Trigonometric Functions

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SYLLABUS (6042) REQUIREMENTS

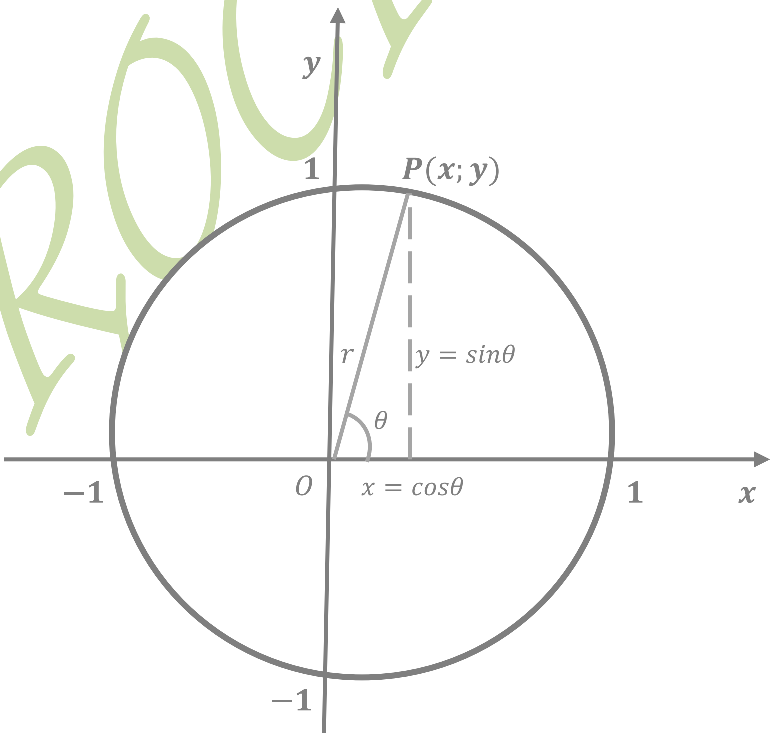
- Sketch the graphs of trigonometrical functions
- Transform the graphs of trigonometrical functions
- Sketch graphs of inverse trigonometrical relations
- Solve trigonometrical equations
- Prove trigonometrical identities
- Solve problems using trigonometrical identities

TRIGONOMETRIC FUNCTIONS

Definition: Trigonometric functions are elementary functions, the argument of which is an angle. Trigonometric functions are simply functions of angles. They are used to relate the angles.

Notes

- Trigonometric functions describe the relation between the sides and angles of a right angle.
- Trigonometric functions are also called circular functions
- Trigonometric functions include the following six functions: sine, cosine, tangent, cotangent, secant and cosecant.
- Trig functions can be defined using unit circles (A circle with a radius of 1)



The diagram above shows a circle with centre (0,0) and radius $r = 1$. $P(x, y)$ is a point on the circle and θ is the angle between the positive direction of x-axis and vector $\overrightarrow{OP}$.

NOW:

- The Sine of an angle θ is the ratio of y coordinate of the point $P(x, y)$ to the radius r:
  $Sin\theta = \frac{y}{r}$
  Since $r = 1$, this is equal to y coordinate of the point $P(x, y)$ i.e. $Sin\theta = y$

- The Cosine of an angle θ is the ratio of x coordinate of the point $P(x, y)$ to the radius r:

$$Cos\theta = \frac{x}{r}$$

- The Tangent of an angle θ is the ratio of x coordinate of the point $P(x, y)$ to the x coordinate:

$$Tan\theta = \frac{y}{x}; \qquad x \neq 0$$

- The Cotangent of an angle θ is the ratio of x coordinate of the point $P(x, y)$ to the y coordinate:

$$Cot\theta = \frac{x}{y}; \qquad y \neq 0$$

NB:

$$Cot\theta = \frac{1}{Tan\theta} \equiv \frac{Cos\theta}{Sin\theta}$$

- The Secant of an angle θ is the ratio of the radius to the x coordinate of the point $P(x, y)$:

$$Sec\theta = \frac{r}{x} = \frac{1}{x}; \qquad x \neq 0$$

NB:

$$Sec\theta \equiv \frac{1}{Cos\theta}$$

- The Cosecant of an angle θ is the ratio of the radius to the y coordinate of the point $P(x, y)$:

$$Cosec\theta = \frac{r}{y} = \frac{1}{y}; \qquad y \neq 0$$

NB:

$$Cosec\theta \equiv \frac{1}{Sin\theta}$$

NOTE

For $Cosec\theta$ and $Sec\theta$ there is a 'C' 'S' relationship i.e. $Sin\theta$ and $Cos\theta$ are reciprocal functions

$$Cosec\theta \equiv \frac{1}{Sin\theta} \quad \text{and} \quad Sec\theta \equiv \frac{1}{Cos\theta}$$

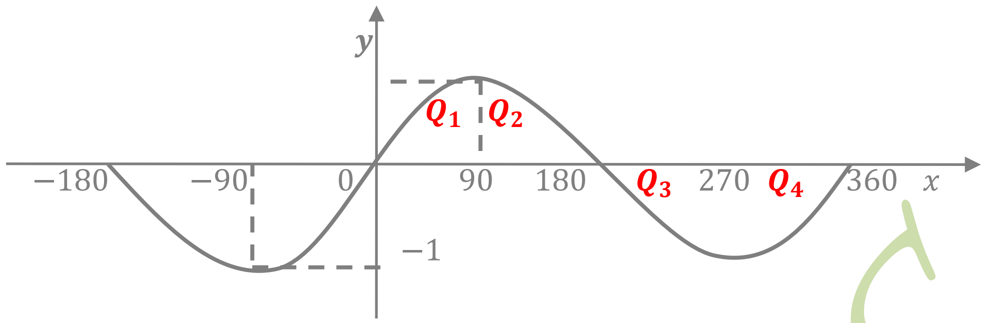
Graphs of Basic Trigonometric Functions

The six basic trigonometric functions are $sinx$, $cosx$, $tanx$, $cotx$, $secx$ and $cosecx$.

Amplitude and Period

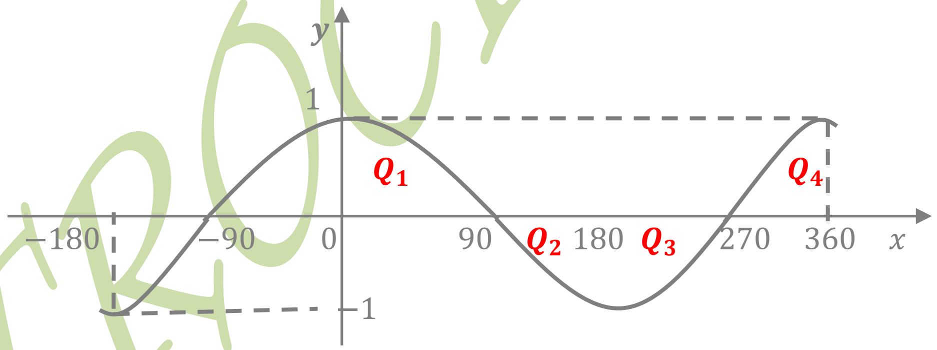
(a) Amplitude refers to the maximum value of y or simple maximum ordinate value
(b) Period refers to the length of the smallest domain interval which corresponds to a complete cycle of values of the function
(c) Domain refers to the set of all possible input values (x values) over which the function has defined outputs
(d) Range refers to the set of all actual outputs (y values)

Sine Function



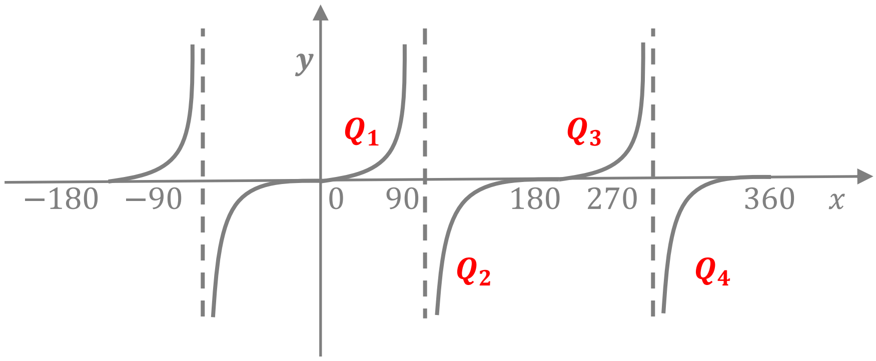
- Amplitude $= 1$
- Period $= 360°$ or $2\pi\ radians$
- Domain $= -\infty < x < +\infty$
- Range $= -1 \leq y \leq +1$
- $Sinx \equiv -Sin(-x)$ i.e. it is odd or asymmetric

Cosine Function



- Amplitude $= 1$
- Period $= 360°$ or $2\pi\ radians$
- Domain $= -\infty < x < +\infty$
- Range $= -1 \leq y \leq +1$
- $Cosx \equiv Cos(-x)$ i.e. it is even or symmetric

Tangent Function



- Amplitude = None
- Period = 180° or π *radians*
- Domain = $-\infty < x < +\infty$ except for $x = \left\{\frac{n\pi}{2}\right\}: n$ is an odd integer $[n \in (odd\ \mathbb{Z})]$
- Range = $-\infty < y < +\infty$
- $Tanx \equiv -Tan(-x)$ i.e. it is odd or asymmetric

NOTE:

$$Tanx = \frac{Sinx}{Cosx};$$

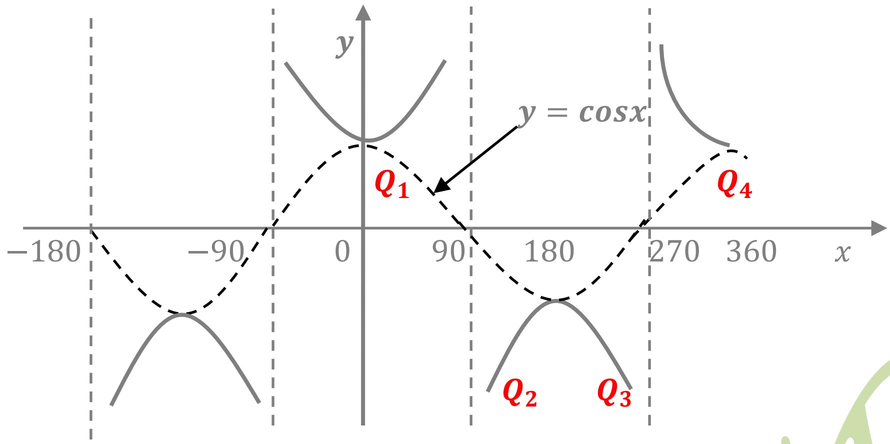
- It is defined for $x \in \mathfrak{R}: Cosx \neq 0$
- So when $Cosx = 0$ we have the asymptotes for the $Tanx$ graph i.e. $x \neq \pm 90°$, $\pm 270°$ *etc*
- The vertical asymptotes must be positioned at $x = \pm 90°$; $x = \pm 270°$ *etc*

  NB: $Tanx$ is undefined for $x = \left\{(2n+1)\frac{\pi}{2}\right\}: n \in \mathbb{Z}$ or

  $x = \left\{\frac{n\pi}{2}\right\}: n \in (odd\ \mathbb{Z})$

  or $x = \{(2n+1)90^{\circ}\}: n \in \mathbb{Z}$ or $x = \{90^{\circ}\ n\}: n \in (odd\ \mathbb{Z})$

Secant Function



- Period = 360° or 2π *radians*
- Domain = $-\infty < x < +\infty$ except for $\frac{n\pi}{2}$:where n is an odd integer [$n \in (odd\ \mathbb{Z})$]
- Range = $-\infty < y < +\infty$ except for $-1 < y < 1$
- $Secx \equiv Sec(-x)$ i.e. it is even or symmetric

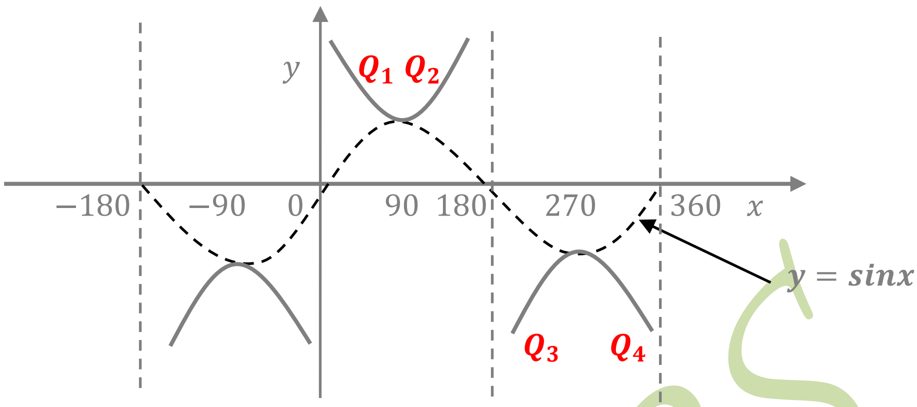
NOTE:

$$Secx = \frac{1}{Cosx};$$

- It is defined for $x \in \Re$: $Cosx \neq 0$
- So when $Cosx = 0$ we have the asymptotes for the $Secx$ graph i.e. $x \neq \pm 90°$, $\pm 270°$ *etc*
- The vertical asymptotes must be positioned at $x = \pm 90°$; $x = \pm 270°$ *etc*

NB: $Secx$ is undefined for $x = \left\{\frac{n\pi}{2} : n \in (odd\ \mathbb{Z})\right\}$ or $x = \{90n° : n \in (odd\ \mathbb{Z})\}$

Cosecant Function



- Period $= 360°$ or $2\pi\ radians$
- Domain $= -\infty < x < +\infty$ except for $n\pi: n$ is an integer ($n \in \mathbb{Z}$)
- Range $= -\infty < y < +\infty$ except for $-1 < y < 1$
- $Cosecx \equiv -Cosec(-x)$ i.e. it is odd or asymmetric

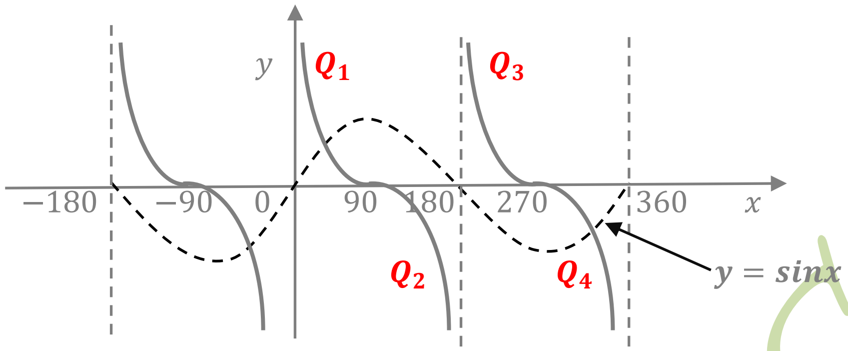
NOTE:

$$Cosecx = \frac{1}{Sinx};$$

- It is defined for $x \in \Re: Sinx \neq 0$
- So when $Sinx = 0$ we have the asymptotes for the $Cosecx$ graph i.e. $x = 0°$, $\pm180°, \pm360°\ etc$
- The vertical asymptotes must be positioned at $x = 0°;\ x = \pm180°;\ x = \pm360°\ etc$

  NB: $Cosecx$ is undefined for $x = \{n\pi: n \in \mathbb{Z}\}$ or $x = \{180^\circ\ n: n \in \mathbb{Z}\}$

Cotangent Function



- Period $= 180°$ or $\pi\ radians$
- Domain $= -\infty < x < +\infty$ except for $n\pi: n$ is an integer $(n \in \mathbb{Z})$
- Range $= -\infty < y < +\infty$
- $Cotx \equiv -Cot(-x)$ i.e. it is odd or asymmetric

NOTE:

$$Cotx = \frac{Cosx}{Sinx};$$

- It is defined for $x \in \Re$: $Sinx \neq 0$
- $x \neq 0°, \pm 180°, \pm 360°\ etc$
- The vertical asymptotes must be positioned at $x = 0°$; $x = \pm 180°$; $x = \pm 360°\ etc$

NB: $Cotx$ is undefined for $x = \{n\pi: n \in \mathbb{Z}\}$ or $x = \{180°\ n: n \in \mathbb{Z}\}$

Graphs of Inverse Trigonometrical Relations

- The six inverse trigonometric functions are $sin^{-1}x$; $cos^{-1}x$; $tan^{-1}x$; $cot^{-1}x$; $sec^{-1}x$ and $cosec^{-1}x$.
- To sketch the graphs of inverses simply reflect the graphs of the original functions along the line $y = x$.

