Transformations – Matrices

Compiled by: Nyasha P. Tarakino (Trockers)

+263772978155/+263717267175

ntarakino@gmail.com

23 JULY 2020

SYLLABUS (6042) REQUIREMENTS

- construct 2 x 2 matrices to represent enlargement, rotation, reflection, stretch and shear transformations
- use a 2 x 2 matrix to represent certain geometrical transformation in the x – y plane
- use the composite transformation AB and A^n in problem solving
- derive the relationship between the area scale-factor of a transformation and the determinant of the corresponding matrix

TRANSFORMATIONS

- To transform is to change
- A transformation changes the position and shape size of an object. That means a shape is moving from one place to another.
- A pre-image point (shape) is the original point (shape) before any transformation takes place.
- An image point (shape) is the resultant/final point (shape) after a certain transformation has been performed.
- Congruent means same shape and size.
- Similar means same shape but different size.
- Invariant means unchanging
- An invariant line of a transformation is one where every point on the line is mapped to a point on the line - possibly the same point.
- An invariant point on a line or a shape is a point which does not move when a specific transformation is applied i.e. it is mapped onto itself. It is its own image
- Transformations in planes occur when an object point$P(x, y)$ is mapped to an image point $P_1(x_1, y_1)$ in the same plane.
- There are six basic transformations namely Translation, Reflection, Rotation, Enlargement, Shear and Stretch.
- To illustrate the nature each transformation we consider a unit square $OPQR$ with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ respectively.

TRANSLATION

- To translate means to move an object from one location to another.
- The Image is congruent to the object.
- The image and the object move in a straight line
- The image and the object have the same sitting position.
- The matrix of the translation vector is $\begin{pmatrix} a \\ b \end{pmatrix}$
- The equation of a translation is given by:

$$Image = Object + Translation\,Vector(t.v)$$

Description

State that it is a translation and give the coordinates of the translation Vector

Solved Problem

Find the image of $OPQR$ under a translation with translation vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Suggested Solution

The image is $O_1P_1Q_1R_1$

Now using the equation

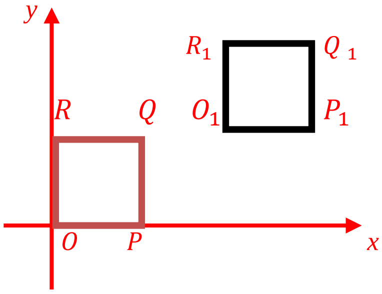
$Image = Object + Translation\,Vector$

$O_1 = O + t.v. = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

$P_1 = P + t.v. = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

$Q_1 = Q + t.v. = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

$R_1 = R + t.v. = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$



The vertices of $OPQR$ are $(0,0), (1,0), (1,1)$ and $(0,1)$ respectively and the vertices of $O_1P_1Q_1R_1$ are $(2,1), (3,1), (3,2)$ and $(2,2)$

REFLECTION

- Reflection means to turn an object into a mirror image of itself using some arbitrary axis or plane of reflection.
- The Image is congruent to the object.
- The image and the object face opposite directions.
- The line joining the image and the object is perpendicular to the mirror line.
- The distance of the object from the mirror line is equal to the distance of the image behind the mirror line

Matrices

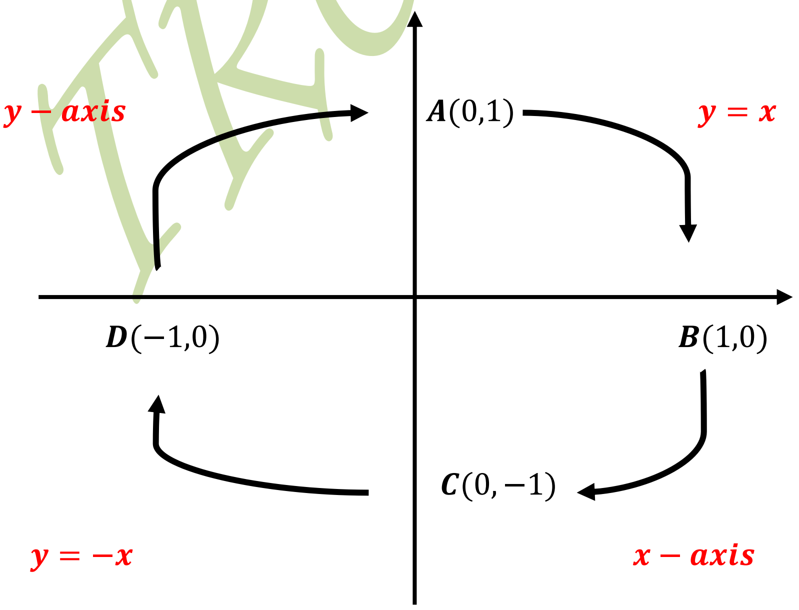
(i) Reflection along the $y-axis$ or $x=0$: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) Reflection along the $x-axis$ or $y=0$: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii)Reflection along the line $y=x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(iv)Reflection along the line $y=-x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Tips to remember these matrices



The matrix is written as:

$$\begin{pmatrix} x - Starting\ point & x - Ending\ point \\ y - Starting\ point & y - Ending\ point \end{pmatrix}$$

Now:

- For a reflection on the $x - axis$ or $y = 0$ we start from $B(1,0)$ to $C(0,-1)$. Thus the matrix becomes: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- For a reflection on the $y - axis$ or $x = 0$ we start from $D(-1,0)$ to $B(0,1)$. Thus the matrix becomes: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- For a reflection on the line $y = x$ we start from $A(0,1)$ to $B(1,0)$. Thus the matrix becomes: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- For a reflection on the line $y = -x$ we start from $C(0,-1)$ to $D(-1,0)$. Thus the matrix becomes: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Description

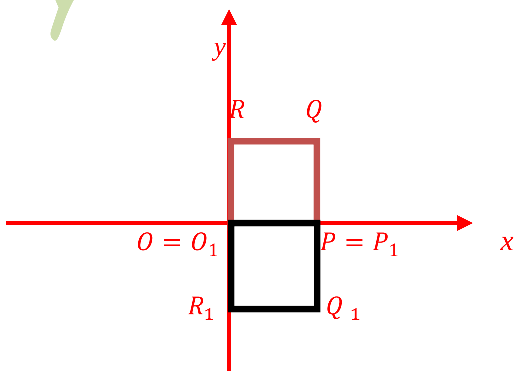
State that it is a reflection and give the equation of the mirror line

Solved Problems

1. Find the image of $OPQR$ under a reflection in the line $y = 0$.
2. Construct the matrix which represents this transformation.

Suggested Solution

1. The image is $O_1P_1Q_1R_1$



The vertices of $OPQR$ are $(0,0), (1,0), (1,1)$ and $(0,1)$ respectively and the vertices of $O_1P_1Q_1R_1$ are $(0,0), (1,0), (1,-1)$ and $(0,-1)$ respectively.

Constructing the (2×2) matrices of a Reflection along the lines $x = 0$ and $y = 0$

Let us consider the set of points $R(0;1)$ and $R_1\,(0;-1)$ and the set of points $Q(1;1)$ and $Q_1\,(1;-1)$

Along the lines $y = 0$ or $x - axis$

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow b = 0$ and $d = -1$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow a + b = 1 \Rightarrow a + 0 = 1 \therefore a = 1$ and

$c + d = -1 \Rightarrow c - 1 = -1 \therefore c = 0$

Hence the matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

<u>Follow up questions</u>

1. Find the image of $OPQR$ under a reflection in the line
   a) $y = 0$ or $y - axis$,
   b) $y = x$.
2. Find the matrices for reflection along:
   a) the $y - axis$,
   b) the line $y = x$.

