 \qquad \Rightarrow f'(1) = 2$

Substitute $y = 0$ and $\frac{dy}{dx} = 2$ into the equation $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$ (i):

$\frac{d^2y}{dx^2} + (0)(2) = 1 \qquad \therefore \frac{d^2y}{dx^2} = 1 \qquad \Rightarrow f''(1) = 1$

Differentiate equation (i) to calculate $\frac{d^3y}{dx^3}$

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x \quad \text{(i)}$$

$$\Rightarrow \frac{d^3y}{dx^3} + \left[y\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx}\left(1\frac{dy}{dx}\right)\right] = 1 \qquad \text{(ii)}$$

Substitute $y = 0$, $\frac{dy}{dx} = 2$ and $\frac{d^2y}{dx^2} = 1$ into the equation (ii):

$\frac{d^3y}{dx^3} + 0(1) + 2(2) = 1$ $\therefore \frac{d^3y}{dx^3} = -3$ $\Rightarrow f'''(1) = -3$

Now:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\Rightarrow f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \text{ since } a = 1$$

$$\Rightarrow f(x) = 0 + 2(x-1) + \frac{1}{2!}(x-1)^2 + \frac{-3}{3!}(x-1)^3$$

$$\therefore f(x) = 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(x-1)^3 + \cdots$$

SOLVED PAST EXAMINATION QUESTIONS

Question 1

UCLES NOVEMBER 1992 PAPER 1

The variable x and y are related by $\frac{dy}{dx} = 2xy - 1$, and $y = 1$ at $x = 0$.

(a) Show that at $x = 0$, $\frac{d^3y}{dx^3} = -4$ and $\frac{d^4y}{dx^4} = 12$.

(b) Find Maclaurin's series for y up to and including the term in x^4, and hence find an approximation to the value of y at $x = 0.1$, giving your answer to an appropriate accuracy.

Suggested Solution

(a) $\frac{dy}{dx}_{x=0} = 2(0)y - 1 = -1$

$\frac{dy}{dx} = 2xy - 1$

$\frac{d^2y}{dx^2} = 2x\frac{dy}{dx} + 2y \quad (Use\ Implicit\ differentiation)$

$\frac{d^2y}{dx^2}_{x=0} = 2(0)(-1) + 2(1) = 2$

Differentiate $\frac{d^2y}{dx^2}$ to get $\frac{d^3y}{dx^3}$.

$\frac{d^3y}{dx^3} = 2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2\frac{dy}{dx}$

$\Rightarrow \frac{d^3y}{dx^3} = 2x\frac{d^2y}{dx^2} + 4\frac{dy}{dx}$

Now:

$\frac{d^3y}{dx^3}_{x=0} = 2(0)(2) + 4(-1)$

$= -4\ (As\ required)$

Differentiate $\frac{d^3y}{dx^3}$ to get $\frac{d^4y}{dx^4}$.

$$\frac{d^4y}{dx^4} = 2x\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 4\frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^4y}{dx^4} = 2x\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2}$$

Now:

$$\frac{d^4y}{dx^4}_{x=0} = 2(0)(-4) + 6(2)$$

$$= 12\ (As\ required)$$

(b) $f(0) = y_{x=0} = 1$

$$f'(0) = \frac{dy}{dx}_{x=0} = -1$$

$$f''(0) = \frac{d^2y}{dx^2}_{x=0} = 2$$

$$f'''(0) = \frac{d^3y}{dx^3}_{x=0} = -4$$

$$f'^{v}(0) = \frac{d^4y}{dx^4}_{x=0} = 12$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f'^{v}(0)}{4!}x^4$$

$$= 1 + (-1)x + \frac{2}{2\times 1}x^2 + \frac{-4}{3\times 2\times 1}x^3 + \frac{12}{4\times 3\times 2\times 1}x^4$$

$$= 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{2}x^4$$

Now:

$$f(x) = 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{2}x^4$$

$$\Rightarrow f(0.1) = 1 - 0.1 + (0.1)^2 - \frac{2}{3}(0.1)^3 + \frac{1}{2}(0.1)^4$$

$$= \frac{54563}{60\,000}$$

$$= 0.90938333333333$$

$$= 0.91\ (to\ 2\ significant\ figures)$$

Question 2

ZIMSEC NOVEMBER 2003 PAPER 1

Given that $y = In\left(e^{-x} - \frac{1}{2}\right)$, show that $\frac{dy}{dx} = -\frac{1}{2}\left(e^{-y} + 2\right)$.