$sin^{-1}x$ Function

It is a reflection of the graph of $y = sinx$ along the line $y = x$. The domain of $y = sinx$ must be restricted so that it becomes a one-one function.

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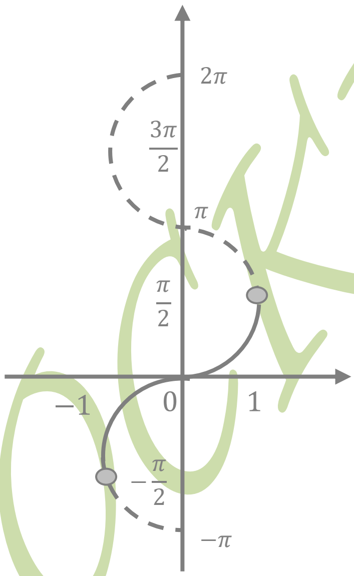

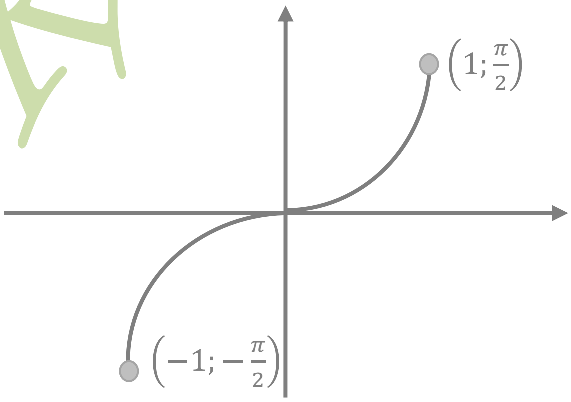

Domain: $[-1; 1]$ or $-1 \leq x \leq +1$

Range: $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ or $-\frac{\pi}{2} \leq y \leq +\frac{\pi}{2}$

$cos^{-1}x$ Function

It is a reflection of the graph of $y = cosx$ along the line $y = x$. The domain of $y = cosx$ must be restricted so that it becomes a one-one function.

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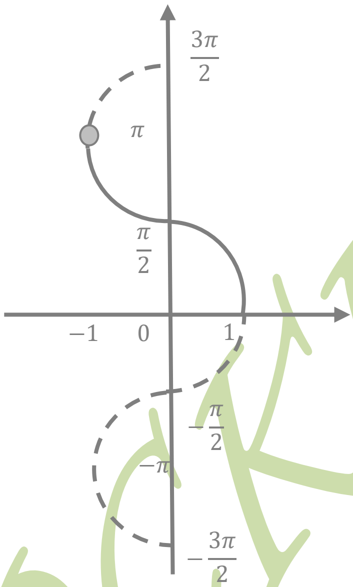
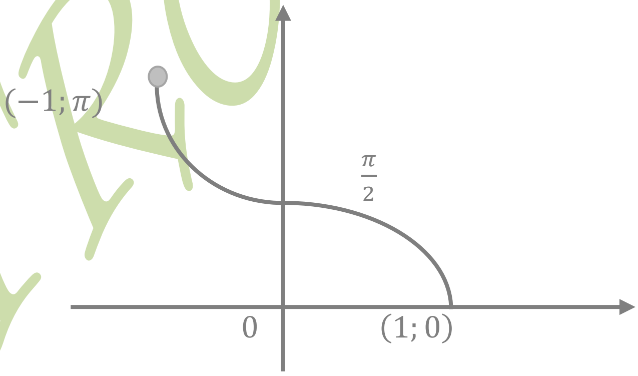


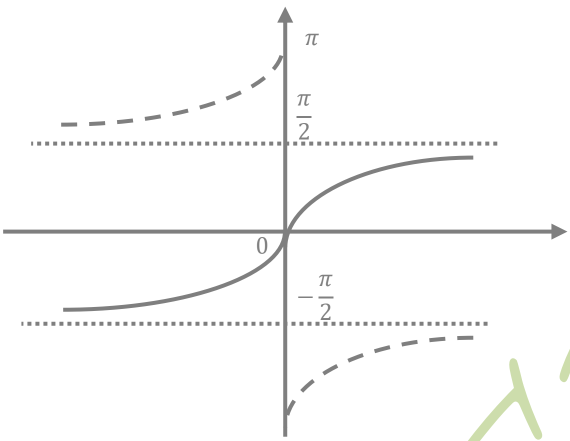


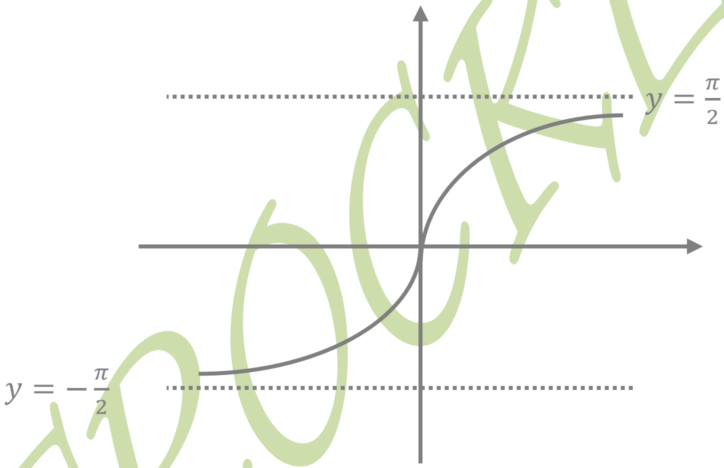
Domain: $[-1;1]$ or $-1 \leq x \leq +1$

Range: $[0;\pi]$ or $0 \leq y \leq +\pi$

$tan^{-1}x$ Function

It is a reflection of the graph of $y = tanx$ along the line $y = x$. The domain of $y = tanx$ must be restricted so that it becomes a one-one function.



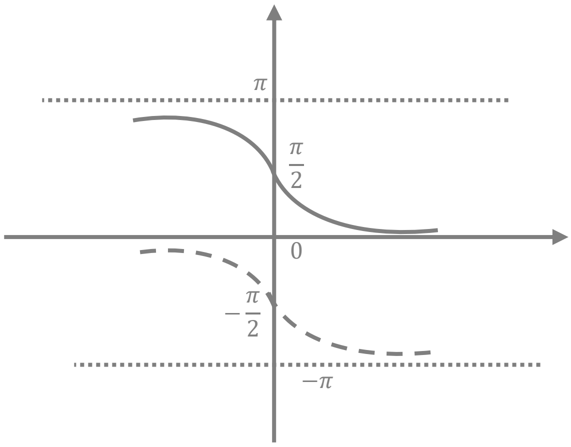


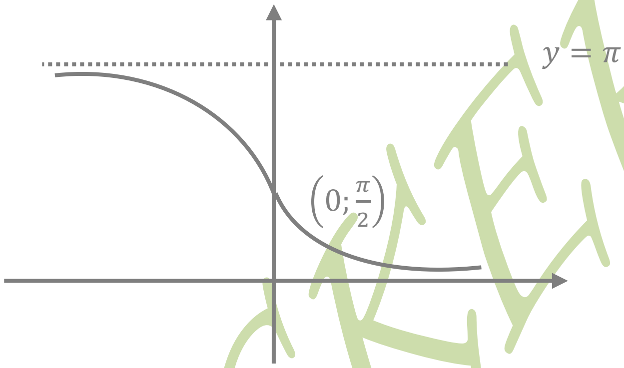
Domain:$[-\infty; \infty]$ or $-\infty \leq x \leq +\infty$

Range: $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ or $-\frac{\pi}{2} < y < +\frac{\pi}{2}$

$cot^{-1}x$ Function

It is a reflection of the graph of $y = cotx$ along the line $y = x$. The domain of $y = cotx$ must be restricted so that it becomes a one-one function.



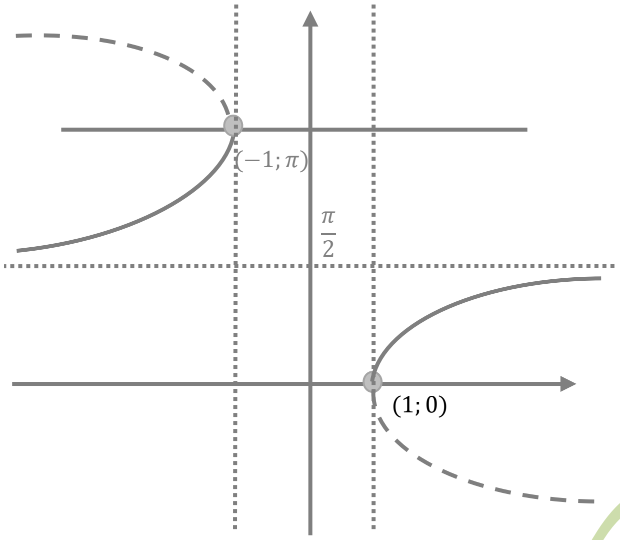


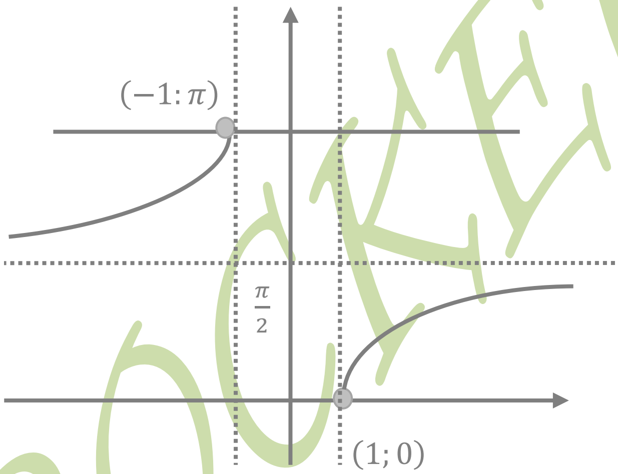
Domain:$[-\infty; \infty]$ or $-\infty \leq x \leq +\infty$

Range: $(0;\ \pi)$ or $0 < y < +\pi$

$sec^{-1}x$ Function

It is a reflection of the graph of $y = secx$ along the line $y = x$. The domain of $y = secx$ must be restricted so that it becomes a one-one function.



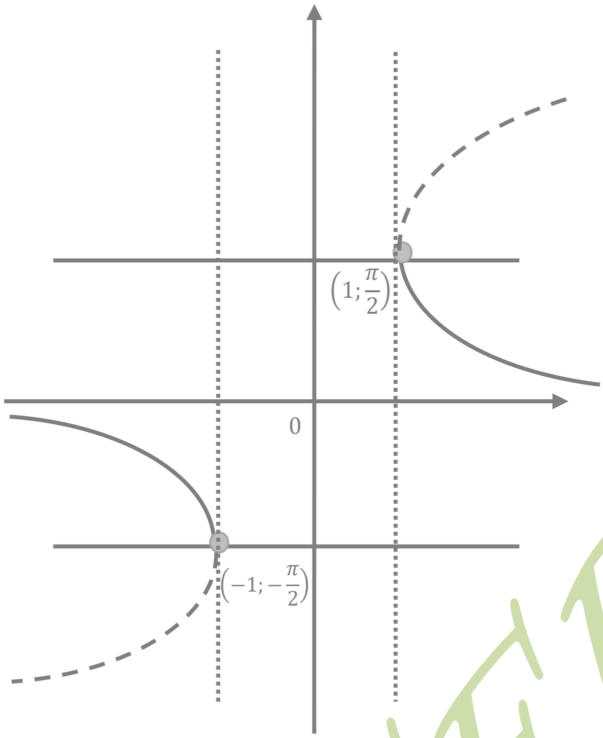


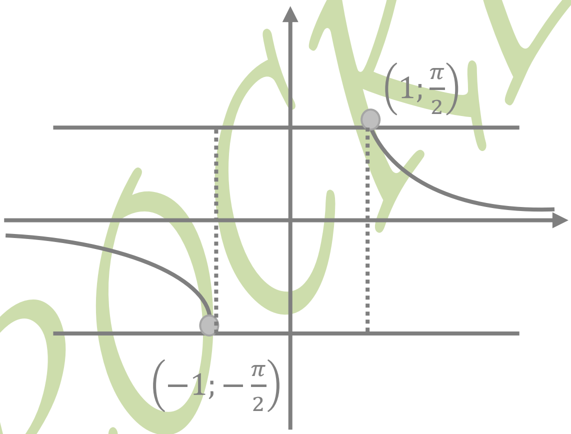
Domain:$[-\infty;\infty]$ except for $(-1;1)$ or $-\infty \leq x \leq +\infty$ except for $-1 < x < 1$

Range: $[0;\pi]$ except for $\frac{\pi}{2}$ or $0 \leq y \leq +\pi: y \neq \frac{\pi}{2}$

$cosec^{-1}x$ Function

It is a reflection of the graph of $y = cosecx$ along the line $y = x$. The domain of $y = cosecx$ must be restricted so that it becomes a one-one function.





Domain$[-\infty;\infty]$ except for $(-1;1)$ or $-\infty \leq x \leq +\infty$ except for $-1 < x < 1$

Range: $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ except for 0 or $-\frac{\pi}{2} \leq y \leq +\frac{\pi}{2}: y \neq 0$

TRANSFORMATIONS

- To transform means to change in shape (size) and position.
- Only three out of the six transformations will be considered i.e. Translation, Reflection and Stretch.

TRANSLATION

(i) x$-\boldsymbol{axis}$

$f(x-a)$: This is translation along the $x-axis$ with translation vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ or simply moving a units to the right side.

$f(x+a)$: This is translation along the $x-axis$ with translation vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ or simply moving a units to the left side.

(ii) y$-\boldsymbol{axis}$

$f(x)-a$: This is translation along the $y-axis$ with translation vector $\begin{pmatrix} 0 \\ -a \end{pmatrix}$ or simply moving a units downwards.

$f(x)+a$: This is translation along the $y-axis$ with translation vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$ or simply moving a units upwards.

REFLECTION

(i) x$-\boldsymbol{axis}$

$-f(x)$: This is reflection along the $x-axis$

(ii) y$-\boldsymbol{axis}$

$f(-x)$: This is reflection along the $y-axis$

STRETCH

(i) x$-\boldsymbol{axis}$

$f(ax)$: This is stretch along the $x-axis$ with factor $\left(\frac{1}{a}\right)$ or simply divide every $x-coordinate$ by a.

(ii) y$-\boldsymbol{axis}$

$af(x)$: This is stretch along the $y-axis$ with factor a or simply multiply every $y-coordinate$ by a.

Worked examples

Question 1

Sketch the graph of $f(x) = Cosx - 1$ for $0 \leq x \leq 2\pi$.

Hence on separate diagrams, sketch the graphs of the following functions and state the resultant transformation.