ROTATION

- Rotation means turning an object through some angle about a point or axis of rotation.
- The Image is congruent to the object.
- The distance of the object from the centre is equal to the distance of the image from the centre.
- The direction is either clockwise or anticlockwise.
- The angle is either $90°, 180°$ or $270°$.
- $90°$ anticlockwise is equal to $270°$ clockwise
- $90°$ clockwise is equal to $270°$ anticlockwise.
- $180°$ anticlockwise is equal to $180°$ clockwise.

Matrices

(i) Rotation centre $(0; 0)$ through $90°$ Anticlockwise or $270°$ clockwise: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(ii) Rotation centre $(0; 0)$ through $90°$ clockwise or $270°$ anticlockwise: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(iii)Rotation centre $(0; 0)$ through$180°$: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Tips to remember these matrices

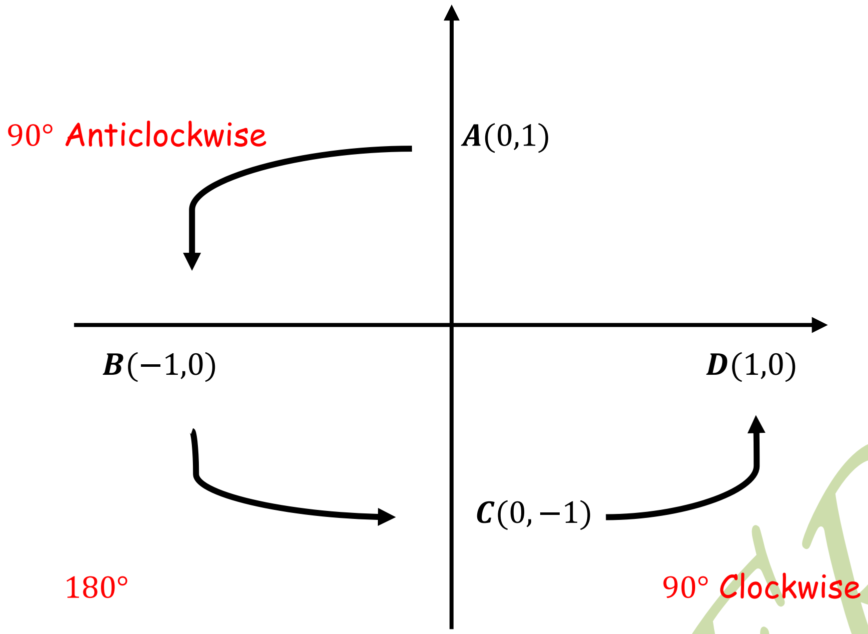
Method 1

We use the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Then:

- We use $\theta = 90°$ or $\theta = -270°$ for a rotation of $90°$ anticlockwise or $270°$ clockwise
- We use $\theta = -90°$ or $\theta = 270°$ for a rotation of $90°$ clockwise or $270°$ anticlockwise
- We use $\theta = 180°$ for a rotation of $180°$.

Method 2



The matrix is written as:

$$\begin{pmatrix} x - Starting\ point & x - Ending\ point \\ y - Starting\ point & y - Ending\ point \end{pmatrix}$$

Now:

- For a rotation of 90° anticlockwise or 270° clockwise we start from $A(0,1)$ to $B(-1,0)$.

  Thus the matrix becomes: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- For a rotation of 90° clockwise or 270° anticlockwise we start from $C(0,-1)$ to $D(1,0)$.

  Thus the matrix becomes: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- For a rotation of 180° we start from $B(-1,0)$ to $C(0,-1)$. Thus the matrix becomes: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Description

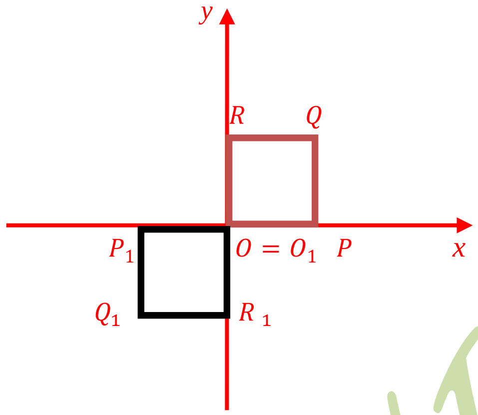
State that it is a rotation; give the coordinates of the centre, angle turned and the direction turned.

Solved Problems

1. Find the image of $OPQR$ under a rotation, 180° about the origin i.e. centre $(0;0)$.
2. Construct the matrix which represents this transformation.

Suggested Solution

1. The image is $O_1P_1Q_1R_1$



The vertices of $OPQR$ are $(0,0),(1,0),(1,1)$ and $(0,1)$ respectively and the vertices of $O_1P_1Q_1R_1$ are $(0,0),(-1,0),(-1,-1)$ and $(0,-1)$ respectively.

Constructing the (2×2) matrices of a Rotation through 180° about the origin

Let us consider the set of points $R(0;1)$ and $R_1\,(0;-1)$ and the set of points $Q(1;1)$ and $Q_1\,(-1;-1)$

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow b = 0$ and $d = -1$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow a + b = -1 \Rightarrow a + 0 = -1 \therefore a = -1$ and

$c + d = -1 \Rightarrow c - 1 = -1 \therefore c = 0$

Hence the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Follow up questions

1. Find the image of $OPQR$ under a rotation
   a) $90°$ clockwise or $270°$ anticlockwise about the origin
   b) $90°$ anticlockwise or $270°$ clockwise about the origin
2. Find the matrices for rotation
   a) $90°$ clockwise or $270°$ anticlockwise about the origin
   b) $90°$ anticlockwise or $270°$ clockwise about the origin

ENLARGEMENT

- Enlargement, scaling or dilation means making an object bigger or smaller while maintaining its shape i.e. keeping all of its dimensions in the same proportion to one another.
- The Image is similar to the object.
- When the image and the object are to one side of the centre, the scale factor is positive
- When the centre is between the image and the object, the scale factor is negative
- If the scale factor is greater than 1, the image is larger than the object
- If the scale factor is less than -1, the image is larger than the object but on the opposite side of the centre
- If the scale factor is between 0 and 1, the image is smaller than the object
- If the scale factor is ± 1, the image is the same as the object

Matrix

Enlargement centre $(0;0)$ with enlargement factor k is: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Description

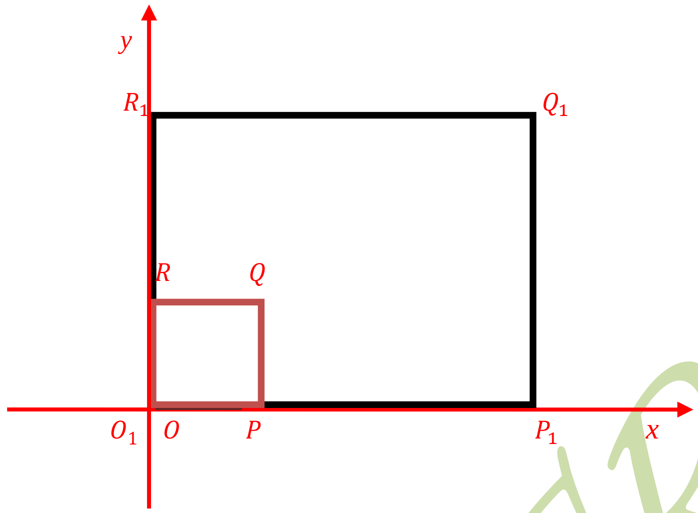
State that it is an enlargement; give the coordinates of the centre and the enlargement factor.

Solved Problems

1. Find the image of $OPQR$ under an enlargement with centre $(0;0)$ and factor 3.
2. Construct the matrix which represents this transformation.

<u>Suggested Solution</u>

1. The image is $O_1P_1Q_1R_1$



The vertices of $OPQR$ are $(0,0), (1,0), (1,1)$ and $(0,1)$ respectively and the vertices of $O_1P_1Q_1R_1$ are $(0,0), (3,0), (3,3)$ and $(0,3)$ respectively.