By finding the second derivative, or otherwise, show that the series expansion of y in ascending powers of x, up to and including the term in x^2, is $-In2 - 2x - x^2$. **[5]**

Suggested Solution

$$y = In\left(e^{-x} - \frac{1}{2}\right)$$

$$e^{y} = e^{In\left(e^{-x} - \frac{1}{2}\right)}$$

$$e^{y} = e^{-x} - \frac{1}{2} \qquad \text{(i)}$$

Now:

$$e^{y}\frac{dy}{dx} = -e^{-x}$$

$$\frac{dy}{dx} = -\frac{e^{-x}}{e^{y}} \qquad \text{(ii)}$$

From (i):

$$e^{-x} = e^{y} + \frac{1}{2} \qquad \text{(iii)}$$

Substituting (iii) into (ii):

$$\frac{dy}{dx} = -\frac{e^{-x}}{e^{y}}$$

$$= -\frac{e^{y} + \frac{1}{2}}{e^{y}}$$

$$= -\left(\frac{e^y}{e^y} + \frac{\frac{1}{2}}{e^y}\right)$$

$$= -\left(1 + \frac{1}{2e^y}\right)$$

$$= -\left(1 + \frac{e^{-y}}{2}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}(e^{-y} + 2) \qquad (As\ required)$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}e^{-y}\frac{dy}{dx}$$

$$= -\frac{1}{2}(e^{-y} + 2)\left(\frac{1}{2}e^{-y}\right)$$

$$= -\frac{1}{4}(e^{-y} + 2)(e^{-y})$$

Evaluating $y = f(x)$ at $x = 0$:

$$\Rightarrow f(0) = In\left(e^0 - \frac{1}{2}\right) = In\left(\frac{1}{2}\right) = In(2^{-1}) = -In2$$

Evaluating $\frac{dy}{dx} = f'(x)$ at $x = 0$:

$$Aside: \frac{dy}{dx} = -\frac{1}{2}(e^{-y} + 2) = -\frac{1}{2}\left(\frac{1}{e^y} + 2\right) = -\frac{1}{2}\left(\frac{1}{\frac{2e^{-x}-1}{2}} + 2\right)$$

$$= -\frac{1}{2}\left(\frac{2}{2e^{-x} - 1} + 2\right)$$

$$\Rightarrow f'(0) = -\frac{1}{2}\left(\frac{2}{2e^0 - 1} + 2\right) = -\frac{1}{2}\left(\frac{2}{1} + 2\right) = -\frac{1}{2}(4) = -2$$

Evaluating $\frac{d^2y}{dx^2} = f''(x)$ at $x = 0$:

Aside: $\frac{d^2y}{dx^2} = -\frac{1}{4}e^{-y}(e^{-y}+2) = -\frac{1}{4}\left(\frac{1}{e^y}\right)\left(\frac{1}{e^y}+2\right)$

$$= -\frac{1}{4}\left(\frac{1}{\frac{2e^{-x}-1}{2}}\right)\left(\frac{1}{\frac{2e^{-x}-1}{2}}+2\right)$$

$$= -\frac{1}{4}\left(\frac{2}{2e^{-x}-1}\right)\left(\frac{2}{2e^{-x}-1}+2\right)$$

$$\Rightarrow f''(0) = -\frac{1}{4}\left(\frac{2}{2e^0-1}\right)\left(\frac{2}{2e^0-1}+2\right) = -\frac{1}{4}\left(\frac{2}{2-1}\right)\left(\frac{2}{2-1}+2\right)$$

$$= -\frac{1}{4}\left(\frac{2}{1}\right)\left(\frac{2}{1}+2\right) = -2$$

Now:

$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2$

$\Rightarrow\ f(x) = -In2 - 2(x) + \frac{-2}{2}(x)^2$

$\therefore\ y = f(x) = -In2 - 2x - x^2 \quad (As\ required)$

<u>Alternative Method</u>

$y = In\left(e^{-x} - \frac{1}{2}\right)$

$e^y = e^{In\left(e^{-x}-\frac{1}{2}\right)}$

$e^y = e^{-x} - \frac{1}{2} \qquad$ (i)

Now:

$e^y\frac{dy}{dx} = -e^{-x}$

$$\frac{dy}{dx} = -\frac{e^{-x}}{e^{y}} \qquad \text{(ii)}$$

From (i):

$$e^{-x} = e^{y} + \frac{1}{2} \qquad \text{(iii)}$$

Substituting (iii) into (ii):

$$\frac{dy}{dx} = -\frac{e^{-x}}{e^{y}}$$

$$= -\frac{e^{y} + \frac{1}{2}}{e^{y}}$$

$$= -\left(\frac{e^{y}}{e^{y}} + \frac{\frac{1}{2}}{e^{y}}\right)$$

$$= -\left(1 + \frac{1}{2e^{y}}\right)$$

$$= -\left(1 + \frac{e^{-y}}{2}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}(e^{-y} + 2) \qquad (As\ required)$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}e^{-y}\frac{dy}{dx}$$

Evaluating $y = f(x)$ at $x = 0$:

$$\Rightarrow f(0) = In\left(e^{0} - \frac{1}{2}\right) = In\left(\frac{1}{2}\right) = In(2^{-1}) = -In2$$

Evaluating $\frac{dy}{dx} = f^{'}(x)$ at $x = 0$:

$$\frac{dy}{dx} = -\frac{1}{2}(e^{-y} + 2)$$

$\Rightarrow f'(0) = -\frac{1}{2}\left(e^{-(-In2)} + 2\right) = -\frac{1}{2}(2+2) = -\frac{1}{2}(4) = -2$

Evaluating $\frac{d^2y}{dx^2} = f''(x)$ at $x = 0$:

$$\frac{d^2y}{dx^2} = \frac{1}{2}e^{-y}\frac{dy}{dx} = \frac{1}{2}e^{-(-In2)}(-2)$$

$$= \frac{1}{2}(2)(-2)$$

$$= -2$$

Now:

$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2$

$\Rightarrow f(x) = -In2 - 2(x) + \frac{-2}{2}(x)^2$

$\therefore y = f(x) = -In2 - 2x - x^2 \quad (As\ required)$

Question 3

ZIMSEC NOVEMBER 2006 PAPER 1

(i) By first expanding $cos(2x + x)$, prove that $cos3x = 4cos^3x - 3cosx$. **[3]**

(ii) Use the series expansion of $cosx$ given in the list of formulae to express $3cosx$ and $cos3x$ in terms of powers of x, up to and including the term in x^6.

Hence show that

$$cos^3x = 1 - \frac{3}{2}x^2 + \frac{7}{8}x^4 - \frac{61}{240}x^6 + \cdots \qquad [4]$$

(iii) Calculate the relative error in using $1 - \frac{3}{2}x^2 + \frac{7}{8}x^4 - \frac{61}{240}x^6$ as an approximation for cos^3x when $x = \frac{1}{3}\pi$, giving your answer to 1 significant figure. **[3]**

Suggested Solution

(i) $cos(2x + x) = cos2xcosx - sin2xsinx$

$$= cosx(2cos^2x - 1) - sinx(2sinxcosx)$$

$$= 2cos^3x - cosx - 2sin^2xcosx$$

$$= 2cos^3x - cosx - 2cosx(1 - cos^2x)$$

$$= 2cos^3x - cosx - 2cosx + 2cos^3x$$

$$= 4cos^3x - 3cosx \ (as\ required)$$

(ii) From tables: $Cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Now:

$$3cosx = 3\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= 3 - \frac{3}{2!}x^2 + \frac{3}{4!}x^4 - \frac{3}{6!}x^6 + \cdots$$

$$= 3 - \frac{3}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{240}x^6 + \cdots$$

$$cos3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots$$

$$= 1 - \frac{9}{2!}x^2 + \frac{81}{4!}x^4 - \frac{729}{6!}x^6 + \cdots$$

$$= 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6 + \cdots$$

$$cos3x = 4cos^3x - 3cosx$$

$$\Rightarrow cos^3x = \frac{cos3x + 3cosx}{4}$$

Now:

$$cos^3x = \frac{\left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6 + \cdots\right) + \left(3 - \frac{3}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{240}x^6 + \cdots\right)}{4}$$

$$= \frac{4 - 6x^2 + \frac{28}{8}x^4 - \frac{61}{60}x^6 + \cdots}{4}$$

$$= 1 - \frac{3}{2}x^2 + \frac{7}{8}x^4 - \frac{61}{240}x^6 + \cdots \quad (as\ required)$$