(i) $y = f\left(\frac{1}{2}x\right)$

(ii) $y = f\left(x - \frac{\pi}{2}\right)$

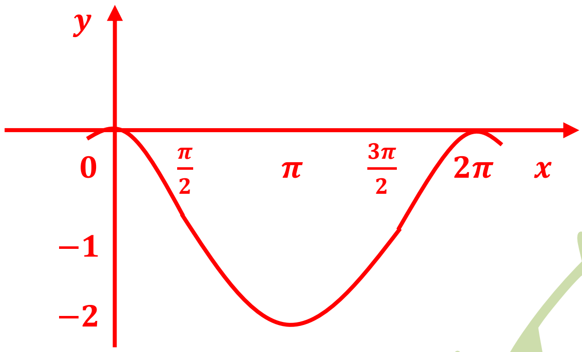
(iii)$y = 2f(x)$

(iv)$y = \left|f\left(x - \frac{\pi}{2}\right) + 1\right|$

Suggested solution

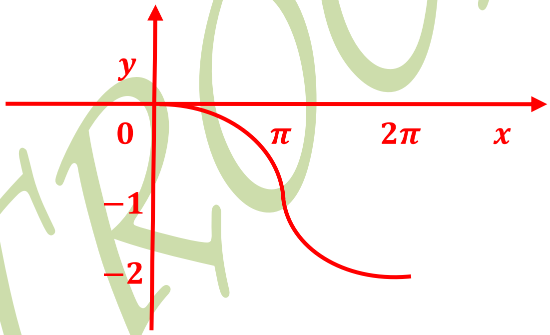
NB: $f(x)$ is a graph of $Cosx$ which has been translated 1 unit below the $y-axis$.

$y = f(x)$



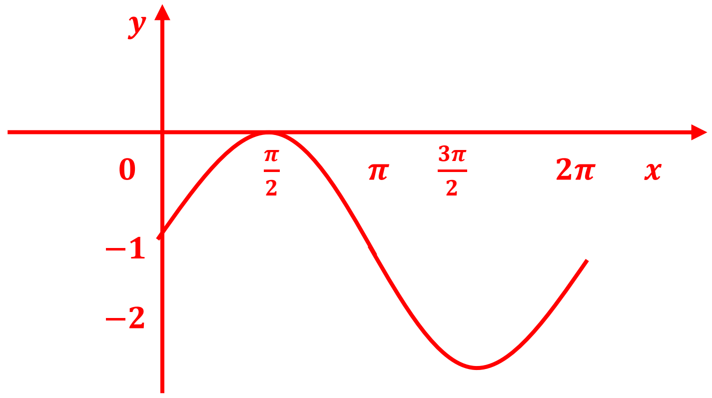
(i) $y = f\left(\frac{1}{2}x\right)$

Stretch along the $x-axis$ with factor $\frac{1}{2}$ or multiply every $x-coordinate$ by 2 or divide every $x-coordinate$ by $\frac{1}{2}$.



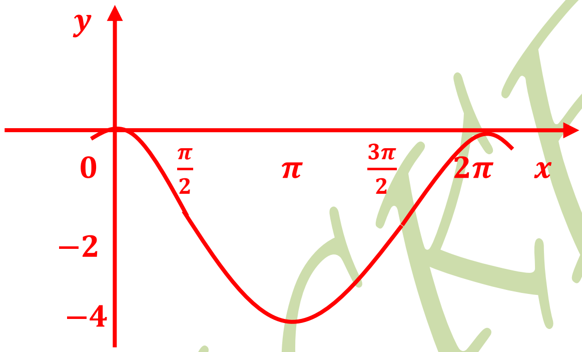
(ii) $y = f\left(x - \frac{\pi}{2}\right)$

Translation along the $x-axis$ with translation vector $\begin{pmatrix}\frac{\pi}{2}\\0\end{pmatrix}$ or moving $\frac{\pi}{2}$ $units$ rightwards.



(iii)$y = 2f(x)$

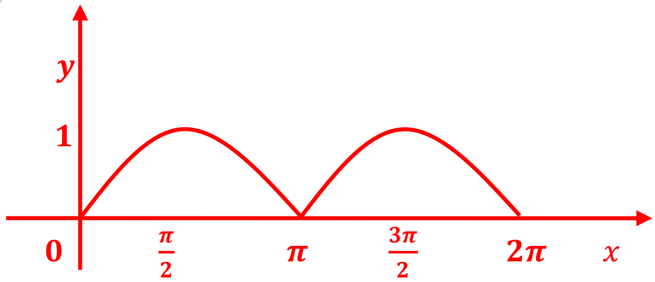
Stretch along the $y-axis$ with factor 2 or multiplying every $y-coordinate$ by 2.



(iv)$y = \left|f\left(x-\frac{\pi}{2}\right)+1\right|$

(a) Translation along the $x-axis$ with translation vector $\begin{pmatrix}\frac{\pi}{2}\\0\end{pmatrix}$ or moving $\frac{\pi}{2}$ $units$ rightwards.

(b) Translation along the $y-axis$ with translation vector $\begin{pmatrix}0\\1\end{pmatrix}$ or moving 1 $unit$ upwards.



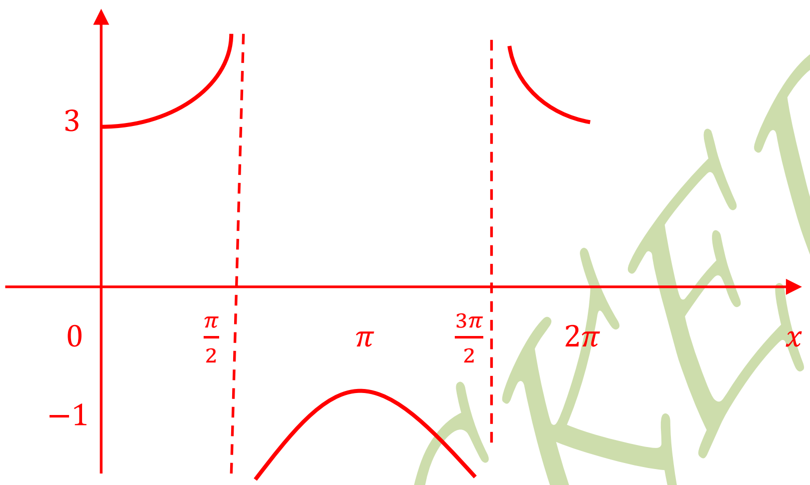
Question 2

Sketch the graph of $y = 2secx + 1$, for the interval $0 \leq x \leq 2\pi$.

Suggested Solution

(a) Stretch along the $y - axis$ with factor 2 or multiply every y −coordinate by 2

(b) Translation along the $y - axis$ with translation vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or moving 1 $unit$ upwards.



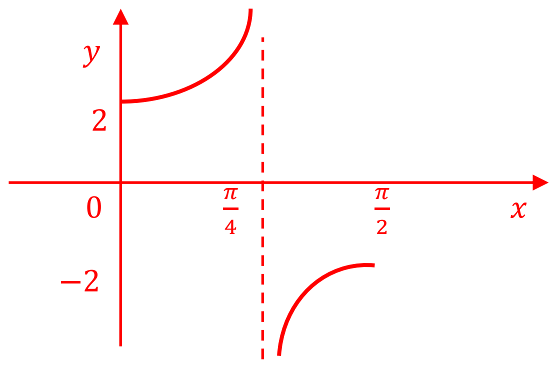
Question 3

Sketch the graph of $y = 2sec2x$, for the interval $0 \leq x \leq \frac{\pi}{2}$.

Suggested Solution

(a) Stretch along the $x - axis$ with factor $\frac{1}{2}$ or divide every x −coordinate by 2

(b) Stretch along the $y - axis$ with factor 2 or multiply every y −coordinate by 2



Question 4

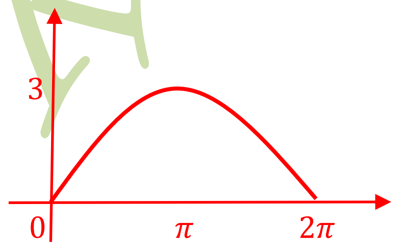
a) Sketch the graph of $f(x) = 3Sin\frac{1}{2}x$ for the interval $0 \leq x \leq 2\pi$

b) Given that $f: x \mapsto sinx$, for the domain $\frac{\pi}{2} \leq x \leq p$, find the largest value of p for which f has an inverse.

Suggested Solution

a) (i) Stretch along the $x - axis$ with factor 2 or divide every x −coordinate by $\frac{1}{2}$

(ii) Stretch along the $y - axis$ with factor 3 or multiply every y −coordinate by 3



b) If f has an inverse then it must be a one to one function thus the largest value of p in the given interval is $\frac{3\pi}{2}$.

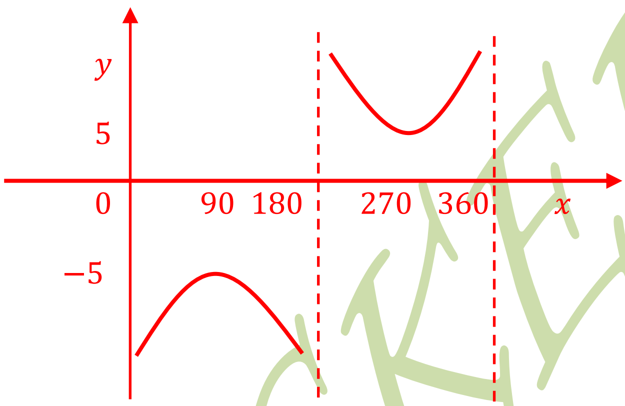
Question 5

Sketch the graph of $y = 5cosec(x - \pi)$, for the interval $0 \leq x \leq 2\pi$.

Suggested Solution

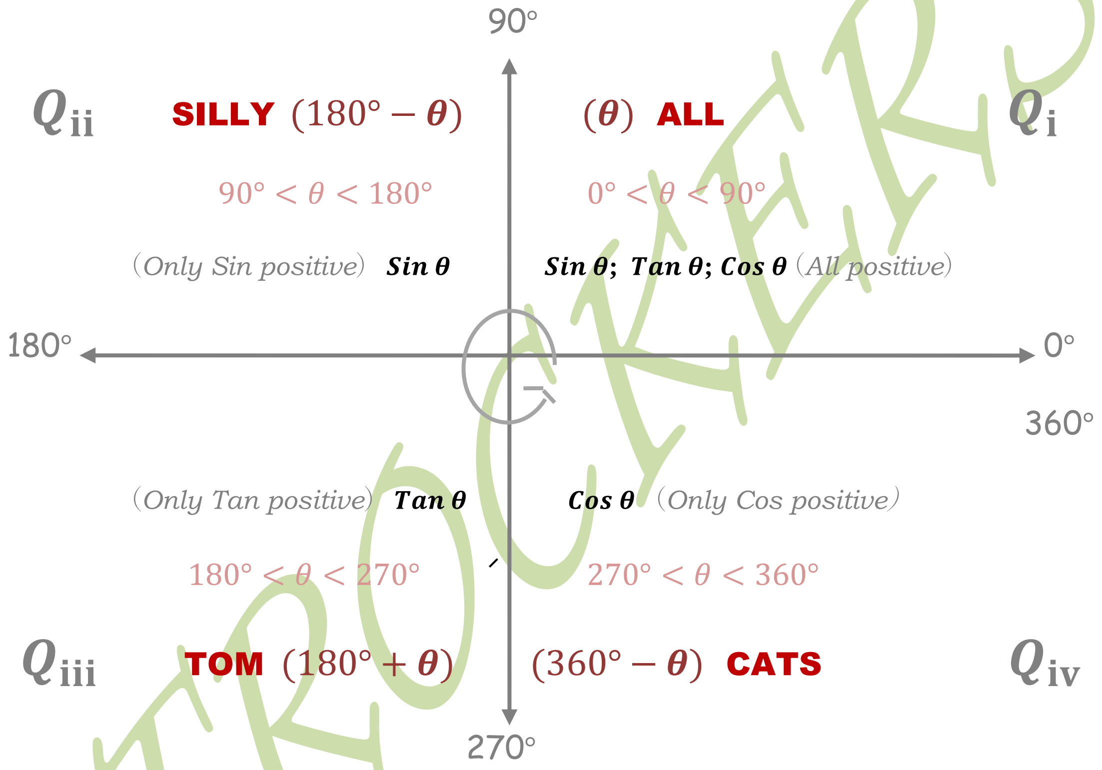
(a) Translation along the $x - axis$ with translation vector $\binom{\pi}{0}$ or moving π $units$ rightwards.

(b) Stretch along the $y - axis$ with factor 5 or multiply every y −coordinate by 5.



Quadrants/CAST Diagram

- A quadrant is a quarter of a coordinate plane
- There are four quadrants namely (i), (ii), (iii) and (iv)
- The coordinates in each respective quadrant are of the form $(+;\ +)$, $(-;\ +)$, $(-;\ -)$ and $(+;\ -)$ respectively
- Quadrants move in an anticlockwise direction from the first quadrant which is the positive quadrant i.e. both x and y have positive coordinates.

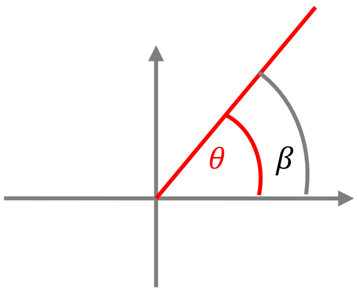


- There are two angles within the first $360°$ that have the same value.
- The first angle is called the principal angle/value and is found using the calculator and the second one is called the secondary angle/ value
- $\theta°$ is the principal angle
- The principal angle is the anti-clockwise angle between the initial arm and the terminal arm of an angle in standard position.
- The value of thee principal angle is between $0°$ and $360°$.

Reference Angles

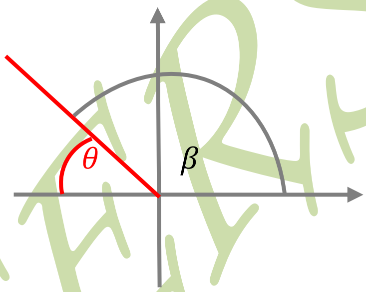
- $\theta°$ is the reference angle: The positive acute and smallest angle representing an angle of any measure and it is made from the terminal side of a given angle with the x-axis. It is never negative
- $\beta°$ is the principal/standard angle

Quadrant 1



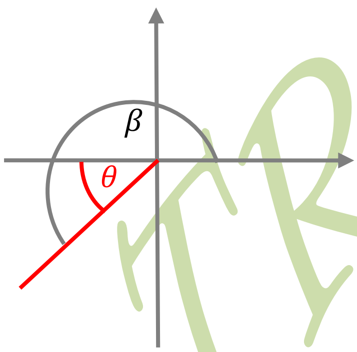
$\boldsymbol{\theta}$

Quadrant 2



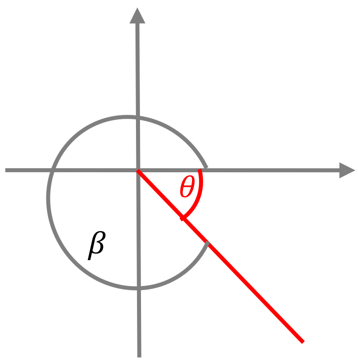
$\mathbf{180° - \boldsymbol{\theta}}$

Quadrant 3



$\mathbf{180° + \boldsymbol{\theta}}$

Quadrant 4



$\mathbf{360° - \boldsymbol{\theta}}$

General Solutions

- A general solution refers to all angles that satisfy the trigonometric equation

Sine function

If $sin\theta = k$, then the general solution is given by:

$$n\pi + (-1)^n\alpha \text{ or } 180°n + (-1)^n\alpha$$

Cosine function

If $cos\theta = k$, then the general solution is given by:

$$2n\pi \pm \alpha \text{ or } 360°n \pm \alpha$$

Tangent function

If $tan\theta = k$, then the general solution is given by:

$$n\pi + \alpha \text{ or } 180°n + \alpha$$

NB: α is the principal angle/value and n is period.