2.

Constructing the (2×2) matrices of an enlargement with centre $(0; 0)$ and factor 3

Let us consider the set of points $R(0; 1)$ and $R_1\,(0; 3)$ and the set of points $Q(1; 1)$ and $Q_1\,(3; 3)$

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow b = 0$ and $d = 3$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \Rightarrow a + b = 3 \Rightarrow a + 0 = 3 \therefore a = 3$ and

$c + d = 3 \Rightarrow c + 3 = 3 \therefore c = 0$

Hence the matrix is given by: $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

Thus in general the matrix of an enlargement with centre $(0; 0)$ and scale factor $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

Follow up questions

1. Find the image of $OPQR$ under an enlargement
   a) centre $(0;0)$ and scale factor -2.
   b) centre $(0;0)$ and scale factor $\frac{3}{2}$.
2. Find the matrices for an enlargement with
   a) centre $(0;0)$ and scale factor -2.
   b) centre $(0;0)$ and scale factor $\frac{3}{2}$.

SHEAR

- Shearing means distorting an object by pushing it over in a particular direction. It is simply a transformation in which all points along a given line remain fixed while other points are shifted parallel to that line by a distance proportional to their perpendicular distance from the same line
- Shearing a plane figure does not change its area.
- A shear moves parallel to the invariant line.

Matrices

(i) Shear $x-axis$ invariant with factor k: $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

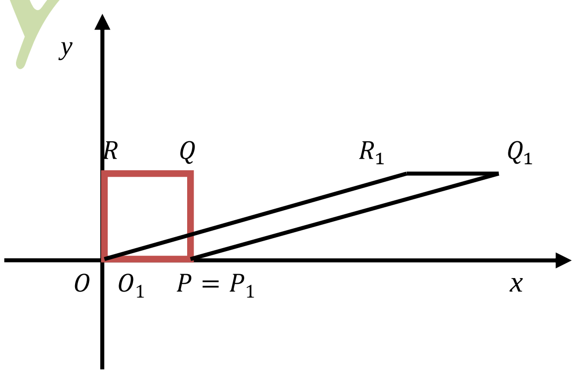
(ii) Shear $y-axis$ invariant with factor k: $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

Description

State that it is a shear; give the shear factor and the invariant line

Solved Problem

In the diagram, $O_1P_1Q_1R_1$ is the image of $OPQR$ under an shear $x-axis$ invariant and factor 3. Construct the matrix which represents this transformation.



The vertices of $OPQR$ are $(0,0), (1,0), (1,1)$ and $(0,1)$ respectively and the vertices of $O_1P_1Q_1R_1$ are $(0,0), (1,0), (4,1)$ and $(3,1)$ respectively.

Suggested Solution

Constructing the (2×2) matrices of a Shear $x - axis$ invariant with factor k

Let us consider the set of points $R(0;1)$ and $R_1\,(3;1)$ and the set of points $Q(1;1)$ and $Q_1\,(4;1)$

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow b = 3$ and $d = 1$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow a + b = 4 \Rightarrow a + 3 = 4 \therefore a = 1$ and

$c + d = 1 \Rightarrow c + 1 = 1 \therefore c = 0$

Hence the matrix is given by: $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Thus in general the matrix of a Shear $x - axis$ invariant with factor $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

Follow up question

$O_1P_1Q_1R_1$ is the image of $OPQR$ under a shear $y - axis$ invariant and factor 2. The vertices of $O_1P_1Q_1R_1$ are $(0{,}0)$, $(1{,}2)$, $(1{,}3)$ and $(0{,}1)$. Construct the matrix which represents this transformation.

STRETCH

- Stretching means enlarging the object only in one direction i.e. enlarges all distances in a particular direction by a constant factor but does not affect distances in the perpendicular direction.
- It is also known as dilation or scaling.
- This transformation is characterised by an invariant line and a scale factor (one way stretch) or two invariant lines and corresponding scale factors (two way stretch)
- For a two way stretch, if scale factors are uniform then it is an enlargement.
- A stretch moves perpendicular to the invariant line.

Matrices

(i) Stretch $y - axis$ invariant (parallel to the $x - axis$) with factor h is $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$

(ii) Stretch $x - axis$ invariant (parallel to the $y - axis$) with factor k is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

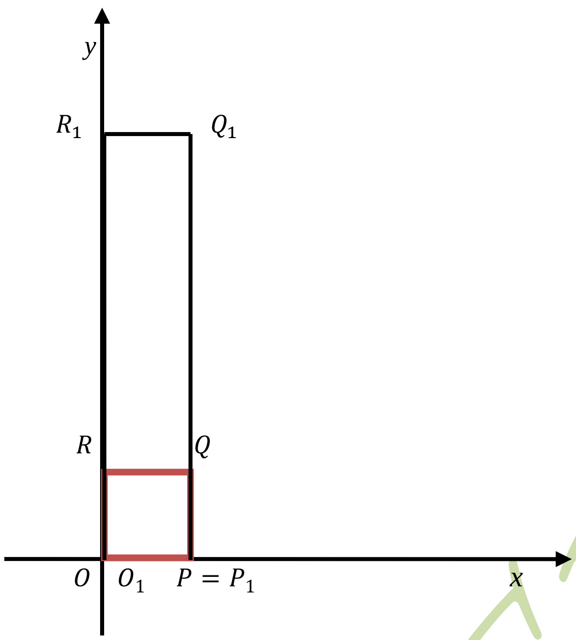
(iii)Two way stretch with factor k, $x - axis$ invariant and factor h, $y - axis$ invariant is $\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$

Description

- For (i) and (ii) state that it is one way a stretch; give the stretch factor and the invariant line.
- For (iii) state that it is a 2 way stretch with factor k, $x - axis$ invariant and factor h, $y - axis$ invariant.

Solved Problem

In the diagram, $O_1P_1Q_1R_1$ is the image of $OPQR$ under a stretch $x - axis$ invariant and factor 4. Construct the matrix which represents this transformation.



The vertices of $OPQR$ are $(0,0), (1,0), (1,1)$ and $(0,1)$ respectively and the vertices of $O_1P_1Q_1R_1$ are $(0,0), (1,0), (1,4)$ and $(0,4)$ respectively.

Suggested Solution

Constructing the (2×2) matrices of a Shear $x - axis$ invariant with factor k

Let us consider the set of points $R(0;1)$ and $R_1\,(0;4)$ and the set of points $Q(1;4)$ and $Q_1\,(4;1)$

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \Rightarrow b = 0$ and $d = 4$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Rightarrow a + b = 1 \Rightarrow a + 0 = 1 \therefore a = 1$ and

$c + d = 4 \Rightarrow c + 4 = 4 \therefore c = 0$

Hence the matrix is given by: $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Thus in general the matrix of a Stretch $x - axis$ invariant with factor k is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

Follow up questions

(i) $O_1P_1Q_1R_1$ is the image of $OPQR$ under a stretch $y-axis$ invariant and factor -2. The vertices of $O_1P_1Q_1R_1$ are $(0,0),(-2,0),(-2,1)$ and $(0,1)$. Construct the matrix which represents this transformation.

(ii) $O_1P_1Q_1R_1$ is the image of $OPQR$ under a two way stretch with factor 3, $y-axis$ invariant and factor -2, $x-axis$ invariant. The vertices of $O_1P_1Q_1R_1$ are $(0,0)$, $(3,0)$,$(3,-2)$ and $(0,-2)$. Construct the matrix which represents this transformation.

2 x 2 matrices representing certain geometrical transformation in the x – y plane

The aim of this section is to:

- construct matrices which represent certain transformations
- find images of given vertices using given matrices
- find original points/ object points for given images and matrices.