(iii) $Relative\ error = \dfrac{|Actual - Estimate|}{Actual}$

<u>Aside</u>

$$Actual = \left(cos\frac{1}{3}\pi\right)^3 = \frac{1}{8}$$

$$Estimate = 1 - \frac{3}{2}\left(\frac{1}{3}\pi\right)^2 + \frac{7}{8}\left(\frac{1}{3}\pi\right)^4 - \frac{61}{240}\left(\frac{1}{3}\pi\right)^6$$

$$= 0.072135213$$

Now:

$$Relative\ error = \frac{\left|\frac{1}{8} - 0.072135213\right|}{\frac{1}{8}}$$

$$= 0.422918294$$

$$= 0.40\ (to\ 1\ s.f.)$$

Question 4

ZIMSEC JUNE 2014 PAPER 1

Given that $f(x) = e^{3x}sinx$ and $f'(x) = e^{3x}cosx + 3e^{3x}sinx$,

(i) find $f''(x), f'''(x)$ and hence write down the Maclaurin series of $f(x)$ up to the term in x^3. **[7]**

(ii) evaluate $f(0.5)$ giving your answer correct to 4 decimal places. **[1]**

(iii)Use Maclaurin series obtained in (i) to estimate the value of $f(0.5)$ and state the absolute error in using the Maclaurin as an approximation. **[2]**

Suggested Solution

$f(x) = e^{3x}sinx$ and $f'(x) = e^{3x}cosx + 3e^{3x}sinx$

(i) Differentiate $f'(x)$ to obtain $f''(x)$:

$$f''(x) = e^{3x}(-sinx) + cosx(3e^{3x}) + 3(e^{3x}cosx + 3e^{3x}sinx)$$
$$= -e^{3x}sinx + 3e^{3x}cosx + 3e^{3x}cosx + 9e^{3x}sinx$$
$$= 6e^{3x}cosx + 8e^{3x}sinx$$

Differentiate $f''(x)$ to obtain $f'''(x)$:

$$f'''(x) = 6(-e^{3x}sinx + 3e^{3x}cosx) + 8(e^{3x}cosx + 3e^{3x}sinx)$$
$$= -6e^{3x}sinx + 18e^{3x}cosx + 8e^{3x}cosx + 24e^{3x}sinx$$
$$= 26e^{3x}cosx + 18e^{3x}sinx$$

Evaluating $y = f(x)$ at $x = 0$:

$\Rightarrow f(0) = e^0 sin0 = 0$

Evaluating $f'(x)$ at $x = 0$:

$\Rightarrow f'(0) = e^0 cos0 + 3e^0 sin0 = 1$

Evaluating $f''(x)$ at $x = 0$:

$\Rightarrow f''(0) = 6e^0 cos0 + 8e^0 sin0 = 6$

Evaluating $f'''(x)$ at $x = 0$:

$\Rightarrow f'''(0) = 26e^0 cos0 + 18e^0 sin0 = 26$

Now:

$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3$

$\Rightarrow\ f(x) = 0 + 1(x) + \frac{6}{2!}(x)^2 + \frac{26}{3!}(x)^3$

$\therefore\ y = f(x) = x + 3x^2 + \frac{13}{3}x^3$

(ii) $f(x) = e^{3x} sinx$

$f(0.5) = e^{3(0.5)} sin0.5$

$= 2.148636196$

$= 2.1486$

(iii)$f(x) = x + 3x^2 + \frac{13}{3}x^3$

$f(0.5) = 0.5 + 3(0.5)^2 + \frac{13}{3}(0.5)^3$

$= 1.7916666667$

$= 1.7917$

$Absolute\ error = |Actual - Estimate|$

$= |2.1486 - 1.7917|$

$= |0.3569|$

$= 0.3569$

__Question 5__

__ZIMSEC NOVEMBER 2015 PAPER 1__

Given that $Iny = xy$, where $y > 0$,

(i) 1. Show that $\frac{dy}{dx} = \frac{y^2}{1-xy}$

2. Find $\frac{d^2y}{dx^2}$. **[5]**

(ii) Hence, or otherwise, obtain the Maclaurin series for y up to and including the term in x^2. **[4]**