Basic Trigonometrical Equations

Steps

- Set the equation equal to zero
- Factor if possible
- Convert to one trigonometric function if possible
- If the angle has a coefficient greater than one (multiple variables e.g. $2x$, $3x$ etc), then go around the circle that many times to find all possible angles.
- Divide the resulting angles by the coefficient e.g. for $2x$ divide by 2.

NOTE

- The other solutions are found by adding or subtracting multiples of the period i.e. for $sinx$ and $cosx$ the period is $2n\pi$ or $360°n$ and for $tanx$ the period is $n\pi$ or $180°n$ BUT be guided by the given range/interval.
- Do not divide the equation by a trig function e.g. for $sinxcos2x = sinx$ never divide by $sinx$ BUT put all functions to one side and factor out $sinx$.
- If you square both sides, always remember to check the solutions at the end e.g. for $sinx = 2 - cosx$, we get $(sinx)^2 = (2 - cosx)^2$ BUT after application of necessary identities and solving this equation, remember to check if all the solutions satisfy the original equation $sinx = 2 - cosx$. Drop/ignore all solutions which do not satisfy this equation.
- If the range is given in radians e.g. $0 \leq x \leq 2\pi$ make sure to give your solutions in radians and if it is given in degrees e.g. $0° \leq x \leq 360°$ always express your answers in degrees.

NB: To enter sin^2x in a calculate we enter $(sinx)^2$

Solved Problems

Question 1

Solve the equation $4sinx = 2$ for $0° \leq x \leq 360°$

Suggested Solution

$4sinx = 2$

$sinx = \frac{2}{4}$

$x = sin^{-1}\left(\frac{1}{2}\right)$

$x = 30°\,;180° - 30°$

$\therefore x = 30°; 150°$

OR

Using the general solution:

$4sinx = 2$

$sinx = \frac{2}{4}$

$x = sin^{-1}\left(\frac{1}{2}\right)$

$x = 30°$

$x = 180°(0) + (-1)^0 30°; \quad 180°(1) + (-1)^1 30°$

$x = 30°; \quad 180° - 30°$

$\therefore x = 30°; 150°$

Question 2

Solve the equation $3sin^2x = cos^2x$ for $0° \leq x \leq 360°$

Suggested Solution

Method 1

$3sin^2x = cos^2x$

$sin^2x = \frac{1}{3}cos^2x$

$\frac{sin^2x}{cos^2x} = \frac{\sqrt{3}}{3}$

$tan^2x = \frac{\sqrt{3}}{3}$

$tanx = \pm\frac{\sqrt{3}}{3}$

$x = tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ or $tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

$x = 30°;\ \ 180° + 30°$ or $- 30°;\ 180° + (-30°)$

$x = 30°;\ \ 210°$ or $x = -30°;\ 150°; 360 + (-30°)$

$\therefore x = 30°;\ \ \ 150°;\ \ \ 210°;\ \ \ 330°$

<u>Method 2</u>

$3sin^2x = cos^2x$

$3(1 - cos^2x) = cos^2x$

$3 - 3cos^2x = cos^2x$

$cos^2x + 3cos^2x = 3$

$4cos^2x = 3$

$cos^2x = \frac{3}{4}$

$cos = \pm\frac{\sqrt{3}}{2}$

$x = cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ or $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$x = \pm 30°$ or $x = \pm 150° + (-30°)$

$\Rightarrow x = 30°;\ \ 360° - 30°$ or $x = 150°;\ 360° - 150°$

$x = 30°;\ 210°;\ \ 150°;\ 330°$

Method 3

$3sin^2x = cos^2x$

$3sin^2x = 1 - sin^2x$

$3sin^2x + sin^2x = 1$

$sin^2x = \frac{1}{4}$

$sin^2x = \frac{1}{4}$

$sin = \pm\frac{1}{2}$

$x = sin^{-1}\left(\frac{1}{2}\right)$ or $sin^{-1}\left(-\frac{1}{2}\right)$

$x = 30°$ or $x = -30°$

$\Rightarrow x = 30°;\ 180° - 30°$ or $x = -30°;\ 180° - (-30);\ 360° + (-30°)$

$x = 30°;\quad 150°;\ 210°;\ 330°$

Question 3

Solve the equation $3 - 3sinx = 2cos^2x$ for $0° < x \leq 180°$

Suggested Solution

$3 - 3sinx = 2cos^2x$

$3 - 3sinx = 2(1 - sin^2x)$

$3 - 3sinx = 2 - 2sin^2x$

$2sin^2x - 3sinx + 1 = 0$

$2sin^2x - 2sinx - sinx + 1 = 0$

$2sinx(sinx-1)-1(sinx-1)=0$

$(2sinx-1)(sinx-1)=0$

$2sinx-1=0$ or $sinx-1=0$

Now:

$2sinx-1=0$

$sinx=\frac{1}{2}$

$x=sin^{-1}\left(\frac{1}{2}\right)$

$x=30°$

Also:

$sinx-1=0$

$sinx=1$

$x=sin^{-1}(1)$

$x=90°$

$\therefore x=30°;\ 90°$

<u>Question 4</u>

Solve the equation $tan3x=1$ for $0° \leq x \leq 360°$

<u>Suggested Solution</u>

$tan3x=1$

$3x=tan^{-1}(1)$

$3x=45°;180°+\ 45°$

$3x = 45°; 225°; 360° + 45°; 360° + 225°; 720° + 45°; 720° + 225°$

NB: After calculating the principal angles, we then add multiples of $360°$ (due to periodicity) in order to satisfy the required restriction i.e. $0° \leq x \leq 360°$

$3x = 45°; 225°; 405°; 585°; 765°; 945°$

NB: We only divide after finding all possible angles.

$x = \frac{45°}{3}; \frac{225°}{3}; \frac{405°}{3}; \frac{585°}{3}; \frac{765°}{3}; \frac{945°}{3}$

$\therefore x = 15°;\ 75°;\ 135°;\ 195°;\ 225°;\ 315°$

Question 5

Solve the equation $3 - 3cos\theta = 2sin^2\theta$ for $-180° \leq \theta \leq 180°$.

$3 - 3cos\theta = 2sin^2\theta$

$3 - 3cos\theta = 2(1 - cos^2\theta)$

$2cos^2\theta - 3cos\theta + 3 - 2 = 0$

$2cos^2\theta - 3cos\theta + 1 = 0$

$2cos^2\theta - 2cos\theta - cos\theta + 1 = 0$

$2cos\theta(cos\theta - 1) - 1(cos\theta - 1) = 0$

$(2cos\theta - 1)(cos\theta - 1) = 0$

$2cos\theta - 1 = 0$ or $cos\theta - 1 = 0$

Aside:

$2cos\theta - 1 = 0$

$cos\theta = \frac{1}{2}$

$\theta = \pm cos^{-1}\left(\frac{1}{2}\right)$

NB: $cos\theta \equiv \cos(-\theta)$

$\theta = \pm 60°$

Also:

$cos\theta - 1 = 0$

$cos\theta = 1$

$\theta = \pm cos^{-1}(1)$
$\theta = 0°$

$\therefore \theta = -60°, 0°$ or $60°$

Question 6

Cambridge Question

a) Solve the equation $2cos^2\theta = 3sin\theta$ for $0° \leq \theta \leq 360°$.

b) The smallest positive solution of the equation $2cos^2(n\theta) = 3sin(n\theta)$, where n is a positive integer is $10°$. State the value of n and hence find the largest solution of the equation in the interval $0° \leq \theta \leq 360°$.

Suggested Solution

a) $2cos^2\theta = 3sin\theta$

$2(1 - sin^2\theta) = 3sin\theta$

$2 - 2sin^2\theta = 3sin\theta$

$2sin^2\theta + 3sin\theta - 2 = 0$

$2sin^2\theta + 4sin\theta - sin\theta - 2 = 0$

$2sin\theta(sin\theta + 2) - 1(sin\theta + 2) = 0$

$(sin\theta + 2)(2sin\theta - 1) = 0$

$sin\theta + 2 = 0$ or $2sin\theta - 1 = 0$

$sin\theta = -2$ or $sin\theta = \frac{1}{2}$

$sin\theta = -2$ has no solutions or $\theta = sin^{-1}\left(\frac{1}{2}\right)$

$\theta = 30°; 180° - 30°$

$\theta = 30°; 150°$

b) $n\theta = 30° \Rightarrow \theta = \frac{30°}{n} \Rightarrow \frac{30°}{n} = 10°$

$$\frac{30°}{n} = 10°$$

$$\frac{30°}{10°} = n$$

$$\Rightarrow n = 3$$

$$\therefore \theta = \frac{150° + 720°}{3} = 290°$$

Question 7

Given that $7tan\theta + cot\theta = 5sec\theta$, derive a quadratic equation for $sin\theta$.
Hence, or otherwise, find all values of θ in the interval $0° \leq \theta \leq 180°$ which satisfy the given equation.

Suggested Solution

$7tan\theta + cot\theta = 5sec\theta$

$$\Rightarrow \frac{7sin\theta}{cos\theta} + \frac{cos\theta}{sin\theta} = \frac{5}{cos\theta}$$

$$\Rightarrow (sin\theta cos\theta)\frac{7sin\theta}{cos\theta} + (sin\theta cos\theta)\frac{cos\theta}{sin\theta} = (sin\theta cos\theta)\frac{5}{cos\theta}$$

$$\Rightarrow 7sin^2\theta + cos^2\theta = 5sin\theta$$

$$\Rightarrow 7sin^2\theta + (1 - sin^2\theta) = 5sin\theta$$

$$\Rightarrow 7sin^2\theta + 1 - sin^2\theta - 5sin\theta = 0$$

$$\Rightarrow 6sin^2\theta - 5sin\theta + 1 = 0 \quad (As\ required)$$

Now

$7ta\theta + cot\theta = 5sec\theta$

$\Rightarrow sin^2\theta - 5sin\theta + 1 = 0$

$6sin^2\theta - 3sin\theta - 2sin\theta + 1 = 0$

$\Rightarrow 3sin\theta(2sin\theta - 1)\ - 1(2sin\theta - 1) = 0$

$\Rightarrow (3sin\theta - 1)(2sin\theta - 1) = 0$

$\Rightarrow 3sin\theta - 1 = 0 \qquad or \qquad 2sin\theta - 1 = 0$

$\Rightarrow sin\theta = \frac{1}{3} \qquad or \qquad sin\theta = \frac{1}{2}$

$\Rightarrow \theta = sin^{-1}\left(\frac{1}{3}\right) \qquad or \qquad \theta = sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \theta = 19.471221°; 160.528779° \qquad or \qquad \theta = 30°; 150°$

$\therefore \theta = 19°; 30°; 150°; 161°$

Proving Trigonometric Identities

Definition: Trigonometric identities are equations involving the trigonometric functions that are true for every value of the variables involved.

Steps

- Determine the more complex side
- Work each side separately. Do NOT move across the equal sign
- Break down the sines and cosines
- When there are two terms on one side sometimes it is necessary to find the common denominator
- Sometimes you need to factor i.e. sum and difference of cubes, difference of two squares
- Use substitutions like $sin2A$ or $cos2A$
- $sin^2A + cos^2A = 1$ is often used
- Sometimes you need to multiply by a conjugate

The Reciprocal identities

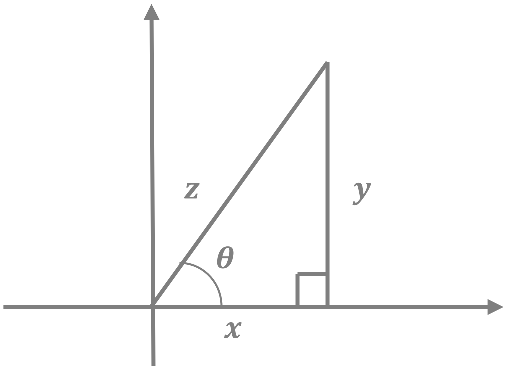
They are of the form $\frac{1}{Trigonometric\ function}$

$Cosec\theta = \frac{1}{Sin\theta}$ (1) $\quad Sec\theta = \frac{1}{Cos\theta}$ (2) $\quad Cot\theta = \frac{1}{Tan\theta}$ (3)

$Cos\theta = \frac{1}{Sec}$ (4) $\quad Sin\theta = \frac{1}{Cosec\theta}$ (5) $\quad Tan\theta = \frac{1}{Cot\theta}$ (6)

NB: For (1), (2), (4) and (5) there is a 'C' 'S' relationship

The Quotient identities



From the diagram above:

(i) $Sin\theta = \frac{y}{z}$ (ii) $Cos\theta = \frac{x}{z}$ (iii) $Tan\theta = \frac{y}{x}$

Now:

$$\frac{Sin\theta}{Cos\theta} = \frac{y}{z} \div \frac{x}{z} = \frac{y}{x} \equiv Tan\theta$$

$$\therefore Tan\theta = \frac{Sin\theta}{Cos\theta} \qquad (7)$$

Using identity (3):

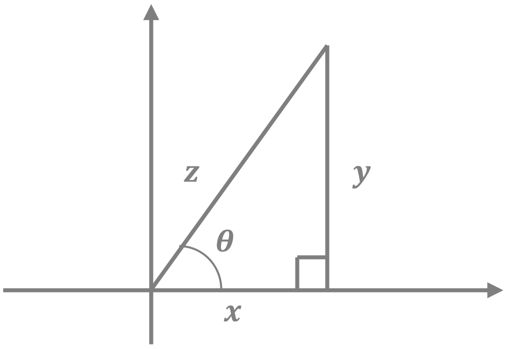
$$Cot\theta = \frac{1}{Tan\theta}$$

$$= \frac{1}{\left(\frac{Sin\theta}{Cos\theta}\right)}$$

$$= 1 \times \frac{Cos\theta}{Sin\theta}$$

$$\therefore Cot\theta = \frac{Cos\theta}{Sin\theta} \qquad (8)$$

The Pythagorean identities



From the diagram above:

(i) $Sin^2\theta = \frac{y^2}{z^2}$ (ii) $Cos^2\theta = \frac{x^2}{z^2}$

$\Rightarrow Sin^2\theta + Cos^2\theta = \frac{y^2 + x^2}{z^2}$ (iii)

But from the triangle above $z^2 = y^2 + x^2$ $(Pythagoras'theorem)$ (iv)

Substituting (iv) into (iii):

$\Rightarrow Sin^2\theta + Cos^2\theta = \frac{z^2}{z^2} = 1$

$\therefore Sin^2\theta + Cos^2\theta = 1$ (9)

NB: (i) $Cos^2\theta = 1 - Sin^2\theta$ (10)

(ii) $Sin^2\theta = 1 - Cos^2\theta$ (11)

Using identity (9):

$Sin^2\theta + Cos^2\theta = 1$

$\frac{Sin^2\theta}{Sin^2\theta} + \frac{Cos^2\theta}{Sin^2\theta} = \frac{1}{Sin^2\theta}$

$1 + Cot^2\theta = Cosec^2\theta$ (12)

Also:

$$Sin^2\theta + Cos^2\theta = 1$$

$$\frac{Sin^2\theta}{Cos^2\theta} + \frac{Cos^2\theta}{Cos^2\theta} = \frac{1}{Cos^2\theta}$$

$$Tan^2\theta + 1 = Sec^2\theta \qquad (13)$$

NB: We manipulate the identities given above to obtain other similar identities

Worked Problems

Question 1

Prove the identity

$$\frac{1 - 2Cos^2A}{SinACosA} \equiv TanA - CotA$$

Suggested Solution

$$\frac{1 - 2Cos^2A}{SinACosA} \equiv TanA - CotA$$

LHS:

$$\frac{1 - 2Cos^2A}{SinACosA} = -\frac{(2Cos^2A - 1)}{SinACosA}$$

$$= -\frac{Cos^2A - Sin^2A}{SinACosA}$$

$$= -\left[\frac{Cos^2A}{SinACosA} - \frac{Sin^2A}{SinACosA}\right]$$

$$= \frac{SinA}{CosA} - \frac{CosA}{SinA}$$

$$= TanA - CotA \ (as\ required)$$

Question 2

Prove that $\tan\theta + \cot\theta \equiv \sec\theta \, \mathrm{cosec}\theta$.

