The Method of Differences

Compiled by: Nyasha P. Tarakino (Trockers)

+263772978155/+263717267175

ntarakino@gmail.com

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SYLLABUS (6042) REQUIREMENTS

- use the method of differences to obtain the sum of finite series

EXAMPLES

NOTES

When working with sequences and series, sometimes partial fractions are needed to solve the problem. The first step is to recognize what types of sequence or series problems require the use of partial fractions. Solving the partial fraction will help set up a new series that enables to solve using limits. The first two examples require the use of partial fractions.

Question 1

Find

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)}$$

Suggested Solution

Step 1

Express $\frac{1}{(r+2)(r+3)}$ into Partial Fractions.

Let $\frac{1}{(r+2)(r+3)} \equiv \frac{A}{(r+2)} + \frac{B}{(r+3)}$

$\Rightarrow A(r+3) + B(r+2) = 1$

When $r = -3$; $-B = 1 \Rightarrow B = -1$.

When $r = -2$; $A = 1$

$\therefore \frac{1}{(r+2)(r+3)} \equiv \frac{1}{(r+2)} - \frac{1}{(r+3)}$

Step 2

Write out the first few terms of the series and notice what is happening with the values. The second value is always cancelling with the exception of the last term, along with the first value after the initial term. This is a Telescoping Series.

$$r = 1 \quad r = 2 \quad r = 3 \quad r = 4 \quad \ldots \quad r = n$$

$$\therefore \sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} \equiv \sum_{r=1}^{n} \frac{1}{(r+2)} - \frac{1}{(r+3)}$$

Now

$$\sum_{r=1}^{n} \frac{1}{(r+2)} - \frac{1}{(r+3)} = \left[\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{5}\right] + \left[\frac{1}{5} - \frac{1}{6}\right] + \cdots + \left[\frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$\therefore \sum_{r=1}^{n} \frac{1}{(r+2)} - \frac{1}{(r+3)} \equiv \frac{1}{2} - \frac{1}{n+3}.$$

Question 2

Find if the series converges or diverges. If convergent then find its sum.

$$\sum_{r=1}^{\infty} \frac{1}{(r+2)(r+3)}$$

Suggested Solution

$$\sum_{r=1}^{\infty} \frac{1}{(r+2)} - \frac{1}{(r+3)} \equiv \frac{1}{2} - \frac{1}{n+3}$$

$$= Lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{n+3}\right]=\frac{1}{2}-\frac{1}{\infty}=\frac{1}{2}.$$

Since it is possible to solve for the limit, it is known that the series converges, and the sum of the series is $\frac{1}{2}$.

$$\therefore \sum_{r=1}^{\infty}\frac{1}{(r+2)}-\frac{1}{(r+3)}\equiv\frac{1}{2}$$

Show that

$$\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\equiv\frac{2}{r(r+1)(r+2)}$$

Hence, or otherwise find a simplified expression for

$$\sum_{r=1}^{n}\frac{2}{r(r+1)(r+2)}$$

NB: The above question is the same as ZIMSEC NOVEMBER 2018 QUESTION.

NOTES

There is no need for partial fractions as we must show the difference between the two fractions is the SINGLE fraction given. Partial fractions would be needed if working the other way round. i.e. Show that the single fraction can be written as the difference between the two fractions

Suggested Solution

a)

$$\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}=\frac{1(r+2)-1(r)}{r(r+1)(r+2)}=\frac{r+2-r}{r(r+1)(r+2)}=\frac{2}{r(r+1)(r+2)}$$

b)

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} \equiv \sum_{r=1}^{n} \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

Write out the first few terms of the series and notice what is happening with the values. The second value is always cancelling with the exception of the last term, along with the first value after the initial term. This is a Telescoping Series.

$$r=1 \quad r=2 \quad r=3 \quad r=4 \quad \ldots \quad r=n$$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} =$$

$$= \left[\frac{1}{(1)(2)} - \frac{1}{(2)(3)}\right] + \left[\frac{1}{(2)(3)} - \frac{1}{(3)(4)}\right] + \left[\frac{1}{(3)(4)} - \frac{1}{(4)(5)}\right] + \cdots + \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$$

Now

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$

ZIMSEC 2018

- a) FIND

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}$$

[6]

- b) Hence, or otherwise deduce the value of

$$\sum_{r=1}^{\infty} \frac{2}{r(r+1)(r+2)}$$

[2]

Suggested Solution

a)

Step 1

Express $\frac{2}{r(r+1)(r+2)}$ into Partial Fractions.

Let $\frac{2}{r(r+1)(r+2)} \equiv \frac{A}{r} + \frac{B}{(r+1)} + \frac{C}{(r+2)}$

$\Longrightarrow A(r+1)(r+2) + B(r)(r+2) + C(r)(r+1) = 2$

When $r = 0;\ 2A = 2 \Longrightarrow A = 1$

When $r = -1;\ -B = 2 \Longrightarrow B = -2$

When $r = -2;\ 2C = 2 \Longrightarrow C = 1$

$\therefore \frac{2}{r(r+1)(r+2)} \equiv \frac{1}{r} - \frac{2}{(r+1)} + \frac{1}{(r+2)}$

Step 2

Express terms as differences using Step 1 and we should do this for at least first 4 terms and at least last 3 terms so that we can show relevant cancelling clearly.