Suggested Solution

(i) 1. $Iny = xy$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{dy}{dx} + y(1)$$

$$\frac{dy}{dx} = xy\frac{dy}{dx} + y^2$$

$$\frac{dy}{dx} - xy\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx}(1 - xy) = y^2$$

$$\frac{dy}{dx} = \frac{y^2}{1 - xy} \quad (As\ required)$$

2. $\frac{dy}{dx} = \frac{y^2}{1 - xy}$

$$\frac{d^2y}{dx^2} = \frac{(1 - xy)2y\frac{dy}{dx} - y^2\left[-\left(x\frac{dy}{dx} + y\right)\right]}{(1 - xy)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y\frac{dy}{dx} - 2xy^2\frac{dy}{dx} + xy^2\frac{dy}{dx} + y^3}{(1 - xy)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y\frac{dy}{dx} - xy^2\frac{dy}{dx} + y^3}{(1 - xy)^2}$$

(ii) $Iny = xy$

$e^{Iny} = e^{xy}$ $\quad\quad \Rightarrow y = e^{xy}$

Evaluating $y = f(x)$ at $x = 0$:

$\Rightarrow f(0) = e^0 = 1$

Evaluating $\frac{dy}{dx} = f^{'}(x)$ at $x = 0$:

$\Rightarrow f'(0) = \frac{1^2}{1-0} = \frac{1}{1} = 1$

Evaluating $\frac{d^2y}{dx^2} = f''(x)$ at $x = 0$:

$\Rightarrow f''(0) = \frac{2(1)(1) - (0)(1)(1) + 1^3}{(1-0)^2} = \frac{2+1}{1} = \frac{3}{1} = 3$

Now:

$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2$

$\Rightarrow \ f(x) = 1 + 1(x) + \frac{3}{2!}(x)^2$

$\therefore \ y = f(x) = 1 + x + \frac{3}{2}x^2$

Question 6

ZIMSEC NOVEMBER 2018 PAPER 2

a) Use the series expansion of $Cosx$ to expand $(1+x)^2Cosx$ up to and including the term in x^3. **[4]**

b) Given that $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = 0$, where $y = 1$ at $x = 0$ and $\frac{dy}{dx} = 2$ at $x = 0$.

Use Taylor's series to find the first three terms of $(1+x)^2Cosx$. **[4]**

Suggested Solution

a) From tables: $Cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Now $(1+x)^2Cosx = (1 + 2x + x^2)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

$= 1 - \frac{x^2}{2} + 2x - x^3 + x^2$

$= 1 + 2x + \frac{1}{2}x^2 - x^3$

b) At $x = 0$

$y = 1 \qquad \Rightarrow f(0) = 1$

$\frac{dy}{dx} = 2 \qquad \Rightarrow f'(0) = 2$

Substituting $y = 1$ and $\frac{dy}{dx} = 2$ into the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = 0$ (i):

$\frac{d^2y}{dx^2} + (2) + 3(1) = 0 \qquad \therefore \frac{d^2y}{dx^2} = -5 \qquad \Rightarrow f''(0) = -5$

Now

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$\Rightarrow f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2$ since $a = 0$.

$\Rightarrow f(x) = 1 + 2(x) + \frac{(-5)}{2!}(x)^2$

$\therefore f(x) = 1 + 2x - \frac{5}{2}x^2$

Question 7

ZIMSEC JUNE 2019 PAPER 2

Use Taylor's series expansion in ascending powers of $(x-1)$ to find $x^2 Inx$ up to and including the term in $(x-1)^4$. **[5]**

Suggested Solution

$f(x) = x^2 Inx \qquad \Rightarrow f(1) = 1^2 In1 = 0$

$f'(x) = x^2\left(\frac{1}{x}\right) + 2xInx = x + 2xInx \qquad \Rightarrow f'(1) = 1 + 2(1)In1 = 1$

$f''(x) = 1 + 2x\left(\frac{1}{x}\right) + 2Inx = 3 + 2Inx \qquad \Rightarrow f''(1) = 3 + 2In1 = 3$

$f'''(x) = \frac{2}{x}$ $\Rightarrow$ $f'''(1) = \frac{2}{1} = 2$

$f'^{v}(x) = -\frac{2}{x^2}$ $\Rightarrow$ $f'^{v}(1) = -\frac{2}{1^2} = -2$

Now

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f'^{v}(a)}{4!}(x-a)^4$$

$$\Rightarrow f(x) = \ x^2 Inx = 0 + 1(x-1) + \frac{3}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-2}{4!}(x-1)^4$$

$$\therefore\ f(x) = x^2 Inx = (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4$$