Suggested Solution

$\tan\theta + \cot\theta \equiv \sec\theta \, \mathrm{cosec}\theta$

LHS:

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{\cos\theta\sin\theta}$$

$$= \frac{1}{\cos\theta} \times \frac{1}{\sin\theta}$$

$$= \sec\theta \, \mathrm{cosec}\theta$$

Compound angle/Addition formulae

Definition: A compound angle is an algebraic sum of two or more angles

Cosine

(i) $Cos(\alpha + \beta) = Cos\alpha Cos\beta - Sin\alpha Sin\beta \quad (1)$

(ii) $Cos(\alpha - \beta) = Cos\alpha Cos\beta + Sin\alpha Sin\beta \quad (2)$

Sine

(iii) $Sin(\alpha + \beta) = Sin\alpha Cos\beta + Cos\alpha Sin\beta \quad (3)$

(iv) $Sin(\alpha - \beta) = Sin\alpha Cos\beta - Cos\alpha Sin\beta \quad (4)$

Tangent

$$(\text{v})Tan(\alpha+\beta)=\frac{Sin(\alpha+\beta)}{Cos(\alpha+\beta)}$$

$$=\frac{Sin\alpha Cos\beta+Cos\alpha Sin\beta}{Cos\alpha Cos\beta-Sin\alpha Sin\beta}$$

$$=\frac{\left(\frac{Sin\alpha Cos\beta}{Cos\alpha Cos\beta}\right)+\left(\frac{Cos\alpha Sin\beta}{Cos\alpha Cos\beta}\right)}{\left(\frac{Cos\alpha Cos\beta}{Cos\alpha Cos\beta}\right)-\left(\frac{Sin\alpha Sin\beta}{Cos\alpha Cos\beta}\right)}$$

$$\therefore\ Tan(\alpha+\beta)\ =\frac{Tan\alpha+Tan\beta}{1-Tan\alpha Tan\beta}\qquad(5)$$

$$(\text{vi})Tan(\alpha-\beta)=\frac{Sin(\alpha-\beta)}{Cos(\alpha-\beta)}$$

$$=\frac{Sin\alpha Cos\beta-Cos\alpha Sin\beta}{Cos\alpha Cos\beta+Sin\alpha Sin\beta}$$

$$=\frac{\left(\frac{Sin\alpha Cos\beta}{Cos\alpha Cos\beta}\right)-\left(\frac{Cos\alpha Sin\beta}{Cos\alpha Cos\beta}\right)}{\left(\frac{Cos\alpha Cos\beta}{Cos\alpha Cos\beta}\right)+\left(\frac{Sin\alpha Sin\beta}{Cos\alpha Cos\beta}\right)}$$

$$\therefore\ Tan(\alpha-\beta)\ =\frac{Tan\alpha-Tan\beta}{1+Tan\alpha Tan\beta}\qquad(6)$$

Double/ Multiple angle formulae

Definition: A multiple angle is an algebraic sum of two or more identical angles

Cosine

$Cos(\alpha+\alpha)=Cos\alpha Cos\alpha-Sin\alpha Sin\alpha$

$$\therefore Cos2\alpha \ = Cos^2\alpha - Sin^2\alpha \qquad (1)$$

NB: $Cos^2\alpha + Sin^2\alpha = 1$

$$\Rightarrow Cos2\alpha \ = Cos^2\alpha - Sin^2\alpha$$

$$\Rightarrow Cos2\alpha \ = Cos^2\alpha - (1 - Cos^2\alpha)$$

$$\therefore Cos2\alpha \ = 2Cos^2\alpha - 1 \qquad \text{(i)}$$

OR

$$\Rightarrow Cos2\alpha \ = 1 - 2Sin^2\alpha$$

$$\Rightarrow Cos2\alpha \ = 1 - Sin^2\alpha \ - Sin^2\alpha$$

$$\therefore Cos2\alpha \ = 1 - 2Sin^2\alpha \qquad \text{(ii)}$$

Sine

$$Sin(\alpha + \alpha) = Sin\alpha Cos\alpha + Cos\alpha Sin\alpha$$

$$\therefore Sin2\alpha \ = 2Sin\alpha Cos\alpha \qquad (2)$$

Tangent

$$Tan(\alpha + \alpha) \ = \frac{Tan\alpha + Tan\alpha}{1 - Tan\alpha Tan\alpha}$$

$$\therefore Tan2\alpha \ = \frac{2Tan\alpha}{1 - Tan^2\alpha} \qquad (3)$$

Solved Problems

Question 1

Find, without using tables/ calculators, exact value for:

a) $Sin\ 75°$
b) $Cos\ 105°$
c) $Tan\ 15°$

Suggested Solution

a) $Sin\ 75°$

$$\Rightarrow Sin\ 75° = Sin\ (45° + 30°)$$
$$= Sin\ 45° Cos30° + Cos45° Sin30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

b) $Cos\ 105°$

$$\Rightarrow Cos105° = Cos\ (60° + 45°)$$
$$= Cos\ 60° Cos45° - Sin60° Sin45°$$
$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

c) $Tan\ 15°$

$$\Rightarrow Tan15° = Tan\ (60° - 45°)$$
$$= \frac{Tan60° - Tan45°}{1 + Tan60° Tan45°}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}(1)}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$= \frac{(\sqrt{3}-1)(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})}$$

$$= \frac{-(1-\sqrt{3})(1-\sqrt{3})}{(1)^2-(\sqrt{3})^2}$$

$$= \frac{-(1-\sqrt{3})^2}{1-3}$$

$$= \frac{-(1-2\sqrt{3}+3)}{-2}$$

$$= \frac{4-2\sqrt{3}}{2}$$

$$= 2-\sqrt{3}$$

Question 2

Prove the following identity:

$$Tan\theta + Cot\theta \equiv 2Cosec2\theta$$

Suggested Solution

$$Tan\theta + Cot\theta = \frac{sin\theta}{cos\theta} + \frac{cos\theta}{sin\theta}$$

$$= \frac{sin^2\theta + cos^2\theta}{sin\theta cos\theta}$$

$$= \frac{1}{sin\theta cos\theta}$$

$$= \frac{2}{2sin\theta cos\theta}$$

$$= \frac{2}{sin2\theta}$$

$$\equiv 2Cosec2\theta \quad (as\ required)$$

Question 3

Prove the identity

$$\frac{Cosx}{1 + Sinx} + \frac{1 + Sinx}{Cosx} \equiv 2Secx$$

Suggested Solution

$$\frac{Cosx}{1 + Sinx} + \frac{1 + Sinx}{Cosx} = \frac{Cos^2x + (1 + Sinx)^2}{Cosx(1 + Sinx)}$$

$$= \frac{Cos^2x + 1 + 2Sinx + Sin^2x}{Cosx(1 + Sinx)}$$

$$= \frac{Cos^2x + Sin^2x + 1 + 2Sinx}{Cosx(1 + Sinx)}$$

$$= \frac{1 + 1 + 2Sinx}{Cosx(1 + Sinx)}$$

$$= \frac{2 + 2Sinx}{Cosx(1 + Sinx)}$$

$$= \frac{2(1 + Sinx)}{Cosx(1 + Sinx)}$$

$$= \frac{2}{Cosx}$$

$$\equiv 2Secx \quad (as\ required)$$

Question 4

Cambridge Question

(i) Prove that $sin^2 2\theta(cosec^2\theta - sec^2\theta) \equiv 4\,cos\,2\theta$.

(ii) Hence

a) solve for $0° \leq \theta \leq 180°$, the equation $sin^2 2\theta(cosec^2\theta - sec^2\theta) = 3$,

b) find the exact value of $cosec^2 15° - sec^2 15°$.

Suggested Solution

(i) $sin^2 2\theta(cosec^2\theta - sec^2\theta) \equiv 4\,cos\,2\theta$

LHS

$$sin^2 2\theta(cosec^2\theta - sec^2\theta) = (2sin\theta cos\theta)^2\left[\frac{1}{sin^2\theta} - \frac{1}{cos^2\theta}\right]$$

$$= 4sin^2\theta cos^2\theta\left[\frac{cos^2\theta - sin^2\theta}{sin^2\theta cos^2\theta}\right]$$

$$= 4(cos^2\theta - sin^2\theta)$$

$$= 4\,cos\,2\theta \equiv RHS$$

(ii) $sin^2 2\theta(cosec^2\theta - sec^2\theta) = 3$

$\Rightarrow 4\,cos\,2\theta = 3$

$cos\,2\theta = \frac{3}{4}$

$2\theta = cos^{-1}\left(\frac{3}{4}\right)$

$2\theta = 41.4096221093°; \quad 318.5903778907°$

$\theta = 20.7048110547°; \quad 159.2951889454°$

$\therefore \theta = 21°; \quad 159°$.

(iii) $sin^2 2\theta(cosec^2\theta - sec^2\theta) \equiv 4\,cos\,2\theta$

$$cosec^2\theta - sec^2\theta \equiv \frac{4\,cos\,2\theta}{sin^2 2\theta}$$

$$\Rightarrow cosec^2 15° - sec^2 15° \equiv \frac{4\,cos(2\times 15°)}{[sin(2\times 15°))^2]}$$

$$= \frac{4\,cos\,30°}{(sin30°)^2}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2}$$

$$= 8\sqrt{3}$$

Question 5

Solve the equation $cos2\theta = cos\theta$ for $0 \leq \theta < 2\pi$.

Suggested Solution

$cos2\theta = cos\theta$

$\Rightarrow 2cos^2\theta - 1 = cos\theta$

$2cos^2\theta - cos\theta - 1 = 0$

$2cos^2\theta - 2cos\theta + cos\theta - 1 = 0$

$2cos\theta(cos\theta - 1) + 1(cos\theta - 1) = 0$

$(cos\theta - 1)(2cos\theta + 1) = 0$

$(cos\theta - 1) = 0$ or $(2cos\theta + 1) = 0$

Aside:

$cos\theta - 1 = 0$

$cos\theta = 1$

$\theta = \cos^{-1}(1)$

$\theta = 0; 2\pi$

Also:

$2cos\theta + 1 = 0$

$cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\theta = \frac{2\pi}{3}; \frac{4\pi}{3}$

$\therefore \theta = 0; \frac{2\pi}{3}; \frac{4\pi}{3}$

Question 6

Cambridge Question

Showing all necessary working, solve the equations

(a) $sec\ \alpha\ cosec\ \alpha = 7$ for $0 < \alpha < 90°$.

(b) $\sin(\beta + 20°) + sin(\beta - 20°) = 6\cos\beta$ for $0° < \beta < 90°$.

Suggested Solution

(a) $\sec\alpha\ cosec\ \alpha = 7$

$\frac{1}{cos\ \alpha} \times \frac{1}{sin\alpha} = 7$

$\frac{1}{sin\alpha\ cos\ \alpha} = 7$

But $2sin\alpha\,cos\,\alpha \;=\; sin2\alpha$

Now multiplying both the numerator and the denominator by 2:

$$\frac{2}{2sin\alpha\,cos\,\alpha} \;=\; 7$$

$$\Rightarrow \frac{2}{sin2\alpha} \;=\; 7$$

$$\Rightarrow \frac{2}{7} \;=\; sin2\alpha$$

$$\Rightarrow \frac{2}{7} \;=\; sin2\alpha$$

$$\Rightarrow 2\alpha = sin^{-1}\left(\frac{2}{7}\right)$$

$$\Rightarrow 2\alpha = 16.6015486°; \quad 180° - 16.6015486°$$

$$\Rightarrow 2\alpha = 16.6015486°; \quad 163.3984504°$$

$$\Rightarrow \alpha = 8.3007748°; \quad 81.6992252°$$

$$\therefore \alpha = 8°; \quad 82°\ (to\ the\ nearest\ degree)$$

(b) $sin(\beta + 20°) + \;sin(\beta - 20°) = \;6\,cos\,\beta$

Remember that $sin(A \pm B) = sinAcosB \pm cosAsinB$

Now:

$$(sin\,\beta cos20° + \;cos\,\beta sin20°) + (sin\,\beta cos20° - \;cos\,\beta sin20°) = \;6\cos\beta$$

$$\Rightarrow 2\,sin\,\beta cos20° \;=\; 6\cos\beta$$

$$\Rightarrow \frac{sin\,\beta}{cos\,\beta} = \frac{6}{2cos20°}$$

$$\Rightarrow \tan\beta = \frac{6}{2cos20°}$$

$$\Rightarrow \;\beta \;=\; tan^{-1}\left(\frac{6}{2cos20°}\right)$$

$$\Rightarrow \;\beta \;=\; 72.60783295°$$

$$\therefore \;\beta \;=\; 73°$$

Question 7

Show that $\frac{sin2\theta}{1+cos2\theta} \equiv tan\theta$. Hence find the exact value of $cot\,\frac{\pi}{8}$.

Suggested Solution

$$\frac{sin2\theta}{1 + cos2\theta} \equiv tan\theta$$

LHS:

$$\frac{sin2\theta}{1 + cos2\theta} = \frac{2sin\theta cos\theta}{1 + (2\cos^2\theta - 1)}$$

$$= \frac{2sin\theta cos\theta}{2\cos^2\theta}$$

$$= \frac{sin\theta}{\cos\theta}$$

$$= tan\theta \equiv RHS \quad (Proven)$$

$$cot\theta = \frac{1}{tan\theta}$$

$$= \frac{1}{\left(\frac{sin2\theta}{1 + cos2\theta}\right)}$$

$$= \frac{1 + cos2\theta}{sin2\theta}$$

$$\Rightarrow cot\frac{\pi}{8} = \frac{1 + cos2\left(\frac{\pi}{8}\right)}{sin2\left(\frac{\pi}{8}\right)}$$

$$= \frac{1 + cos\left(\frac{\pi}{4}\right)}{sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$= \frac{2 + \sqrt{2}}{2} \times \frac{2}{\sqrt{2}}$$

$$= \frac{2 + \sqrt{2}}{\sqrt{2}}$$

$$= \frac{2 + \sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$= \frac{2\sqrt{2} + (\sqrt{2})^2}{2}$

$= \frac{2 + 2\sqrt{2}}{2}$

$= 1 + \sqrt{2}$

Harmonic Identities/ R-Formulae

- Given an expression $Acos\theta + Bsin\theta$, express it in the form $Rcos(\theta - \alpha)$ i.e.

$$Acos\theta + Bsin\theta \equiv Rcos(\theta - \alpha)$$

Thus

$Acos\theta + Bsin\theta \equiv Rcos(\theta - \alpha)$

$= Rcos\theta cos\alpha - Rsin\theta sin\alpha$ $(Use\ of\ Compound\ angles\ formula)$

Now:

$Rcos\alpha = A \Rightarrow cos\alpha = \frac{A}{R}$ (i)

$Rsin\alpha = B \Rightarrow sin\alpha = \frac{B}{R}$ (ii)

Method 1

Using $sin^2\alpha + cos^2\alpha = 1$

$\Rightarrow \left(\frac{A}{R}\right)^2 + \left(\frac{B}{R}\right)^2 = 1$

$\Rightarrow \frac{A^2}{R^2} + \frac{B^2}{R^2} = 1$

$\Rightarrow \frac{A^2 + B^2}{R^2} = 1$

$\Rightarrow R^2 = A^2 + B^2$

$\therefore R = \sqrt{A^2 + B^2}$ i.e. $R > 0$

Using $Tan\alpha = \frac{Sin\alpha}{Cos\alpha}$

$\Rightarrow Tan\alpha = \frac{\frac{B}{R}}{\frac{A}{R}}$

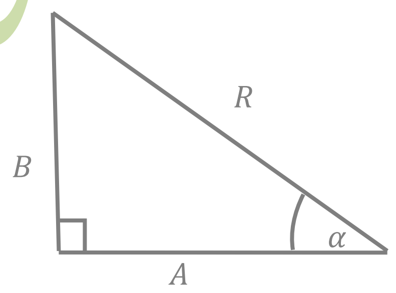
$\Rightarrow Tan\alpha = \frac{B}{A}$

$\therefore \alpha = Tan^{-1}\left(\frac{B}{A}\right)$

<u>Method 2</u>

$Rcos\alpha = A \;\Rightarrow\; cos\alpha = \frac{A}{R}$ (i)

$Rsin\alpha = B \;\Rightarrow\; sin\alpha = \frac{B}{R}$ (ii)



From the triangle:

$Tan\alpha = \frac{B}{A}$

$\therefore \alpha = Tan^{-1}\left(\frac{B}{A}\right)$

Using the Pythagoras' theorem

$Hypotenuse^2 = Adjacent^2 + Opposite^2$

$\Rightarrow R^2 = A^2 + B^2$

$\therefore R = \sqrt{A^2 + B^2}$ i.e. $R > 0$

<u>NOTE</u>

(i) Express $Acos\theta - Bsin\theta$ in the form $Rcos(\theta + \alpha)$

(ii) Express $Asin\theta - Bcos\theta$ in the form $Rsin(\theta - \alpha)$

(iii)Express $Asin\theta + Bcos\theta$ in the form $Rsin(\theta + \alpha)$

Harmonic Equations

- They are of the form $Rcos(\theta \pm \alpha) = C$ and $Rsin(\theta \pm \alpha) = C$
- Given the equation, say, $Acos\theta + Bsin\theta = C$ express the LHS in the form $Rcos(\theta - \alpha)$ and solve the equation $Rcos(\theta - \alpha) = C$.