Note

(i) If a matrix T transforms a shape ABC to its image shape $A_1B_1C_1$ then to find the image shape $A_1B_1C_1$:

- multiply the vertices of ABC by the matrix T or simply
- set up a coordinate matrix in which the coordinates of the points A, B and C form the columns
- Pre-multiply the coordinate matrix by the matrix T.

(ii) To find the matrix which transforms the shape ABC to its image shape $A_1B_1C_1$:

- choose any two set of points
- set up a full matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- pre-multiply the coordinate matrix by this matrix
- solve simultaneous equations to determine the values of a, b, c and d.

(iii) If a matrix T transforms a shape ABC to its image shape $A_1B_1C_1$ then the inverse matrix T^{-1} transforms the image shape $A_1B_1C_1$ back to the object/original shape ABC.

(iv) To find the transformation matrix which will map a point from one position to another:

- set up a diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
- pre-multiply the point by it and solve for a and b.

Solved Problems

Question 1

ZIMSEC NOVEMBER 2018 PAPER 2

Under a 2×2 matrix M the images of $A(1,0)$ and $B(0,1)$ are $A_1(3,5)$ and $B_1(5,9)$ respectively.

Find the

(i) matrix M, **[2]**
(ii) image of point $(2; -5)$ under matrix M. **[2]**

Suggested Solution

(i) Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \Rightarrow a = 3$ and $c = 5$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} \Rightarrow b = 5$ and $d = 9$

Hence the matrix is given by: $\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -19 \\ -35 \end{pmatrix}$ $\therefore$ the image of point $(2; -5)$ under matrix M is $(-19; -35)$.

Question 2

The image of A under a 2×2 matrix $M = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$ is $A_1(-8,6)$, find A.

Suggested Solution

Let $A = (x; y)$.

$\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$

Aside

Finding M^{-1}.

$\det(M) = ad - bc = 4(2) - 2(2) = 8 - 4 = 4$

$\therefore M^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$

Now:

$$\frac{1}{4}\begin{pmatrix}4 & -2\\ -2 & 2\end{pmatrix}\times\begin{pmatrix}2 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{4}\begin{pmatrix}4 & -2\\ -2 & 2\end{pmatrix}\times\begin{pmatrix}-8\\ 6\end{pmatrix}$$

Remember that $M^{-1}M \equiv MM^{-1} \equiv I$, where $I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

$$\Rightarrow I\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{4}\begin{pmatrix}4 & -2\\ -2 & 2\end{pmatrix}\times\begin{pmatrix}-8\\ 6\end{pmatrix}$$

Remember that $IM \equiv MI \equiv M$.

$$\Rightarrow \begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{4}\begin{pmatrix}4 & -2\\ -2 & 2\end{pmatrix}\times\begin{pmatrix}-8\\ 6\end{pmatrix}=\frac{1}{4}\begin{pmatrix}-44\\ 28\end{pmatrix}=\begin{pmatrix}-11\\ 7\end{pmatrix}\Rightarrow (x;y)=(-11;7)$$

$\therefore A = (-11;7)$.

Alternatively we can use the following method:

$$\begin{pmatrix}2 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-8\\ 6\end{pmatrix}$$

$2x + 2y = -8$ (i)

$2x + 4y = 6$ (ii)

(ii)−(i) $2y = 14$

$\Rightarrow y = 7$

$2x + 4y = 6$ (ii)

$2x + 4(7) = 6$

$2x = 6 - 28$

$2x = -22$

$\Rightarrow x = -11$

$\therefore A = (-11;7)$.

Question 3

Find the 2×2 matrix for which the image of $A(-1,2)$ is $A_1(7,-12)$.

Suggested Solution

Let the matrix be $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$

Now:

$$\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}-1\\ 2\end{pmatrix}=\begin{pmatrix}7\\ -12\end{pmatrix}\Rightarrow -a = 7 \therefore a = -7 \text{ and } 2b = -12 \Rightarrow b = -6$$

$\therefore$ The matrix is $\begin{pmatrix} -7 & 0 \\ 0 & -6 \end{pmatrix}$

Follow up exercise

Question 1

Under a 2×2 matrix A the images of $P(1, -1)$ and $Q(2,1)$ are $P_1(2,1)$ and $Q_1(0, -2)$ respectively.

Find the

(i) matrix A,

(ii) image of point $(-1; 3)$ under matrix A.

Question 2

Find the 2×2 matrix for which the image of $A(0,1)$ is $A_1(-1,2$ and the image of $B(-1,0)$ is $B_1(0, -2)$under the same matrix.

Question 3

The transformation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps the line AB, where $A = (1, -3)$ and $B = (-2, 1)$, to the line $A_1 B_1$, with $A_1 = (1, 3)$ and $B_1 = (2, 1)$.

Find a, b, c and d, using simultaneous equations and, hence, write out the matrix T.

Question 4

Find the image of $A(-2,3)$ under a 2×2 matrix $T = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$.

Question 5

The image of P under a 2×2 matrix $N = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ is $P_1(-4,7)$. Find P.

Question 6

The triangle ABC has vertices $(-2, 1), (0, 3)$ and $(1, 0)$ respectively.

(a) Under a reflection in the line $y = 0$, the image of triangle ABC is $A_1B_1C_1$.

(i) Draw triangles ABC and $A_1B_1C_1$ on squared paper.

(ii) Write down the $2\,' \times 2$ matrix associated with this transformation.

(b) Under the transformation defined by the matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

the image of triangle $A_1B_1C_1$ is $A_2B_2C_2$.

(i) Find the vertices of triangle $A_2B_2C_2$ and draw it on your diagram.

(ii) Describe in geometrical terms the transformation defined by M.

(c) Find the $2\,' \times 2$ matrix N under which the image of triangle $A_2B_2C_2$ is triangle ABC.

Describe in geometrical terms the transformation defined by N.

Question 7

Find the 2×2 matrix for which the image of $M(3, -2)$ is $M_1(-15, 2)$.

Question 8

The square $ABCD$ has vertices $(2, 1), (5, 4), (2, 7)$ and $(-1, 4)$ respectively.

(a) Under a reflection in the line $y = -x$, the image of square $ABCD$ is $A_1B_1C_1D_1$.

(i) Draw squares $ABCD$ and $A_1B_1C_1D_1$ on squared paper.

(ii) Write down the $2\,' \times 2$ matrix associated with this transformation.

(b) Under the transformation defined by the matrix: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the image of $A_1B_1C_1D_1$ is $A_2B_2C_2D_2$.

(i) Find the vertices of square $A_2B_2C_2D_2$ and draw it on your diagram.

(ii) Describe in geometrical terms the transformation defined by R.

(c) The matrix D maps $A_2B_2C_2D_2$ back to $ABCD$. Find D.

Describe in geometrical terms the transformation defined by D.

Question 9

The matrix M is given by $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$

a) Calculate the inverse of M.

b) It is given that $\boldsymbol{M}$ maps A, B and C onto $A'(0; -1; 1)$, $B'(1; 3; -2)$ and $C'(2; 1; -5)$, respectively. Find

   (i) the coordinates of A, B and C,

   (ii) the area of triangle ABC.

Question 10

The matrix T is given by $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

a) Interpret geometrically the transformation represented by matrix T.

b) Triangle $P'Q'R'$ has vertices $P'(-2; 0)$, $Q'(0; 4)$ and $R'(8; 0)$ and is the image of triangle PQR under T. Find the coordinates of P, Q and R.

Question 11

The matrix M is given by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

a) Interpret geometrically the transformation represented by matrix M.

b) Triangle ABC has vertices $A(-2; 0)$, $B(0; 4)$ and $C(8; 0)$. Find the vertices of triangle $A'B'C'$, the image of triangle ABC under M.

Question 12

The matrix P is given by $\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$.

a) Interpret geometrically the transformation represented by matrix P.

b) The point A has coordinates $A(-2; 4)$. Find the coordinates of A', the image of point A under P.