$i.e.$ for $r = 1 \quad r = 2 \quad r = 3 \quad r = 4 \quad r = 5 \quad \ldots \; r = n-2 \quad r = n-1 \quad r = n$

$$\therefore \sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} \equiv \sum_{r=1}^{n} \frac{1}{r} - \frac{2}{(r+1)} + \frac{1}{(r+2)}$$

Now

$$\sum_{r=1}^{n} \frac{1}{r} - \frac{2}{(r+1)} + \frac{1}{(r+2)} =$$

$$= \left[\frac{1}{1} - \frac{2}{2} + \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] + \left[\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right] + \left[\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right] + \left[\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right] + \cdots$$

$$\ldots + \left[\frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}\right] + \left[\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right] + \left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right]$$

After performing relevant cancelling we should obtain

$$\sum_{r=1}^{n} \frac{1}{r} - \frac{2}{(r+1)} + \frac{1}{(r+2)} = \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2} = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$\therefore \sum_{r=1}^{n} \frac{1}{r} - \frac{2}{(r+1)} + \frac{1}{(r+2)} = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2} \text{ or } \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$

b)

$$\sum_{r=1}^{\infty} \frac{2}{r(r+1)(r+2)} \equiv \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$

$$= Lim_{n\to\infty} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{2} - \frac{1}{\infty} = \frac{1}{2}.$$

$\therefore$ The value of $\displaystyle\sum_{r=1}^{\infty} \frac{2}{r(r+1)(r+2)} = \frac{1}{2}$

PRACTICE QUESTIONS

QUESTION 1

a) Show that $\frac{1}{r} - \frac{3}{r+1} + \frac{2}{r+2} \equiv \frac{2-r}{r(r+1)(r+2)}$

b) Hence show that

$$\sum_{r=1}^{n} \frac{2-r}{r(r+1)(r+2)} = \frac{n}{(r+1)(n+2)}$$

c) Find the value of

$$\sum_{r=2}^{\infty} \frac{2-r}{r(r+1)(r+2)}$$

QUESTION 2

a) Show that $\frac{1}{3r-1} - \frac{3}{3r+2} \equiv \frac{3}{(3r-1)(3r+2)}$

b) Hence show that

$$\sum_{r=1}^{2n} \frac{1}{(3r-1)(3r+2)} = \frac{n}{2(3n+1)}$$

QUESTION 3

a) Show that $\frac{1}{r} - \frac{1}{r+2} \equiv \frac{2}{r(r+2)}$

b) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{2}{r(r+2)}$$

c) Given that

$$\sum_{r=N+1}^{\infty} \frac{2}{r(r+2)} = \frac{11}{30}, \text{ find the vaue of } N.$$

QUESTION 4

a) Show that $\frac{1}{r-1} - \frac{1}{r+1} \equiv \frac{2}{r^2-1}$

b) Hence find an expression, in terms of n, for

$$\sum_{r=2}^{n} \frac{2}{r^2-1}$$

c) Find the value of

$$\sum_{r=1000}^{\infty} \frac{2}{r^2-1}$$

QUESTION 5

a) Show that $\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$

b) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$$

c) Find the value of

$$\sum_{r=2}^{\infty} \frac{2r+1}{r^2(r+1)^2}$$

QUESTION 6

a) Show that $\frac{1}{2r-3} - \frac{1}{2r+1} \equiv \frac{4}{4r^2-4r-3}$

b) Hence find an expression, in terms of n, for

$$\sum_{r=2}^{n} \frac{4}{4r^2 - 4r - 3}$$

c) Show that

$$\sum_{r=2}^{\infty} \frac{4}{4r^2 - 4r - 3} = \frac{4}{3}$$

QUESTION 7

a) Show that $\frac{2}{r} - \frac{1}{r+1} - \frac{1}{r+2} \equiv \frac{3r+4}{r(r+1)(r+2)}$

b) Hence:

(i) Find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{3r + 4}{r(r + 1)(r + 2)}$$

(ii) Write down the value of

$$\sum_{r=1}^{\infty} \frac{3r + 4}{r(r + 1)(r + 2)}$$

c) Given that

$$\sum_{r=N+1}^{\infty} \frac{3r + 4}{r(r + 1)(r + 2)} = \frac{7}{10}, \text{ find the vaue of } N.$$

QUESTION 8

a) Show that $\frac{1}{r!} - \frac{1}{(r+1)!} \equiv \frac{r}{(r+1)!}$

b) Hence find an expression, in terms of n, for

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{r}{(r+1)!}$$

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ENJOY

Nyasha P. Tarakino (Trockers)

+263772978155/+263717267175

ntarakino@gmail.com