Question 8

ZIMSEC JUNE 2020 PAPER 2

(i) Use Taylor's theorem to obtain a series expansion for $cos\left(x+\frac{\pi}{4}\right)$ about $x = \frac{\pi}{4}$ up to and including the term in x^3. **[6]**

(ii) Hence evaluate $cos 46°$ leaving the answer in exact form. **[2]**

Suggested Solution

(i)NB: $f\left(x+\frac{\pi}{4}\right) = cos\left(x+\frac{\pi}{4}\right)$

$a = \frac{\pi}{4}$

$\Rightarrow (x-a) = \left[\left(x+\frac{\pi}{4}\right) - a\right] = x$

i.e. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

$$f\left(x+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x+\frac{\pi}{4}-\frac{\pi}{4}\right) + \frac{f''(a)}{2!}\left(x+\frac{\pi}{4}-\frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x+\frac{\pi}{4}-\frac{\pi}{4}\right)^3$$

$$\Rightarrow f\left(x+\frac{\pi}{4}\right)=f\left(\frac{\pi}{4}\right)+f'\left(\frac{\pi}{4}\right)x+\frac{f''(a)}{2!}x^2+\frac{f'''\left(\frac{\pi}{4}\right)}{3!}x^3$$

$$f(x)=cos(x)$$
$$f'(x)=-sin(x)$$
$$f''(x)=-cos(x)$$
$$f'''(x)=sin(x)$$

Now:

$$f\left(\frac{\pi}{4}\right)=cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$
$$f'\left(\frac{\pi}{4}\right)=-sin\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$
$$f''\left(\frac{\pi}{4}\right)=-cos\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$
$$f'''\left(\frac{\pi}{4}\right)=sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

$$\Rightarrow f\left(x+\frac{\pi}{4}\right)=\ cos\left(x+\frac{\pi}{4}\right)=f\left(\frac{\pi}{4}\right)+f'\left(\frac{\pi}{4}\right)(x)+\frac{f''\left(\frac{\pi}{4}\right)}{2!}(x)^2+\frac{f'''\left(\frac{\pi}{4}\right)}{3!}(x)^3$$

$$=\frac{\sqrt{2}}{2}+\left(-\frac{\sqrt{2}}{2}\right)(x)+\frac{-\frac{\sqrt{2}}{2}}{2!}(x)^2+\frac{\frac{\sqrt{2}}{2}}{3!}(x)^3$$

$$=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{4}x^2+\frac{\sqrt{2}}{12}x^3$$

(ii) $f\left(x+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{4}x^2+\frac{\sqrt{2}}{12}x^3$

Hence means use the above result:

$$\cos 46° = \cos\left(x + \frac{\pi}{4}\right)$$

$$\cos 46° = \cos(x + 45°)$$

$$\cos(1 + 45°) = \cos(x + 45°)$$

$$\Rightarrow x = 1°$$

$$\Rightarrow x = \frac{\pi}{180}$$

Now:

$$\cos 46° = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{4}x^2 + \frac{\sqrt{2}}{12}x^3$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\frac{\pi}{180}\right) - \frac{\sqrt{2}}{4}\left(\frac{\pi}{180}\right)^2 + \frac{\sqrt{2}}{12}\left(\frac{\pi}{180}\right)^3$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}\pi}{360} - \frac{\sqrt{2}\pi^2}{129\,600} + \frac{\sqrt{2}\pi^3}{69\,984\,000}$$

PAST EXAMINATION QUESTIONS

QUESTION 1

The displacement $x\, metres$ of a particle at time $t\, seconds$ is given by the differential equation $\frac{d^2x}{dt^2} + x + Cosx = 0$ when $t = 0, x = 0$ and $\frac{dx}{dt} = \frac{1}{2}$. Find a Taylor's series solution for x the in ascending powers of t up to and including the term in t^3.

QUESTION 2

Given that $\frac{d^2y}{dx^2} = e^x\left[2y\frac{dy}{dx} + y^2 + 1\right]$

a) Show that $\frac{d^3y}{dx^3} = e^x\left[2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + ky\left(\frac{dy}{dx}\right) + y^2 + 1\right]$ where k is a constant to be found.