Question 13

The matrix N is given by $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

a) Interpret geometrically the transformation represented by matrix N.

b) The point P' with coordinates $(3; -1)$ is the image of P and the point Q' with coordinates $(4; 2)$ is the image of Q under the matrix N.

Find the coordinates of P and Q.

Composite transformation AB and A^n

- A composite transformation is made up of one transformation (e.g. reflection in the $y - axis$) followed by a second transformation (e.g. rotation, centre $(0, 0)$ through $180°$.)
- The composite transformation may be performed by one operation, if we multiply together the two matrices describing these transformations, thereby forming a single transformation matrix.
- We then multiply the coordinate matrix of the point (or set of points forming the vertices of the shape to be transformed) by this matrix.
- A composite transformation matrix can be formed by multiplying two transformation matrices together. It is important to remember that the matrix for the first transformation must be pre-multiplied by the matrix for the second transformation.
  - If A is the first and B is the second of the two transformation matrices to be used in transforming a point or a shape, the single transformation matrix combining A and B is BA, not AB.
  - A composite transformation A^n can be formed by multiplying the transformation matrix A n times. For Solved Problem $A^2 \equiv AA$.

Solved Problems

Question 1

A triangle ABC with vertices $(0, -2), (3, 1)$, and $(2, -5)$ respectively is given a composite transformation under $P = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ followed by $Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(a) Write the geometric transformations represented by Transformations P and Q.

(b) Find the composite transformation of P followed by Q.

(c) Hence write down the vertices of the transformed triangle $A_1B_1C_1$.

Suggested Solution

(a) P is an enlargement with centre $(0,0)$ and scale factor 3 and Q is the reflection on the line $y = -x$.

(b) The composite transformation of P followed by Q is QP.

$$QP = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}.$$

(c) $A_1B_1C_1 = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 3 & 2 \\ -2 & 1 & -5 \end{pmatrix} = \begin{pmatrix} 6 & -3 & 15 \\ 0 & -9 & -6 \end{pmatrix}.$

$\therefore$ the vertices of the transformed triangle $A_1B_1C_1$ are $(6,0)$, $(-3,-9)$, and $(15,-6)$ respectively.

Question 2

The point $T(-2,1)$ is transformed to T_1 by a stretch, $y-axis$ invariant with factor 2 followed by a rotation through $90°$ clockwise about the origin. Find the coordinates of T_1 using the composite transformation matrix.

Suggested Solution

The composite transformation is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$

Now $T_1 = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\therefore$ the coordinates of T_1 using the composite transformation matrix $\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ are $(2,2)$.

Question 3

The first shape is transformed to a second shape by a transformation matrix $M = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ and a third shape is obtained from the second shape using the transformation matrix $N = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find the single transformation that would transform the first shape to the third shape directly.

Suggested Solution

The composite transformation is $NM = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$

Question 4

The first shape is transformed to a second shape by a transformation matrix $M = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$ followed by the same transformation $M = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$. Write down the composite transformation M^2.

Suggested Solution

The composite transformation is $M^2 = MM = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 4 & 1 \end{pmatrix}$.

Follow Up Exercise

Question 1

A triangle ABC with vertices $(1,2), (2,-3)$, and $(-1,1)$ respectively is given a composite transformation under $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ followed by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Show that the vertices of $A_1B_1C_1$, the image of ABC following this composite transformation are $(2,-4), (4,6)$ and $(-2,-2)$ respectively.

Question 2

$A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$.

(a) The line PQ, where $P = (2,3)$ and $Q = (1,-2)$, is transformed to $P_1 Q_1$ by the transformation matrix A. Find P_1 and Q_1.

(b) The line $P_1 Q_1$ is transformed to $P_2 Q_2$ by the transformation matrix B. Find P_2 and Q_2.

(c) Find the single transformation matrix that is equivalent to the composite transformation, A followed by B.

Check this by transforming PQ to $P_2 Q_2$, using your single transformation matrix.

Question 3

The point $A(2,3)$ is transformed to A_1 by a rotation through $90°$ anti-clockwise about the origin followed by a reflection in the line $y = x$. Show that coordinates of A_1 using the composite transformation matrix are $(2,-3)$.

Question 4

The first shape is transformed to a second shape by a transformation matrix $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ and a third shape is obtained from the second shape using the transformation $\begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$. Find the single transformation that would transform the first shape to the third shape directly.

Question 5

The matrix M is given by $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$ and the matrix N is given by $\begin{pmatrix} -1 & 1 & -4 \\ 2 & 0 & -1 \\ 0 & 5 & 0 \end{pmatrix}$

a) Calculate the product MN.

b) Triangle OPQ has vertices $O(1;2;-1)$, $P(3;0;1)$ and $Q(-1;-1;1)$, respectively. It is given that OPQ is mapped onto $O'P'Q'$ by transformation N followed by transformation M. Find the coordinates of O', P' and Q' under the composite transformation.

Question 6

The matrix S is given by $\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and the matrix N is given by $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

a) Interpret geometrically the transformations represented by matrices S and N.

b) The point T has coordinates $T(-12;8)$. Find the coordinates of T', the image of point A under transformation S followed by transformation N.

Question 7

The matrix A is given by $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and the matrix B is given by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

a) Interpret geometrically the transformations represented by matrices A and B.

b) The point $T_1(-12;8)$ is the image of point T under transformation A followed by transformation B. Find the coordinates of T under the composite transformation.

Relationship between the area scale-factor of a transformation and the determinant of the corresponding matrix

- If a plane transformation T is represented by a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A)$ represents the scale factor of the area increase produced by T.
- $det(A) = ad - bc$
- Thus if ABC is transformed to $A_1B_1C_1$ by a transformation T which is represented by a 2×2 matrix A, then:

$$\text{The area of } A_1B_1C_1 = |det(A)| \times \text{area of } ABC\ .$$

Solved Problems

Question 1

The first shape is transformed to a second shape by a transformation matrix $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ and a third shape is obtained from the second shape using the transformation $\begin{pmatrix} -1 & 5 \\ -1 & 2 \end{pmatrix}$.

(a) Find the single transformation that would transform the first shape to the third shape directly.

(b) If the first shape has an area of 5 $square\ units$, find the area of the second and third shape.

Suggested Solution

(a) The composite transformation is $\begin{pmatrix} -1 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 18 & 14 \\ 6 & 5 \end{pmatrix}$.

(b) Area of the second Shape

Let $M = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$. Now $\det(M) = ad - bc = (3)(2) - (4)(1) = 2.$

Now the area of the second shape $= |det(A)| \times$ area of the first shape

$$= 2 \times 5 = 10\ square\ units$$

Area of the third Shape

Let $N = \begin{pmatrix} -1 & 5 \\ -1 & 2 \end{pmatrix}$. Now $\det(T) = ad - bc = (2)(-1) - (5)(-1) = 3$

The area of the third shape $= |det(N)| \times$ area of the second shape

$$= 3 \times 10 = 30 square\ units$$

Alternatively we use the direct method:

Let $T = \begin{pmatrix} 18 & 14 \\ 6 & 5 \end{pmatrix}$. Now $\det(T) = ad - bc = (18)(5) - (14)(6) = 6$

The area of the third shape $= |det(T)| \times$ area of the first shape

$$= 6 \times 5 = 30 square\ units$$

Question 2

A triangle MNP is mapped to triangle $M_1N_1P_1$ by the matrix $\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. If the area of triangle MNP $6\ square\ units$, find the area of the transformed triangle $M_1N_1P_1$.

Suggested Solution

Let $T = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Now $\det(T) = ad - bc = (-1)(1) - (2)(2) = -5$.

Now the area of triangle $M_1N_1P_1 = |det(A)| \times$ area of triangle MNP.

$$= |-5| \times 6$$

$$= 5 \times 6 = 30 square\ units$$

Question 3

The transformation matrix M maps a point $P(3;\ 1)$ onto $P_1(12;\ 3)$ and a point $Q(4;\ 3)$ onto $Q_1(16;\ 9)$.