Worked Problem

<u>Question 1</u>

<u>ZIMSEC NOVEMEBER 2003 PAPER 1</u>

Express $\sqrt{6}cos\theta - \sqrt{2}sin\theta$ in the form $Rcos(\theta + \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$ **[2]**

Hence or otherwise find the solution of the equation

$$\left(\sqrt{6}cos\theta - \sqrt{2}sin\theta\right)^2 = 4 \text{ for } 0 \leq \theta \leq 2\pi$$ **[4]**

<u>Suggested Solution</u>

$\sqrt{6}cos\theta - \sqrt{2}sin\theta \equiv Rcos(\theta + \alpha)$

$$= Rcos\theta cos\alpha - Rsin\theta sin\alpha$$

Now:

$Rcos\alpha = \sqrt{6} \Rightarrow cos\alpha = \frac{\sqrt{6}}{R}$ (i)

$Rsin\alpha = \sqrt{2} \Rightarrow sin\alpha = \frac{\sqrt{2}}{R}$ (ii)

Using $sin^2\alpha + cos^2\alpha = 1$

$\Rightarrow \left(\frac{\sqrt{6}}{R}\right)^2 + \left(\frac{\sqrt{2}}{R}\right)^2 = 1$

$\Rightarrow \frac{6}{R^2} + \frac{2}{R^2} = 1$

$\Rightarrow \frac{6+2}{R^2} = 1$

$\Rightarrow R^2 = 8$

$\therefore R = \sqrt{8}$

Using $Tan\alpha = \frac{Sin\alpha}{Cos\alpha}$

$\Rightarrow Tan\alpha = \frac{\left(\frac{\sqrt{2}}{R}\right)}{\left(\frac{\sqrt{6}}{R}\right)}$

$\Rightarrow Tan\alpha = \sqrt{\frac{2}{6}}$

$\Rightarrow \alpha = Tan^{-1}\left(\sqrt{\frac{1}{3}}\right)$

$\Rightarrow \alpha = \frac{\pi}{6}$

$\therefore \sqrt{6}cos\theta - \sqrt{2}sin\theta \equiv \sqrt{8}cos\left(\theta + \frac{\pi}{6}\right)$

Now:

$\left(\sqrt{6}cos\theta - \sqrt{2}sin\theta\right)^2 = 4$

$\Rightarrow \left[\sqrt{8}cos\left(\theta + \frac{\pi}{6}\right)\right]^2 = 4$

$\Rightarrow \sqrt{8}cos\left(\theta + \frac{\pi}{6}\right) = \pm\sqrt{4}$

$\Rightarrow \sqrt{8}cos\left(\theta + \frac{\pi}{6}\right) = \pm 2$

$\Rightarrow \sqrt{8}cos\left(\theta + \frac{\pi}{6}\right) = -2$ $\quad or \quad$ $\sqrt{8}cos\left(\theta + \frac{\pi}{6}\right) = 2$

$\Rightarrow cos\left(\theta + \frac{\pi}{6}\right) = -\frac{2}{\sqrt{8}}$ $\quad or \quad$ $cos\left(\theta + \frac{\pi}{6}\right) = \frac{2}{\sqrt{8}}$

$\Rightarrow \theta + \frac{\pi}{6} = cos^{-1}\left(-\frac{2}{\sqrt{8}}\right)$ $\quad or \quad$ $\theta + \frac{\pi}{6} = cos^{-1}\left(\frac{2}{\sqrt{8}}\right)$

$\Rightarrow \theta + \frac{\pi}{6} = \frac{3\pi}{4}$ $\quad or \quad$ $\theta + \frac{\pi}{6} = \frac{\pi}{4}$

$\Rightarrow \theta + \frac{\pi}{6} = \frac{3\pi}{4}; 2\pi - \frac{3\pi}{4}$ $\quad or \quad$ $\theta + \frac{\pi}{6} = \frac{\pi}{4}; 2\pi - \frac{\pi}{4}$

$\Rightarrow \theta + \frac{\pi}{6} = \frac{3\pi}{4}; \frac{5\pi}{4}$ $\quad or \quad$ $\theta + \frac{\pi}{6} = \frac{\pi}{4}; \frac{7\pi}{4}$

$\Rightarrow \theta = \frac{3\pi}{4} - \frac{\pi}{6}; \frac{5\pi}{4} - \frac{\pi}{6}$ $\quad or \quad$ $\theta = \frac{\pi}{4} - \frac{\pi}{6}; \frac{7\pi}{4} - \frac{\pi}{6}$

$\Rightarrow \theta = \frac{7\pi}{12}; \frac{13\pi}{12}$ $\quad or \quad$ $\theta = \frac{\pi}{12}; \frac{19\pi}{12}$

$\therefore \theta = \frac{\pi}{12}; \ \frac{7\pi}{12}; \ \frac{13\pi}{12}; \ \frac{19\pi}{12}$

MAXIMUM AND MINIMUM VALUES

Case 1: Expressions of the form $Acos\theta \pm Bsin\theta$ and $Asin\theta \pm Bcos\theta$

(i) -Express $Acos\theta - Bsin\theta$ in the form $Rcos(\theta + \alpha)$

(ii) Express $Acos\theta + Bsin\theta$ in the form $Rcos(\theta - \alpha)$

(iii)Express $Asin\theta - Bcos\theta$ in the form $Rsin(\theta - \alpha)$

(iv)Express $Asin\theta + Bcos\theta$ in the form $Rsin(\theta + \alpha)$

Method 1

The range for the above expressions is given by:

$$-R \leq y \leq R$$

∴The maximum value is R and the minimum value is $-R$

Method 2

Make use of transformations but take note of the following:

(i) The largest/greatest value gives the maximum
(ii) The smallest/least value is the minimum.

Case 2: Expressions of the form $Acos\theta \pm Bsin\theta + K$ and $Asin\theta \pm Bcos\theta + K$

(i) Express $Acos\theta - Bsin\theta + K$ in the form $Rcos(\theta + \alpha) + K$

(ii) Express $Acos\theta + Bsin\theta + K$ in the form $Rcos(\theta - \alpha) + K$

(iii)Express $Asin\theta - Bcos\theta + K$ in the form $Rsin(\theta - \alpha) + K$

(iv)Express $Asin\theta + Bcos\theta + K$ in the form $Rsin(\theta + \alpha) + K$

Method1

The range for the above expressions is given by:

$$-R + K \leq y \leq R + K$$

∴The maximum value is $R + K$ and the minimum value is $-R + K$

Method 2

Make use of transformations but take note of the following:

(i) The largest/greatest value gives the maximum
(ii) The smallest/least value is the minimum.

Case 3: Expressions of the form $\frac{A}{Bcos\theta \pm Csin\theta}$ and $\frac{A}{Bsin\theta \pm Ccos\theta}$

(i) Express $\frac{A}{Bcos\theta - Csin\theta}$ in the form $\frac{A}{Rcos(\theta+\alpha)} \equiv \frac{A}{R} sec(\theta + \alpha)$

(ii) Express $\frac{A}{Bcos\theta + Csin\theta}$ in the form $\frac{A}{Rcos(\theta-\alpha)} \equiv \frac{A}{R} sec(\theta - \alpha)$

(iii) Express $\frac{A}{Bsin\theta - Ccos\theta}$ in the form $\frac{A}{Rsin(\theta-\alpha)} \equiv \frac{A}{R} cosec(\theta - \alpha)$

(iv) Express $\frac{A}{Bsin\theta + Ccos\theta}$ in the form $\frac{A}{Rsin(\theta+\alpha)} \equiv \frac{A}{R} cosec(\theta + \alpha)$

Method 1

The range for the above expressions is given by:

$$y \leq -\frac{A}{R} \text{ or } y \geq \frac{A}{R}$$

∴The maximum value is $-\frac{A}{R}$ and the minimum value is $\frac{A}{R}$

Method 2

Make use of transformations but take note of the following:

(i) The largest/greatest value gives the minimum
(ii) The smallest/least value is the maximum

Case 4: Expressions of the form $\frac{A}{Bcos\theta \pm Csin\theta + K}$ and $\frac{A}{Bsin\theta \pm Ccos\theta + K}$

(i) Express $\frac{A}{Bcos\theta - Csin\theta + K}$ in the form $\frac{A}{Rcos(\theta+\alpha)+K}$

(ii) Express $\frac{A}{Bcos\theta + Csin\theta + K}$ in the form $\frac{A}{Rcos(\theta-\alpha)+K}$

(iii) Express $\frac{A}{Bsin\theta - Ccos\theta + K}$ in the form $\frac{A}{Rsin(\theta-\alpha)+K}$

(iv) Express $\frac{A}{Bsin\theta + Ccos\theta + K}$ in the form $\frac{A}{Rsin(\theta+\alpha)+K}$

Method 1

The range for the above expressions is given by:

$$y \leq \frac{A}{-R+K} \ or \ y \geq \frac{A}{R+K}$$

∴The maximum value is the largest value between $\frac{A}{-R+K}$ and $\frac{A}{R+K}$ and the minimum value is the smallest value between $\frac{A}{-R+K}$ and $\frac{A}{R+K}$

Method 2

Make use of transformations but take note of the following:

(i) The largest/greatest value gives the maximum
(ii) The smallest/least value is the minimum.

Solved Problems

Question 1

Express $\sqrt{3}cos\theta - sin\theta$ in the form $Rsin(\theta + \alpha)$, where R is positive and $0 < \alpha < 2\pi$.

Hence,

(i) sketch the graph of $\sqrt{3}cos\theta - sin\theta$ for $-\frac{5\pi}{3} \leq \theta \leq \frac{\pi}{3}$.

(ii) solve the equation $\sqrt{3}cos\theta - sin\theta = 1$ for $-\pi < \theta < \pi$,

(iii)determine

(a) the value of θ for which $\sqrt{3}cos\theta - sin\theta$ is maximum,

(b) the minimum and maximum values of $\frac{1}{\sqrt{3}cos\theta - sin\theta + 12}$.

Suggested Solution

$\sqrt{3}cos\theta - sin\theta \equiv Rsin(\theta + \alpha)$

$\equiv Rsin\theta cos\alpha + Rcos\theta sin\alpha$

$Rcos\alpha = -1$ (i) $\quad Rsin\alpha = \sqrt{3}$ (ii)

$$\frac{Rsin\alpha}{Rcos\alpha} = \frac{\sqrt{3}}{-1}$$

$tan\alpha = -\sqrt{3}$

$\alpha = tan^{-1}(-\sqrt{3})$

$$\alpha = -\frac{\pi}{3}$$

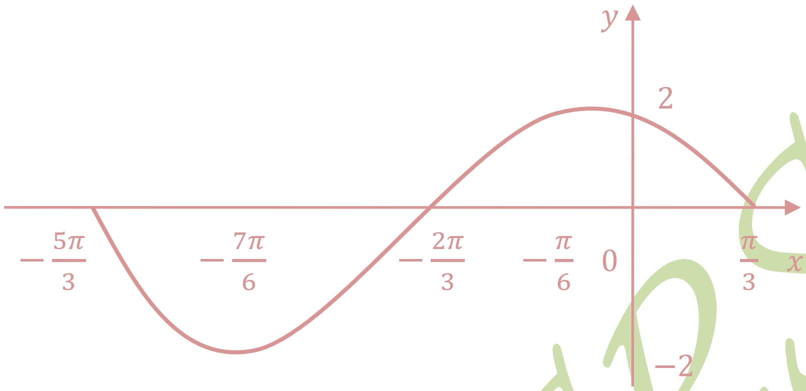
$$\therefore \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Also:

$$R = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$\therefore \sqrt{3}cos\theta - sin\theta \equiv 2sin\left(\theta + \frac{2\pi}{3}\right)$

(i) Sketching the graph of $\sqrt{3}cos\theta - sin\theta$ for $-\frac{5\pi}{3} \leq \theta \leq \frac{\pi}{3}$.



(ii) solving the equation $\sqrt{3}cos\theta - sin\theta = 1$ for $-\pi < \theta < \pi$

$2sin\left(\theta + \frac{2\pi}{3}\right) = 1$

$\Rightarrow sin\left(\theta + \frac{2\pi}{3}\right) = \frac{1}{2}$

$\Rightarrow \left(\theta + \frac{2\pi}{3}\right) = sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \theta + \frac{2\pi}{3} = \frac{\pi}{6}; \frac{5\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{6} - \frac{2\pi}{3}; \frac{5\pi}{6} - \frac{2\pi}{3}$

$\therefore \theta = -\frac{\pi}{2}; \frac{\pi}{6}$

(iii)Determining

(a) the value of θ for which $\sqrt{3}cos\theta - sin\theta$ is maximum,

The maximum value is 2

$$2 = 2sin\left(\theta + \frac{2\pi}{3}\right)$$

$$\left(\theta + \frac{2\pi}{3}\right) = sin^{-1}(1)$$

$$\theta + \frac{2\pi}{3} = \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6}$$

Alternatively, graphical method or differentiation can be used

(b) the minimum value of $\frac{1}{\sqrt{3}cos\theta - sin\theta + 12}$

$$\frac{1}{\sqrt{3}cos\theta - sin\theta + 12} = \frac{1}{2sin\left(\theta + \frac{2\pi}{3}\right) + 12}$$

Method 1

$$\frac{A}{K - R} = \frac{1}{12 - 2} = \frac{1}{10}$$

Also:

$$\frac{A}{K + R} = \frac{1}{12 + 2} = \frac{1}{14}$$

$\therefore$ The minimum value (smallest value between the two values) is $\frac{1}{14}$ and the maximum value (largest value between the two values) is $\frac{1}{10}$.

Method 2

$$\frac{1}{\sqrt{3}cos\theta - sin\theta + 12} \equiv \frac{1}{2sin\left(\theta + \frac{2\pi}{3}\right) + 12}$$

When $sin\left(\theta + \frac{2\pi}{3}\right) = -1 \Rightarrow \frac{1}{2(-1)+12} = \frac{1}{-2+12} = \frac{1}{10}$.