Given that at $x = 0, y = 1$ and $\frac{dy}{dx} = 2,$.

b) find a series solution for y in ascending powers of x up to and including the term in x^3

QUESTION 3

Given that $x\left(\frac{dy}{dx}\right) = 3x + y^2$

a) Show that $x\frac{d^2y}{dx^2} + (1 - 2y)\left(\frac{dy}{dx}\right) = 3.$

Given that at $y = 1$ at $x = 1,$

b) find a series solution for y in ascending powers of $(x - 1)$ up to and including the term in $(x - 1)^3$

QUESTION 4

$$\frac{d^2y}{dx^2} + 4y - Sinx = 0$$

Given that $y = \frac{1}{2}$ and $\frac{dy}{dx} = \frac{1}{8}$ at $x = 0$, , find a series expansion for y in terms of x, up to and including the term in x^3.

QUESTION 5

Given that

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 5y = 0$$

a) Find $\frac{d^3y}{dx^3}$ in terms of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y

Given that $y = 2$ and $\frac{dy}{dx} = 2$ at $x = 0$,

b) find a series solution for y in ascending powers of x up to and including the term in x^3

QUESTION 6

Given that

$$x\frac{dy}{dx} + y^2 - 3x + 4 = 0, \qquad y = 1 \text{ when } x = 1.$$

Use Taylor's series to find a solution of the differential equation in ascending powers of $(x - 1)$ up to and including the term in $(x - 1)^4$

QUESTION 7

Given that $y = \sqrt{(cosx)}$, show that

a) $4y^2\left(\frac{dy}{dx}\right)^2 = 1 - y^4$,

b) $\frac{d^2y}{dx^2} = -\frac{1}{4}\left(y^{-3} + y\right)$.

By further differentiation, or otherwise, obtain the expansion of y as a series of ascending powers of x up to and including the term in x^4. (JUNE 1980)

QUESTION 8

Given that $\frac{dy}{dx} = y + 2x$ and that $y = 1$ when $x = 0$.

Show that $\frac{d^2y}{dx^2} = 3$ when $x = 0$. Find the first three terms in the Maclaurin series for y.

(NOV 1989)

QUESTION 9

Given that $y = In\left[\frac{1}{2}(1 + e^{-x})\right]$, show that $\frac{d^2y}{dx^2} = \frac{1}{2}e^{-y} - 1$.

By repeated differentiation of this result, or otherwise, show that the series expansion of y in ascending powers of x, up to and including the term in x^4, is $-\frac{1}{2}x + \frac{1}{8}x^2 + kx^4$ where the numerical value of k is to be determined. (JUNE 1988)

QUESTION 10

If $y = In(1 + sinx)$, find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} = \frac{-1}{1+sinx}$. Find the expansion in ascending powers of x of $In(1 + sinx)$ up to and including the term in x^4.

Deduce that, if powers of x above the fourth are neglected,

$$In\left(\frac{1+x}{1+sinx}\right) = \frac{1}{6}x^3(1 - x).$$ (JUNE 1977)

QUESTION 11

If $y = In(secx)$, prove that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 1$. By further differentiation of this result, or otherwise, find the series expansion of y in terms of x up to and including the term in x^4.

(JUNE 1978)

QUESTION 12

Let $y = \sqrt{4 + Sin^2x}$.

a) Show that $y\frac{dy}{dx} = \frac{1}{2}sin2x$

b) By successive differentiation of this result show

$$y\frac{d^4y}{dx^4} + 4\frac{dy}{dx}\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^2 = -4cos2x$$

c) Find the series expansion of y in ascending powers of x up to and including the term in x^4. (NOV 1985)

QUESTION 13

Given that $y = tan^2x$ express $\frac{dy}{dx}$ in terms of $tanx$, and hence show that

$$\frac{dy}{dx} = 2 + 8y + 6y^2.$$

By further differentiation of this result, or otherwise, show that if x is small enough for powers of x above x^6 to be neglected, then

$$tan^2x = x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6.$$

(JUNE 1979)

QUESTION 14

If $y = tan^{-1}x$, prove that

$$(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$

Differentiate three more times, and find the values of the first five derivatives of $tan^{-1}x$ at $x = 0$. Hence or otherwise, find the first three non-zero terms of the Maclaurin expansion of $tan^{-1}x$. (NOV 1974)

QUESTION 15

Let $y = e^x sinx$. Show that $\frac{dy}{dx}$ may be expressed as $ce^x sin(x + \alpha)$ where $c > 0$ and $0 < \alpha < \frac{\pi}{2}$ and state the numerical values of c and a.

Hence, or otherwise, find $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ and $\frac{d^5y}{dx^5}$.