(i) Find the matrix representing the transformation M.

(ii) Describe fully this transformation.

(iii)Given that R is mapped onto R_1 by transformation M, and that the area of triangle PQR is $6cm^2$, find the area of triangle $P_1Q_1R_1$.

Suggested Solution

(i) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$

$\Rightarrow 3a + b = 12$ (i)

$\Rightarrow 3c + d = 3$ (ii)

Also:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 16 \\ 9 \end{pmatrix}$

$\Rightarrow 4a + 3b = 16$ (iii)

$\Rightarrow 4c + 3d = 9$ (iv)

Solving equation (i) and (iii) simultaneously:

$3a + b = 12$ (i)

$4a + 3b = 16$ (iii)

Substituting equation (i) into equation (iii)

$4a + 3b = 16$ (iii)

$\Rightarrow 4a + 3(12 - 3a) = 16$

$\Rightarrow 4a + 36 - 9a = 16$

$\Rightarrow -5a = -20$

$\Rightarrow a = 4$

$b = 12 - 3a$ (i)

$\Rightarrow b = 12 - 3(4)$

$\Rightarrow b = 12 - 12$

$\Rightarrow b = 0$

Also:

Solving equation (ii) and (iv) simultaneously:

$3c + d = 3$ (ii)

$4c + 3d = 9$ (iv)

Substituting equation (ii) into equation (iv)

$4c + 3d = 9$ (iv)

$\Rightarrow 4c + 3(3 - 3c) = 9$

$\Rightarrow 4c + 9 - 9c = 9$

$\Rightarrow -5c = 0$

$\Rightarrow c = 0$

$3c + d = 3$ (ii)

$\Rightarrow d = 3 - 3c$

$\Rightarrow d = 3 - 0$

$\Rightarrow d = 3$

$\therefore M = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

(ii) Two way stretch with factor 4 y-$axis$ invariant and factor 3 x-$axis$ invariant

(iii) $Area\ of\ P_1Q_1R_1 = |det(M)| \times area\ of\ PQR$

$= |4 \times 3 - 0| \times 6cm^2$

$= |12| \times 6cm^2$

$= 12 \times 6cm^2$

$= 72cm^2$

Follow Up Exercise

Question 1

A triangle PQR has vertices $P(1,3), Q(4,3)$ and $R(3,7)$ is transformed by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & -4 \end{pmatrix}$. Show that the area of the

(i) triangle PQR is 6 $square\ units$

(ii) transformed triangle $P_1Q_1R_1$ is 30 $square\ units$.

Question 2

A triangle ABC is mapped to triangle $A_1B_1C_1$ by the matrix $\begin{pmatrix} 2 & -3 \\ 3 & 8 \end{pmatrix}$. If the area of triangle ABC is 2.5 $square\ units$, find the area of the triangle $A_1B_1C_1$.

Question 3

A triangle XYZ has vertices $X(-1,4)$, $Y(2,1)$ and $Z(1,0)$ is transformed to triangle $X_1Y_1Z_1$ by the matrix $M = \begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}$.

(i) Show that the lines XY and YZ are perpendicular.

(ii) Hence find the area of the

(a) triangle XYZ

(b) transformed triangle $X_1Y_1Z_1$ under M.

Question 4

A triangle with vertices $(1,0)$, $(0,1)$ and $(-1,0)$ is transformed by $T = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$. Show that the area of the transformed triangle is $8\ square\ units$.

Question 5

The matrix A is given by $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and the matrix B is given by $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

a) Interpret geometrically the transformations represented by matrices A and B.

b) Calculate the product BA.

c) Triangle KLM has vertices$K(-2;0)$, $L(0;4)$ and $M(8;0)$

(i) Show that the lines KL and LM are perpendicular.

(ii) Find the area of triangle KLM.

(iii)It is given that KLM is mapped onto $K'L'M'$ by transformation A followed by transformation B. Find the coordinates of K', L' and M'.

(iv)Find the area of triangle $K'L'M'$.

Transformation of a Line

- A linear transformation will map the straight line with vector equation $r = \boldsymbol{a} + t\boldsymbol{b}$ onto a straight line with vector $r = \boldsymbol{a}' + t\boldsymbol{b}'$.
- If the straight line with vector equation $r = \boldsymbol{a} + t\boldsymbol{b}$ is transformed by a(2×2) matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the image line is obtained by multiplying the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with the transformation T as follows:

$$\boldsymbol{a}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\boldsymbol{a} \text{ and } \boldsymbol{b}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\boldsymbol{b}.$$

- If the straight line with $y = mx + c$ is transformed by a(2×2) matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the image line is obtained by multiplying the transformation T with a general point of the line as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix}$$

  We form two linear equations and solve them simultaneously by eliminating x.
- Alternatively we can find the image line using the following steps:
  - Choose any two points on the line
  - Transform these points
  - Find the using the gradient approach i.e.
    - Compute the gradient using image points
    - Find the general point and use this to determine the equation of the line

Solved Problems

Question 1

ZIMSEC NOVEMBER 2007 PAPER 2

Find the image of the line $y = 2x + 1$ under the transformation matrix $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, giving your answer in the form $ax + by + c = 0$. **[3]**

Suggested Solution

Method 1

The general point on the line is $\begin{pmatrix} x \\ 2x+1 \end{pmatrix}$

Now:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ 2x+1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x \\ x+6x+3 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x \\ 7x+3 \end{pmatrix}$$

$\Rightarrow x' = 2x \ \Rightarrow \ \dfrac{x'}{2} = x$ (i) and $y' = 7x + 3$ (ii)

Substituting equation (i) into equation (ii) we get:

$$y' = 7\left(\frac{x'}{2}\right) + 3 \ \Rightarrow \ 2y' = 7x' + 6$$

$\therefore$ The image line is $7x - 2y + 6 = 0$.

<u>Method 2</u>

Choosing any two points from the line $y = 2x + 1$

$A(0;1)$ and $B(1;3)$

Now:

$$A' = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$B' = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

Calculating the gradient of the line joining the points $A'(0;3)$ and $B'(2;10)$

$$Gradient = \frac{10-3}{2-0}$$

$$Gradient = \frac{7}{2}$$

Choosing the general point $(x;y)$and the point $A'(0;3)$

$$\frac{y-3}{x-0}=\frac{7}{2}$$

$2(y-3)=7(x)$

$2y-6=7x$

$\therefore$The equation is $7x-2y+6=0$

Question 2

Given that $T=\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 1\\ 1 & 4 & -2\end{pmatrix}$ transform line $r=\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ to a certain image line. Find this image line.

Suggested Solution

Now let $\boldsymbol{a}=\begin{pmatrix}2\\1\\3\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

$\Rightarrow \boldsymbol{a}'=\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 1\\ 1 & 4 & -2\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$ and $\boldsymbol{b}'=\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 1\\ 1 & 4 & -2\end{pmatrix}\begin{pmatrix}-1\\-1\\2\end{pmatrix}=\begin{pmatrix}-2\\3\\-9\end{pmatrix}$

$\therefore$ The image line is $r=\begin{pmatrix}4\\2\\0\end{pmatrix}+t\begin{pmatrix}-2\\3\\-9\end{pmatrix}$.

Question 3

The transformation with matrix $A=\begin{pmatrix}1 & -3\\ 0 & 1\end{pmatrix}$ followed by the transformation with matrix $B=\begin{pmatrix}3 & 0\\ 0 & 5\end{pmatrix}$ map the line $r=\begin{pmatrix}1\\0\end{pmatrix}+t\begin{pmatrix}2\\1\end{pmatrix}$ onto the new image line.

a) Interpret geometrically the transformations represented by matrices A and B.

b) Find the product of BA.

c) Hence, find the image of the line $r=\begin{pmatrix}-1\\2\end{pmatrix}+t\begin{pmatrix}3\\5\end{pmatrix}$ under this transformation.