When $sin\left(\theta + \frac{2\pi}{3}\right) = 1 \Rightarrow \frac{1}{2(1)+12} = \frac{1}{2+12} = \frac{1}{14}$.

$\therefore$ The maximum value is $\frac{1}{10}$ and the minimum value is $\frac{1}{14}$

Method 3

$$\frac{1}{\sqrt{3}cos\theta - sin\theta + 12} = \frac{1}{2sin\left(\theta + \frac{2\pi}{3}\right) + 12}$$

Function	Maximum	Minimum
$\boldsymbol{sin\theta}$	1	-1
$\boldsymbol{2sin\left(\theta + \frac{2\pi}{3}\right)}$	2	-2
$\boldsymbol{2sin\left(\theta + \frac{2\pi}{3}\right) + 12}$	$2 + 12 = 14$	$-2 + 12 = 10$
$\boldsymbol{\frac{1}{2sin\left(\theta + \frac{2\pi}{3}\right) + 12}}$	$\frac{1}{14} = minimum$	$\frac{1}{10} = maximum$

Minimum value $= \frac{1}{14}$

Maximum value $= \frac{1}{10}$

Question 2

Express $\sqrt{3}cos\emptyset + sin\emptyset$ in the form $Rcos(\emptyset - \alpha)$, where $0 \leq \alpha \leq \frac{\pi}{2}$ and $R > 0$.

Hence,

(i) solve the equation $\left(\sqrt{3}cos\emptyset + sin\emptyset\right)^2 = 2$ for $0 \leq \emptyset \leq 2\pi$,

(ii) state the maximum and minimum values of $\frac{4}{\sqrt{3}cos\emptyset + sin\emptyset}$.

Suggested Solution

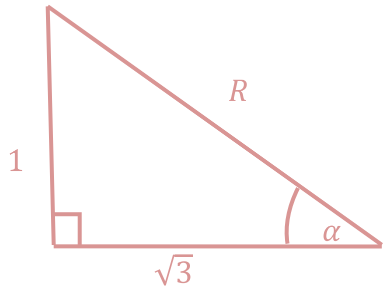
$\sqrt{3}cos\emptyset + sin\emptyset \equiv Rcos(\emptyset - \alpha)$

$= Rcos\emptyset cos\alpha + Rsin\emptyset sin\alpha$

Now:

$Rcos\alpha = \sqrt{3} \Rightarrow cos\alpha = \frac{\sqrt{3}}{R}$ (i)

$Rsin\alpha = 1 \Rightarrow sin\alpha = \frac{1}{R}$ (ii)



From the triangle:

$Tan\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = Tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore \alpha = \frac{\pi}{6}$

Using the Pythagoras' theorem

$\Rightarrow R^2 = \left(\sqrt{3}\right)^2 + 1^2$

$\Rightarrow R = \sqrt{4}$

$\therefore R = 2$

$\therefore \sqrt{3}cos\emptyset + sin\emptyset \equiv 2cos\left(\emptyset - \frac{\pi}{6}\right)$

(ii) $\left(\sqrt{3}cos\emptyset + sin\emptyset\right)^2 = 2$

$\Rightarrow \left[2cos\left(\emptyset - \frac{\pi}{6}\right)\right]^2 = 2$

$\Rightarrow 2cos\left(\emptyset - \frac{\pi}{6}\right) = \pm\sqrt{2}$

$\Rightarrow 2cos\left(\emptyset - \frac{\pi}{6}\right) = -\sqrt{2}$ or $2cos\left(\emptyset - \frac{\pi}{6}\right) = \sqrt{2}$

$$\Rightarrow cos\left(\emptyset-\frac{\pi}{6}\right)=-\frac{\sqrt{2}}{2} \qquad or \qquad cos\left(\emptyset-\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}$$

$$\Rightarrow \emptyset-\frac{\pi}{6}=cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) \qquad or \qquad \emptyset-\frac{\pi}{6}=cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \emptyset-\frac{\pi}{6}=\frac{3\pi}{4} \qquad or \qquad \emptyset-\frac{\pi}{6}=\frac{\pi}{4}$$

$$\Rightarrow \emptyset-\frac{\pi}{6}=\frac{3\pi}{4};2\pi-\frac{3\pi}{4} \qquad or \qquad \emptyset-\frac{\pi}{6}=\frac{\pi}{4};2\pi-\frac{\pi}{4}$$

$$\Rightarrow \emptyset-\frac{\pi}{6}=\frac{3\pi}{4};\frac{5\pi}{4} \qquad or \qquad \emptyset-\frac{\pi}{6}=\frac{\pi}{4};\frac{7\pi}{4}$$

$$\Rightarrow \emptyset=\frac{3\pi}{4}+\frac{\pi}{6};\frac{5\pi}{4}+\frac{\pi}{6} \qquad or \qquad \emptyset=\frac{\pi}{4}+\frac{\pi}{6};\frac{7\pi}{4}+\frac{\pi}{6}$$

$$\Rightarrow \emptyset=\frac{11\pi}{12};\frac{17\pi}{12} \qquad or \qquad \emptyset=\frac{5\pi}{12};\frac{23\pi}{12}$$

$$\therefore \emptyset=\frac{5\pi}{12};\quad \frac{11\pi}{12};\quad \frac{17\pi}{12};\quad \frac{23\pi}{12}$$

(ii) Finding maximum and minimum values of $\frac{4}{\sqrt{3}cos\emptyset+sin\emptyset}$.

Method 1

$$\frac{4}{\sqrt{3}cos\emptyset+sin\emptyset}\equiv\frac{4}{2cos\left(\emptyset-\frac{\pi}{6}\right)}\equiv\frac{2}{cos\left(\emptyset-\frac{\pi}{6}\right)}\equiv 2sec\left(\emptyset-\frac{\pi}{6}\right)$$

Minimum value occurs when $cos\left(\emptyset-\frac{\pi}{6}\right)=-1\Rightarrow\frac{4}{2(-1)}=\frac{4}{-2}=-2.$

Maximum value occurs when $cos\left(\emptyset-\frac{\pi}{6}\right)=1\Rightarrow\frac{4}{2(1)}=\frac{4}{2}=2.$

Method 2

$$\frac{4}{\sqrt{3}cos\emptyset+sin\emptyset}\equiv\frac{4}{2cos\left(\emptyset-\frac{\pi}{6}\right)}$$

Minimum value $=\frac{A}{R}=\frac{4}{2}=2$

Maximum value $= -\frac{A}{R} = -\frac{4}{2} = -2$

Method 3

$$\frac{4}{\sqrt{3}cos\emptyset + sin\emptyset} \equiv \frac{4}{2cos\left(\emptyset - \frac{\pi}{6}\right)}$$

Function	Maximum	Minimum
$cos\emptyset$	1	-1
$2cos\left(\emptyset - \frac{\pi}{6}\right)$	2	-2
$\frac{4}{2cos\left(\emptyset - \frac{\pi}{6}\right)}$	$\frac{4}{-2} = -2$	$\frac{4}{2} = 2$

Minimum value $= 2$

Maximum value $= -2$

Question 3

Solve $\sqrt{3}cosx = \sqrt{2} - sinx$ for $0 \leq x \leq 360°$

Suggested Solution

Method 1

$$\left(\sqrt{3}cosx\right)^2 = \left(\sqrt{2} - sinx\right)^2$$

$$3cos^2x = 2 - 2\sqrt{2}sinx + sin^2x$$

$$3(1 - sin^2x) = 2 - 2\sqrt{2}sinx + sin^2x$$

$$2 - 2\sqrt{2}sinx + sin^2x - 3 + 3sin^2x = 0$$

$$4sin^2x - 2\sqrt{2}sinx - 1 = 0$$

$$\frac{4sin^2x}{4} - \frac{2\sqrt{2}sinx}{4} - \frac{1}{4} = \frac{0}{4}$$

$$\sin^2 x - \frac{\sqrt{2}}{2}\sin x = \frac{1}{4}$$

$$\sin^2 x - \frac{\sqrt{2}}{2}\sin x + \left(-\frac{\sqrt{2}}{4}\right)^2 = \frac{1}{4} + \left(-\frac{\sqrt{2}}{4}\right)^2$$

$$\left(\sin x - \frac{\sqrt{2}}{4}\right)^2 = \frac{1}{4} + \frac{2}{16}$$

$$\left(\sin x - \frac{\sqrt{2}}{4}\right)^2 = \frac{1}{4} + \frac{1}{8}$$

$$\left(\sin x - \frac{\sqrt{2}}{4}\right)^2 = \frac{3}{8}$$

$$\sin x - \frac{\sqrt{2}}{4} = \pm\sqrt{\frac{3}{8}}$$

$$\sin x = \frac{\sqrt{2}}{4} \pm \sqrt{\frac{3}{8}}$$

$$\sin x = \frac{\sqrt{2}}{4} - \sqrt{\frac{3}{8}} \quad or \quad \sin x = \frac{\sqrt{2}}{4} + \sqrt{\frac{3}{8}}$$

$$\sin x = \frac{\sqrt{2}-\sqrt{6}}{4} \quad or \quad \sin x = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$x = \sin^{-1}\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) \quad or \quad x = \sin^{-1}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)$$

$$x = -15°;\ 180° - (-15°);\ \ 360° + (-15°)\ \ or\ \ x = 75°;\ 180° - 75°$$

$$x = -15°;\ 195°;\ \ 345°\ \ or\ \ x = 75°;\ 105°$$

$$x = 75°;\ \ 105°;\ 195°;\ \ 345°$$

Now we have to check if all the solutions satisfy the original equation $\sqrt{3}cosx = \sqrt{2} - sinx$.

$x = 75°$:

$\sqrt{3}cosx + sinx = \sqrt{3}cos75° + sin75° = \sqrt{2}$

$x = 105°$:

$$\sqrt{3}cosx + sinx = \sqrt{3}cos105° + sin105° = \left(\frac{-\sqrt{2}+\sqrt{6}}{4}\right)$$

$x = 195°$:

$$\sqrt{3}cosx + sinx = \sqrt{3}cos195° + sin195° = \left(\frac{-\sqrt{2}-\sqrt{6}}{4}\right)$$

$x = 345°$:

$$\sqrt{3}cosx + sinx = \sqrt{3}cos345° + sin345° = \sqrt{2}$$

$\therefore x = 75°;\quad 345°$

Method 2

$$\sqrt{3}cosx = \sqrt{2} - sinx$$

$$\sqrt{3}cosx + sinx = \sqrt{2}$$

Aside:

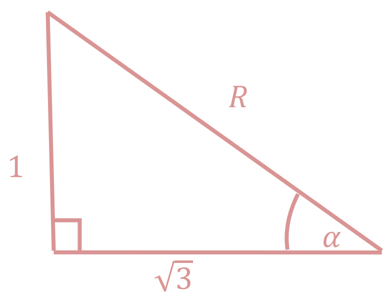
Let $\sqrt{3}cosx + sinx \ \equiv Rcos(x+\alpha)$

$$= Rcosxcos\alpha - Rsinxsin\alpha$$

Now:

$$Rcos\alpha = \sqrt{3} \ \Rightarrow \ cos\alpha = \frac{\sqrt{3}}{R} \qquad \text{(i)}$$

$$Rsin\alpha = 1 \ \Rightarrow \ sin\alpha = \frac{1}{R} \qquad \text{(ii)}$$



From the triangle:

$$Tan\alpha = \frac{\sqrt{3}}{1}$$

$$\Rightarrow \alpha = Tan^{-1}(\sqrt{3})$$

$$\therefore \alpha = 60°$$

Using the Pythagoras' theorem

$$\Rightarrow R^2 = (\sqrt{3})^2 + (1)^2$$

$$\Rightarrow R = \sqrt{4}$$

$$\therefore R = 2$$

$$\therefore \sqrt{3}cosx + sinx \equiv 2cos(\theta + 60°)$$

Now:

$$\sqrt{3}cosx + sinx = \sqrt{2}$$

$$\Rightarrow 2cos(\theta + 60°) = \sqrt{2}$$

$$\Rightarrow cos(\theta + 60°) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \theta + 60° = cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \theta + 60° = 45°$$

$$\Rightarrow \theta + 60° = 45°;\ \ 180° - 45°;\ \ 360° + 45°$$

$$\Rightarrow \theta + 60° = 45°;\ \ 135°;\ \ 405°$$

$$\Rightarrow \theta = 45° - 60°;\ \ 135° - 60°;\ \ 405° - 60°$$

$$\Rightarrow \theta = -15°; 75°;\ \ 345°$$

$$\therefore \theta = 75°;\ \ 345°$$

SOLVED PAST EXAMINATION QUESTIONS

Question 1

ZIMSEC JUNE 2006 PAPER 1

(a) It is given that $tan\alpha = \frac{1}{4}$ and $tan\beta = \frac{3}{5}$, where α and β are acute. Use this information to prove that $(\alpha + \beta) = 45°$. **[3]**

(b) Express $7cos\theta - 24sin\theta$ in the form $Rcos(\theta + \alpha)$, where $R > 0$ and α is acute, giving the value of α correct to the nearest one tenth of a degree. **[2]**

Hence find

(i) values of θ, correct to one tenth of a degree, for $0° \leq \theta \leq 360°$, which satisfy $7cos\theta - 24sin\theta = 16$, **[3]**

(ii) the greatest and least values of $\frac{1}{27 + 7cos\theta - 24sin\theta}$. **[2]**

Suggested Solution

(a) $tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta}$

$$tan(\alpha + \beta) = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4}\left(\frac{3}{5}\right)}$$

$$tan(\alpha + \beta) = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{3}{20}}$$

$$tan(\alpha + \beta) = \frac{\frac{17}{20}}{\frac{17}{20}}$$

$tan(\alpha + \beta) = 1$

$(\alpha + \beta) = tan^{-1}(1)$

$\therefore (\alpha + \beta) = 45°$

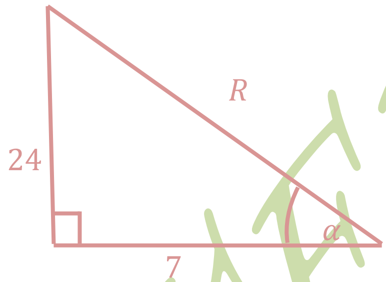
(b) $7cos\theta - 24sin\theta \equiv Rcos(\theta + \alpha)$

$$= Rcos\theta cos\alpha - Rsin\theta sin\alpha$$

Now:

$Rcos\alpha = 7 \Rightarrow cos\alpha = \frac{7}{R}$ (i)

$Rsin\alpha = 24 \Rightarrow sin\alpha = \frac{24}{R}$ (ii)



From the triangle:

$Tan\alpha = \frac{24}{7}$

$\Rightarrow \alpha = Tan^{-1}\left(\frac{24}{7}\right)$

$\alpha = 73.7°$

$\therefore \alpha = 73.7°$

Using the Pythagoras' theorem

$\Rightarrow R^2 = (24)^2 + (7)^2$

$\Rightarrow R = \sqrt{625}$

$\therefore R = 25$

$\therefore 7cos\theta - 24sin\theta \equiv 25cos(\theta + 73.7°)$

(i) $7cos\theta - 24sin\theta = 16$

$\Rightarrow 25cos(\theta + 73.7°) = 16$

$\Rightarrow cos(\theta + 73.7°) = \frac{16}{25}$

$\Rightarrow \theta + 73.7° = cos^{-1}\left(\frac{16}{25}\right)$

$\Rightarrow \theta + 73.7° = 50.2081805°$

$\Rightarrow \theta + 73.7° = 50.2081805°\,; \quad 360° - 50.2081805°\,; \quad 360° + 50.2081805°$

$\Rightarrow \theta + 73.7° = 50.2081805°\,; \quad 309.7918195°\,; \quad 410.2081805°$

$\Rightarrow \theta = 50.2081805° - 73.7°\,; \; 309.7918195° - 73.7°\,; \; 410.2081805° - 73.7°$

$\Rightarrow \theta = -25.5318195°\,; \; 236.0918195°; \; 336.5081805°$

$\therefore \theta = 236.1°; \; 336.5°$

(ii) $\frac{1}{27 + 7cos\theta - 24sin\theta} \equiv \frac{1}{27 + 25cos(\theta + 73.7°)}$

$$\frac{A}{K-R} = \frac{1}{27-25} = \frac{1}{2}$$

$$\frac{A}{K+R} = \frac{1}{27+25} = \frac{1}{52}$$

$\therefore$ The least value $= \frac{1}{52}$ and the greatest value $= \frac{1}{2}$.