Using these results, or otherwise, expand y as a series in ascending powers of x up to and including the term in x^5. (JUNE 1981)

QUESTION 16

Show that, if x is so small that x^6 and higher powers of x may be neglected,

$$In(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

Deduce that, for such values of x,

$$In\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5\right)$$

By giving x a suitable value in this last result, prove that for large N,

$$In\left(\frac{N+1}{N-1}\right) \approx \frac{2}{N} + \frac{2}{3N^3} + \frac{2}{5N^3}$$

Hence show that, for large N, $\left(\frac{N+1}{N-1}\right)^N \approx e^2$.

(JUNE 1982)

QUESTION 17

If $y = tan\left(\frac{\pi}{4} + x\right)$, show that $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$.

By repeated differentiation of this result, or otherwise, find the series expansion of y in ascending powers of x up to and including the term in x^4.

When $x = 0.1$, the value of y, correct to 8 decimal places, is $1.223\,048\,88$. Verify that, in this case, the series expansion up to and including the term in x^4 gives an estimate that is in error by about 0.004%.

(JUNE 1989)

QUESTION 18

ZIMSEC NOVEMBER 2019 PAPER 2

a) (i) Find the Maclaurin series expansion for y up to and including the term in x^3 given that $\dfrac{dy}{dx} = y^3 + x^8$ and $y = 1$ when $x = 0$. **[8]**

(ii) Evaluate y when $x = 1.2$ **[2]**

b) Use Taylor's theorem to obtain series expansion about $(x = 4)$ for $f(x) = e^{\sqrt{x}}$, up to and including the term in $(x - 1)^4$. **[6]**

QUESTION 19

ZIMSEC JUNE 2019 PAPER 1

(i) Obtain, in terms of the constant a, an expression for the first 3 simplified terms of the series expansion for $y = e^{a-x}$. **[3]**

(ii) If the coefficient of x in the series expansion in **(i)** is -4, find the exact value of a. **[2]**

QUESTION 20

ZIMSEC JUNE 2018 PAPER 1

Given that $\dfrac{dy}{dx} = x + 2y$ and that $y = 1$ when $x = 0$.

(a) Show that $\frac{d^2y}{dx^2} = 5$. **[3]**

(b) Obtain the Maclaurin series for y in ascending powers of x as far as the term inx^3. **[3]**

QUESTION 21

ZIMSEC NOVEMBER 2017 PAPER 1

a) Given that $y = e^{2x}sinx$ and $\frac{dy}{dx} = e^{2x}cosx + 2e^{2x}sinx$ find

1. $\frac{d^2y}{dx^2}$, **[2]**
2. $\frac{d^3y}{dx^3}$. **[3]**

b) Hence, or otherwise, obtain the Maclaurin series for $y = e^{2x}sinx$ up to the term in x^3.**[2]**

QUESTION 22

ZIMSEC NOVEMBER 2013 PAPER 1

(i) If $y = (1 + 3e^{-x})^{\frac{1}{2}}$, find $\frac{dy}{dx}$ at $x = 0$. **[3]**

(ii) Show that $= 2y\frac{dy}{dx} = 1 - y^2$. **[2]**

(iii) By further differentiating, use Maclaurin's therom to find the series expansion for y in ascending powers of x, up to and including the term in x^2. **[4]**

QUESTION 23

ZIMSEC JUNE 2007 PAPER 1

Given that $f(x) = xe^{3x}, f'(x) = 3xe^{3x} + e^{3x}$ and $f''(x) = 3e^{3x}(3x + 2)$

(a) Find $f'''(x), f'^{v}(x)$ and hence write down the Maclaurin series of $f(x)$ up to the term in x^4. **[6]**

(b) Find by integration the area in the first quadrant bounded by $f(x)$, the $x - axis$

and the line $x = 1$, giving your answer correct to 3 decimal places. **[4]**

(c) Use Maclaurin series obtained in (**i**) to estimate the area and state the absolute error. **[4]**

QUESTION 24

ZIMSEC NOVEMBER 2003 PAPER 1 SPECIMEN

Given that $= e^{-x}cos^2x$, find $\frac{dy}{dx}$ and deduce that

$$\frac{d^2y}{dx^2} = e^{-x}[2sin2x - 2cos2x + cos^2x]$$

By finding the second derivative, or otherwise, find the series expansion of y in ascending powers of x, up to and including the term in x^3. **[8]**

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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