Suggested Solution

a) The matrix A represents a shear, $x - axis$ invariant with factor -3 and the matrix B represents a two way stretch with factor $3, y - axis$ invariant and factor $5, x - axis$ invariant.

b) $BA = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -9 \\ 0 & 5 \end{pmatrix}$.

c) Now let $\boldsymbol{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\Rightarrow \boldsymbol{a}' = \begin{pmatrix} 3 & -9 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -21 \\ 10 \end{pmatrix}$ and $\boldsymbol{b}' = \begin{pmatrix} 3 & -9 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -36 \\ 25 \end{pmatrix}$

$\therefore$ The image line is $r = \begin{pmatrix} -21 \\ 10 \end{pmatrix} + t\begin{pmatrix} -36 \\ 25 \end{pmatrix}$.

Follow Up Exercise

Question 1

The transformation matrix $T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ maps the line $y + 4x - 1 = 0$ onto the new image line.

a) Interpret geometrically the transformation represented by matrix T.

b) Find the image of the line $y + 4x - 1 = 0$ under the transformation matrix T giving your answer in the form $ax + by + c = 0$.

Question 2

Find the image of the line $y = 3 - x$ under the transformation matrix $M = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, giving your answer in the form $ax + by + c = 0$. Interpret geometrically the transformation which is being represented by the matrix M.

Question 3

The transformation with matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ followed by the transformation with matrix $B = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$ map the line $y = 3x + 2$ onto the new image line.

a) Interpret geometrically the transformations represented by matrices A and B.
b) Find the image of the line $y = 3x + 2$ under the combined transformation.

Question 4

The transformation matrix $Q = \begin{pmatrix} -1 & 0 & 3 \\ 4 & -2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ maps the line $r = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ onto the new image line. Find the image of the line $r = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ under the transformation matrix Q.

Question 5

Find the image of the line $r = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ under the transformation matrix $S = \begin{pmatrix} 1 & -2 \\ 4 & 7 \end{pmatrix}$.

Question 6

The transformation with matrix $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ followed by the transformation with matrix $B = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$ map the line $r = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ onto the new image line.

a) Interpret geometrically the transformation represented by matrix A.
b) Find the image of the line $r = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ under the combined transformation.

Question 7

Given that $=\begin{pmatrix}1 & 0 & 0\\ 2 & -2 & 1\\ 1 & -1 & 3\end{pmatrix}$.

(i) Find the inverse of P.

(ii) The line $r=\begin{pmatrix}1\\-1\\0\end{pmatrix}+t\begin{pmatrix}3\\-1\\1\end{pmatrix}$ the image of line $r=\begin{pmatrix}a\\b\\c\end{pmatrix}+t\begin{pmatrix}d\\e\\f\end{pmatrix}$ under the transformation matrix P, find the values of a, b, c, d, e and f.

Question 8

Given that $N=\begin{pmatrix}1 & 0 & 0\\ 2 & -2 & 1\\ 1 & -1 & 3\end{pmatrix}$ and $T=\begin{pmatrix}1 & 0 & 0\\ 2 & -2 & 1\\ 1 & -1 & 3\end{pmatrix}$

(i) Find the product of NT.

(ii) The line $r=\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ is the image of a certain line under the composite transformation T followed by N. Find the original equation.

Invariant Points and Lines

- Invariant means unchanging
- An invariant (or fixed) point on a line or a shape is a point which does not move when a specific transformation is applied i.e. it is mapped onto itself. It is its own image
- An invariant (or fixed) line of a transformation is one where every point on the line is mapped to a point on the line - possibly the same point.
- For invariance:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- We form two linear equations and solve them simultaneously by replacing x' with x.

Solved Problems

Question 1

ZIMSEC JUNE 2004 PAPER 2

A transformation matrix M is given by $M = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. Show that, under this transformation all points on the line $y = x$ are invariant points. **[2]**

Suggested Solution

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x \end{pmatrix}$

$\Rightarrow x' = 2x - y$ (i) and $y' = x$ (ii)

Substituting equation (ii) into equation (i):

$\Rightarrow x' = 2y' - y$

For invariance $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow x = 2y - y$

$x = y$

$\therefore$ The image line is $y = x$. Thus all points on the line $y = x$ are invariant points.

Alternative method:

The general coordinates of the line $y = x$ are (x, x)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 2x - x \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$$

$\Rightarrow x' = x$ (i) and $y' = x$ (ii)

Substituting equation (ii) into equation (i):

$\Rightarrow x' = y'$

For invariance $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow x = y$

$\therefore$ The image line is $y = x$. Thus all points on the line $y = x$ are invariant points.

Question 2

Find the invariant points of the following transformation: $x' = 2 + y$ and $y' = 2x + 1$.

Suggested Solution

$x' = 2 + y$ (i)

$y' = 2x + 1$ (ii)

For invariance $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow x = 2 + y$ (i)

$\Rightarrow y = 2x + 1$ (ii)

Substituting equation (ii) into equation (ii):

$\Rightarrow x = 2 + 2x + 1 \Rightarrow x = -3$ and $y = 2(-3) + 1 = -5$.

$\therefore$ The the invariant points is $(-3; -5)$

Question 3

Show that $y = -2x$ and $y = x$ are the only invariant lines of $\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$.

Suggested Solution

The general point is $(x; mx + c)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 5x + mx + c \\ 2x + 4mx + 4c \end{pmatrix}$$

Now $y = mx + c$

$\Rightarrow 2x + 4mx + 4c = m(5x + mx + c) + c$

$\Rightarrow 2x + 4mx + 4c = 5mx + m^2x + mc + c$

$\Rightarrow 0 = mx + m^2x - 2x + mc - 3c$

$\Rightarrow 0 = (m + m^2 - 2)x + c(m - 3)$

Clearly for the equation to be true both $(m + m^2 - 2)$ and $c(m - 3)$ must be zeroes

$\Rightarrow m + m^2 - 2 = 0$ and $c(m - 3) = 0$

$\Rightarrow (m + 2)(m - 1) = 0$ and $c(m - 3) = 0$

$\Rightarrow m = -2$ and $m = 1$ (from the first equation) and $c = 0$ and $m = 3$ (from the second equation)

But it is clear that $m \neq 3$ since in this case the coefficient of x, $(m + m^2 - 2)$ would not be zero.

Thus we use $m = -2$ and $c = 0$ and also $m = 1$ and $c = 0$.

Substitute these values into $y = mx + c$

$\Rightarrow y = -2x + 0 = -2x$ and $y = x + 0 = x$.

$\therefore$ The invariant lines are $y = -2x$ and $y = x$.

Question 4

Find the invariant lines of $P = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

Suggested Solution

The equations are of the form $y = mx + c$. Thus the general point is $(x; mx + c)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 3x + 2mx + 2c \\ x + 2mx + 2c \end{pmatrix}$$

Now $y = mx + c$

$\Rightarrow x + 2mx + 2c = m(3x + 2mx + 2c) + c$

$\Rightarrow x + 2mx + 2c = 3mx + 2m^2x + 2mc + c$

$\Rightarrow 0 = mx + 2m^2x - x + 2mc - c$

$\Rightarrow 0 = (2m^2 + m - 1)x + c(2m - 1)$

Clearly for the equation to be true both $(2m^2 + m - 1)$ and $c(2m - 1)$ must be zeroes

$\Rightarrow 2m^2 + m - 1 = 0$ and $c(2m - 1) = 0$

$\Rightarrow (2m - 1)(m + 1) = 0$ and $c(2m - 1) = 0$

$\Rightarrow m = \frac{1}{2}$ and $m = -1$ (from the first equation) and $c = 0$ and $m = \frac{1}{2}$ (from the second equation)

But it is clear that if $m = \frac{1}{2}, c$ will take any value

Thus we use $m = \frac{1}{2}$ and c and also $m = -1$ and $c = 0$.

Substitute these values into $y = mx + c$

$\Rightarrow y = \frac{1}{2}x + c$ and $y = -x + 0 = -x$.

$\therefore$ The invariant lines are $y = \frac{1}{2}x + c$ and $y = -x$.

Question 5

Find the only invariant line and point of $P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$.

Suggested Solution

The general point is $(x; mx + c)$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 3x - mx - c \\ x + mx + c \end{pmatrix}$

Now $y = mx + c$

$\Rightarrow x + mx + c = m(3x - mx - c) + c$

$\Rightarrow x + mx + c = 3mx - m^2x - mc + c$

$\Rightarrow m^2x - 2mx + x + mc = 0$

$\Rightarrow (m^2 - 2m + 1)x + mc = 0$

Clearly for the equation to be true both $(m^2 - 2m + 1)$ and mc must be zeroes

$\Rightarrow m^2 - 2m + 1 = 0$ and $mc = 0$

$\Rightarrow (m - 1)(m - 1) = 0$ and $mc = 0$

$\Rightarrow m = 1$ (from the first equation) and $c = 0$ and $m = 1$ (from the second equation)

Thus we use $m = 1$ and $c = 0$

Substitute these values into $y = mx + c$

$\Rightarrow y = x + 0 = x$

$\therefore$ The only invariant line is $y = x$ and the only invariant point is $(0, 0)$ under this line

Note

$(0; 0)$ must be included as an invariant point for any transformation in the plane which is represented by any 2×2 matrix since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Follow Up Exercise

Question 1

Find the invariant lines of $N = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

Answer: The invariant lines are $y = \frac{1}{2}x + c$ and $y = -x$.

Question 2

Find the invariant points under the transformation given by the matrix $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$.

Answer: All points of the form $(k; k)$ or any point on the line $y = x$

Question 3

Find the invariant points of transformation $T = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$.

Answer: $(0; 0)$.

Question 4

Show that the line $2y + x = 0$ is invariant under the transformation with matrix $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$.

Question 5

Find the invariant lines of the matrix $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

Answer: The invariant lines are $y = -\frac{1}{2}x$ and $y = x$.

Question 7

Show that the transformation matrix $\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$ has no invariant lines

Question 8

Show that the line $x - 2y + 7 = 0$ is invariant under the transformation with matrix $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

Question 9

Show that the line $2y = x$ is invariant under the transformation with matrix $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

Question 10

Show that the line $y + 2x = 0$ is invariant under the transformation with matrix $\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$.

PAST EXAMINATION QUESTIONS

Question 1

ZIMSEC JUNE 2019 PAPER 2

Find the image of the line $y = 5x$ under the transformation matrix $\begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix}$. **[3]**

Answer: $27x + y = 0$

Question 2

ZIMSEC NOVEMBER 2018 PAPER 2

Under a 2×2 matrix M the images of $A(1,0)$ and $B(0,1)$ are $A_1(3,5)$ and $B_1(5,9)$ respectively.

Find the

(i) matrix M, **[2]**

(ii) image of point $(2; -5)$ under matrix M. **[2]**

Answer: $\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$; $(-19; -35)$

Question 3

ZIMSEC JUNE 2016 PAPER 2

The transformation, P, maps a point $A(1,2)$ onto $A_1(3; 2)$ and a point $B(2; 5)$ onto $B_1(7; 5)$.

(i) Find the matrix representing the transformation P. **[3]**

(ii) Describe fully the transformation P. **[3]**

(iii) Given that C is mapped onto C_1 by transformation P, and that the area of triangle ABC is $4cm^2$, find the area of triangle $A_1B_1C_1$. **[2]**

Answer: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; Shear, x-axis invariant with factor 1; $4cm^2$

Question 4

ZIMSEC NOVEMBER 2015 PAPER 2

Find the equation of the line whose image is $y + 14x + 22 = 0$ under the transformation which maps

$\begin{pmatrix} y \\ x \end{pmatrix}$ to $\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. **[4]**

Answer: $y = 3x + 2$

Question 5

ZIMSEC JUNE 2015 PAPER 2

Given that matrix $T = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

(i) interpret geometrically the transformation represented by T,

(ii) find the image of a line $y = 2x + 3$ under transformation T, giving your answer in the form $ax + by + c = 0$. **[5]**

Answer: T-Shear, x-axis invariant with factor -2

$3y + 2x - 3 = 0$.

Question 6

ZIMSEC JUNE 2013 PAPER 2

A transformation matrix M is given by $M = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}$.

(i) Show that the transformation matrix has no invariant line using the equation of line $y = mx$, **[5]**

(ii) All points on the line $y - 3x = -2$ are transformed by the matrix M. Find the equation of the image line. **[4]**

Answer: The equation $m^2 + m + 1$ has real roots i.e $b^2 - 4ac < 0$ thus the transformation has no invariant line; $11x + 5y = 14$

Question 7

ZIMSEC NOVEMBER 2010 PAPER 2

If a matrix $T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, describe completely the single transformation represented by each of the matrices M and T.

Find the coordinate of the point whose image under $(\mathrm{TM})^{-1}$ is $(3; 1)$. **[7]**

Answer: T-Shear, y-axis invariant with factor 2; M- Rotation, centre $(0,0)$, $90°$ clockwise or $270°$ anticlockwise; $(1; -1)$.

Question 8

ZIMSEC NOVEMBER 2008 PAPER 2

Find the equation of the line onto which the line $y + x = 0$ is mapped by the transformation

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. **[4]**

Answer: $3y - 2x = 0$

Question 9

ZIMSEC NOVEMBER 2007 PAPER 2

Find the image of the line $y = 2x + 1$ under the transformation matrix $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, giving your answer in the form $ax + by + c = 0$. **[3]**

Answer: $7x - 2y + 6 = 0$

Question 10

ZIMSEC JUNE 2007 PAPER 2

Interpret geometrically the transformation represented by the matrix $M = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Find a second matrix, N, such that $MN = I$. **[3]**

Answer: M-Shear, x-axis invariant with factor 3

$3y + 2x - 3 = 0$; $N = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ (The inverse)

Question 11

ZIMSEC NOVEMBER 2005 PAPER 2

The matrix C is given by $\begin{pmatrix} 0 & -1 & -1 \\ -2 & 1 & 1 \\ -3 & 0 & 1 \end{pmatrix}$

a) Calculate the inverse of C. **[6]**

b) It is given that $\boldsymbol{C}$ maps E, F and G onto $E'(-1; 1; 0)$, $F'(-1; -5; -9)$ and $G'(-2; -4; -9)$ respectively.

(i) Show that the coordinates of E, F and G are $(0; 1; 0)$, $(3; 1; 0)$ and $(3; 2; 0)$ respectively. **[3]**

(ii) Hence find the area of triangle EFG. **[2]**

Answer: $\frac{3}{2}\ units^2$

Question 12

ZIMSEC JUNE 2004 PAPER 2

A transformation matrix M is given by $M = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

a) Show that, under this transformation

(i) all points on the line $y = x$ are invariant points; **[2]**

(ii) any point on the, line $y = x + c$ where $c \neq 0$, is transformed to another point on that line. **[2]**

b) Prove by induction that, if n is an integer, then $M^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$. **[5]**

Question 13

ZIMSEC NOVEMBER 2003 PAPER 2

All points on the line' $y. = 2x - 5$ are transformed by the matrix $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$. Find the equation of the image line. **[5]**

Answer: $x + 8y = 25$

ASANTE SANA

********THERE IS A LIGHT AT THE END OF EVERY TUNNEL ********

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

Nyasha P. Tarakino (Trockers)

+263772978155/+263717267175

ntarakino@gmail.com