Question 2

ZIMSEC NOVEMBER 2011 PAPER 1

Describe the transformation needed to transform the graph of $y = sinx$ onto the graph of $y = 6\,sin(2x) - \pi$. **[3]**

Suggested Solution

(a) Stretch along the $x-axis$ with factor $\frac{1}{2}$ or divide every x −coordinate by 2

(b) Stretch along the $y-axis$ with factor 6 or multiply every y −coordinate by 6

(c) Translation along the $y-axis$ with translation vector $\binom{0}{\pi}$ or moving π $units$ downwards

Question 3

ZIMSEC JUNE 2016 PAPER 1

(a) Prove the identity

$$tan^2\theta - sin^2\theta \equiv tan^2\theta sin^2\theta$$ **[3]**

(b) Solve the equation

$$4sin^2\theta tan\theta - tan\theta = 0,$$

for $0 \leq \theta \leq 2\pi$. **[6]**

Suggested Solution

(a) $tan^2\theta - sin^2\theta \equiv tan^2\theta sin^2\theta$

RHS:

$$tan^2\theta sin^2\theta = tan^2\theta(1 - cos^2\theta)$$

$$= tan^2\theta - tan^2\theta cos^2\theta$$

$$= tan^2\theta - \left(\frac{sin^2\theta}{cos^2\theta}\right)cos^2\theta$$

$$= tan^2\theta - sin^2\theta \equiv LHS \ \ (Shown)$$

(b) $4sin^2\theta tan\theta - tan\theta = 0$

$tan\theta(4sin^2\theta - 1) = 0$

$tan\theta = 0$ or $4sin^2\theta - 1 = 0$

$\theta = tan^{-1}(0)$ or $sin^2\theta = \frac{1}{4}$

$\theta = 0;\ \ \pi;\ \ 2\pi$ or $sin\theta = \pm\sqrt{\frac{1}{4}}$

$$sin\theta = \pm\frac{1}{2}$$

$$\theta = sin^{-1}\left(\frac{1}{2}\right) \text{ or } sin^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{6};\ \frac{5\pi}{6} \quad \text{or} \quad -\frac{\pi}{6}; \pi - \left(-\frac{\pi}{6}\right); 2\pi + \left(-\frac{\pi}{6}\right)$$

$$-\frac{\pi}{6}; \frac{7\pi}{6}; \frac{11\pi}{6}$$

$\therefore \theta = 0;\ \ \frac{\pi}{6};\ \ \frac{5\pi}{6};\ \ \pi;\ \ \frac{7\pi}{6};\ \ \frac{11\pi}{6};\ \ 2\pi$

Question 4

ZIMSEC NOVEMBER 2017 PAPER 1

(a) Prove the identity

$$cot2\theta \equiv cot\theta - cosec2\theta$$ [4]

(b) Hence, or otherwise solve the equation

$$cot\theta - cosec2\theta = \frac{\sqrt{3}}{2},$$

for $0° \leq \theta \leq 360°$. **[6]**

Suggested Solution

(a) $cot2\theta \equiv cot\theta - cosec2\theta$

RHS:

$cot\theta - cosec2\theta = \frac{cos\theta}{sin\theta} - \frac{1}{sin2\theta}$

$$= \frac{cos\theta}{sin\theta} - \frac{1}{2sin\theta cos\theta}$$

$$= \frac{2cos^2\theta - 1}{2sin\theta cos\theta}$$

NB: $cos2\theta = cos^2\theta - sin^2\theta = 2cos^2\theta - 1$

$$= \frac{cos2\theta}{2sin\theta cos\theta}$$

$$= \frac{cos2\theta}{sin2\theta}$$

$$= cot2\theta \ \equiv LHS \quad (Proven)$$

(b) $cot\theta - cosec2\theta = \frac{\sqrt{3}}{2}$

$$cot2\theta = \frac{\sqrt{3}}{2}$$

$$\frac{1}{tan2\theta} = \frac{\sqrt{3}}{2}$$

$$tan2\theta = \frac{2}{\sqrt{3}}$$

$$2\theta = tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$2\theta = 49.106605°;\ 180° + 49.106605°;\ 360° + 49.106605°\ ;\ 360° + 49.106605°$

$2\theta = 49.1066053508°;\ 229.1066053508°;\ 409.1066053508°;\ 589.1066053508°$

$\theta = 24.553302675°;\ 114.553302675°;\ 204.553302675°;\ 294.553302675°$

$\therefore \theta = 25°;\ 115°;\ 205°;\ 295°$

Question 5

ZIMSEC JUNE 2018 PAPER 1

(a) Show that

$$cos2A \equiv \frac{1 - tan^2 A}{1 + tan^2 A}. \qquad [3]$$

(b) By letting $2A = 45°$, deduce that $tan^2 22.5 = 3 - 2\sqrt{2}$. **[4]**

Suggested Solution

(a) $cos2A \equiv \frac{1 - tan^2 A}{1 + tan^2 A}$

RHS:

$$\frac{1 - tan^2 A}{1 + tan^2 A} = \frac{1 - \left(\frac{sin^2 A}{cos^2 A}\right)}{1 + \left(\frac{sin^2 A}{cos^2 A}\right)}$$

$$= \frac{\frac{cos^2 A - sin^2 A}{cos^2 A}}{\frac{cos^2 A + sin^2 A}{cos^2 A}}$$

$$= \frac{cos^2 A - sin^2 A}{cos^2 A} \times \frac{cos^2 A}{cos^2 A + sin^2 A}$$

$$= \frac{cos^2 A - sin^2 A}{cos^2 A + sin^2 A}$$

$$= \frac{cos^2 A - sin^2 A}{1}$$

$$= cos^2 A - sin^2 A$$

$$= cos2A \equiv LHS\ (Shown)$$

(b) $cos2A \equiv \frac{1 - tan^2 A}{1 + tan^2 A}$

Let $2A = 45°$

$$\Rightarrow cos45° = \frac{1 - tan^2 22.5°}{1 + tan^2 22.5°}$$

$$\Rightarrow \frac{\sqrt{2}}{2} = \frac{1 - tan^2 22.5°}{1 + tan^2 22.5°}$$

$$\Rightarrow \sqrt{2}(1 + tan^2 22.5°) = 2(1 - tan^2 22.5°)$$

$$\Rightarrow \sqrt{2} + \sqrt{2}tan^2 22.5° = 2 - 2tan^2 22.5°$$

$$\Rightarrow \sqrt{2}tan^2 22.5° + 2tan^2 22.5° = 2 - \sqrt{2}$$

$$\Rightarrow tan^2 22.5°(\sqrt{2} + 2) = 2 - \sqrt{2}$$

$$\Rightarrow tan^2 22.5° = \frac{(2 - \sqrt{2})}{(\sqrt{2} + 2)}$$

$$\Rightarrow tan^2 22.5° = \frac{(2 - \sqrt{2})(\sqrt{2} - 2)}{(\sqrt{2} + 2)(\sqrt{2} - 2)}$$

$$\Rightarrow tan^2 22.5° = \frac{2\sqrt{2} - 4 - 2 + 2\sqrt{2}}{(\sqrt{2})^2 - (2)^2}$$

$$\Rightarrow tan^2 22.5° = \frac{4\sqrt{2} - 6}{2 - 4}$$

$$\Rightarrow tan^2 22.5° = \frac{4\sqrt{2} - 6}{-2}$$

$$\Rightarrow tan^2 22.5° = 3 - 2\sqrt{2} \ (As\ required)$$

Question 6

CAMBRIDGE JUNE 2009 PAPER 3

(i) Prove that

$$cosec2\theta + cot2\theta \equiv cot\theta. \qquad [3]$$

(ii) Hence solve $cosec2\theta + cot2\theta = 2$, for $0° \leq \theta \leq 360°$. **[2]**

Suggested Solution

(i) $cosec2\theta + cot2\theta \equiv cot\theta$

LHS:

$$cosec2\theta + cot2\theta = \frac{1}{sin2\theta} + \frac{cos2\theta}{sin2\theta}$$

$$= \frac{1 + cos2\theta}{sin2\theta}$$

$$= \frac{1 + 2cos^2\theta - 1}{2sin\theta cos\theta}$$

$$= \frac{2cos^2\theta}{2sin\theta cos\theta}$$

$$= \frac{cos\theta}{sin\theta}$$

$$= cot\theta \equiv RHS\ (Proven)$$

(ii) $cosec2\theta + cot2\theta = 2$

$cot\theta = 2$

$\frac{1}{tan\theta} = 2$

$tan\theta = \frac{1}{2}$

$\theta = tan^{-1}\left(\frac{1}{2}\right)$

$\theta = 26.565051177°;\quad 180° + 26.565051177°$

$\theta = 26.565051177°;\quad 206.565051177°$

$\therefore \theta = 26.6°;\quad 206.6°$

Question 7

CAMBRIDGE JUNE 2009 PAPER 3

The angle α and β lie in the interval $0° < \theta < 180°$, and are such that

$$tan\alpha = 2\,tan\,\beta \quad \text{and} \quad tan(\alpha+\beta) = 3.$$

Find the possible values of α and β. **[6]**

Suggested Solution

$tan(\alpha+\beta) = 3$

$$\frac{tan\,\alpha + tan\,\beta}{1 - tan\,\alpha\,tan\,\beta} = 3$$

$tan\,\alpha + tan\,\beta = 3(1 - tan\,\alpha\,tan\,\beta)$

$tan\,\alpha + tan\,\beta = 3 - 3\,tan\,\alpha\,tan\,\beta$

$tan\,\alpha + 3\,tan\,\alpha\,tan\,\beta + tan\,\beta - 3 = 0$ (i)

$tan\alpha = 2\,tan\,\beta$ (ii)

Substituting equation (i) into equation (ii):

$2\,tan\,\beta + 3\,tan\,\beta\,(2\,tan\,\beta) + tan\,\beta - 3 = 0$

$2\,tan\,\beta + 6\,tan^2\,\beta + tan\,\beta - 3 = 0$

$6\,tan^2\,\beta + 3\,tan\,\beta - 3 = 0$

$2\,tan^2\,\beta + tan\,\beta - 1 = 0$

$2\,tan^2\,\beta + 2\,tan\,\beta - tan\,\beta - 1 = 0$

$2\,tan\,\beta\,(tan\,\beta + 1) - (tan\,\beta + 1) = 0$

$(2\,tan\,\beta - 1)(tan\,\beta + 1) = 0$

$2\,tan\,\beta - 1 = 0$ or $tan\,\beta + 1 = 0$

$tan\,\beta = \frac{1}{2}$ or $tan\,\beta = -1$

$\beta = tan^{-1}\left(\frac{1}{2}\right)$ or $\beta = tan^{-1}(-1)$

$\beta = 26.565051177°$ or $\beta = -45°$; $180° + (-45°)$

$\therefore \beta = 26.6°$ or $\beta = 135°$

$tan\alpha = 2\,tan\,\beta$ (ii)

$tan\alpha = 2 \times \frac{1}{2}$ or $tan\alpha = 2 \times -1$

$tan\alpha = 1$ or $tan\alpha = -2$

$\alpha = tan^{-1}(1)$ or $\alpha = tan^{-1}(-2)$

$\alpha = 45°$ or $\alpha = -63.434948823°$; $180° + (-63.434948823°)$

$\alpha = 45°$ or $\alpha = -63.434948823°$; $116.565051177°$

$\therefore \alpha = 45°$ or $\alpha = 116.6°$

$\therefore$ $\alpha = 45°$ and $\beta = 26.6°$ or $\alpha = 116.6°$ and $\beta = 135°$.

Question 8

CAMBRIDGE NOVEMBER 2017 PAPER 3

(i) Prove the identity

$$tan\,(45° + x) + tan\,(45° - x) \equiv 2sec2x.$$ **[4]**

(ii) Hence sketch the graph $y = tan\,(45° + x) + tan\,(45° - x)$ for $0° \le \theta \le 90°$. **[3]**

Suggested Solution

(i) $tan\,(45° + x) + tan\,(45° - x) \equiv 2sec2x$

LHS:

$$ta\,n(45° + x) + ta\,n(45° - x) = \frac{tan\,45° + tan\,x}{1 - tan\,45°\,tan\,x} + \frac{tan\,45° - tan\,x}{1 + tan45°\,tan\,x}$$

$$= \frac{1 + tan\,x}{1 - tan\,x} + \frac{1 - tan\,x}{1 + tan\,x}$$

$$= \frac{(1 + \tan x)^2 + (1 - \tan x)^2}{(1 - \tan x)(1 + \tan x)}$$

$$= \frac{1 + 2tanx + tan^2x + 1 - 2tanx + tan^2x}{1 - tan^2x}$$

$$= \frac{2 + 2tan^2x}{1 - tan^2x}$$

$$= \frac{2(1 + tan^2x)}{1 - tan^2x}$$

$$= \frac{2sec^2x}{1 - tan^2x}$$

$$= \frac{2}{cos^2x(1 - tan^2x)}$$

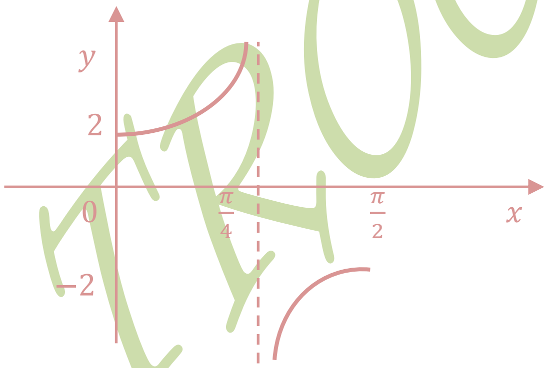
$$= \frac{2}{cos^2x - cos^2x\left(\frac{sin^2x}{cos^2x}\right)}$$

$$= \frac{2}{cos^2x - sin^2x}$$

$$= \frac{2}{cos2x}$$

$$= 2sec2x \equiv RHS\ (Proven)$$

(ii) $y = 2sec2x$



- Stretch along the $x - axis$ with factor $\frac{1}{2}$ or divide every x −coordinate by 2
- Stretch along the $y - axis$ with factor 2 or multiply every y −coordinate by 2

Question 9

CAMBRIDGE JUNE 2018 PAPER 3

(i) By first expanding $(cos^2x + sin^2x)^3$, or otherwise, show that

$$cos^6x + sin^6x = 1 - \frac{3}{4}sin^2 2x \qquad [\mathbf{4}]$$

(ii) Hence solve the equation

$$cos^6x + sin^6x = \frac{2}{3}$$

for $0° < \theta < 180°$. **[4]**

Suggested Solution

(i) $(cos^2x + sin^2x)^3$

$= (cos^2x + sin^2x)^2(cos^2x + sin^2x)$

$= (cos^4x + 2cos^2xsin^2x + sin^4x)(cos^2x + sin^2x)$

$= cos^2x(cos^4x + 2cos^2xsin^2x + sin^4x) + sin^2x(cos^4x + 2cos^2xsin^2x + sin^4x)$

$= cos^6x + 2cos^4xsin^2x + cos^2xsin^4x + sin^2xcos^4x + 2cos^2xsin^4x + sin^6x$

$= cos^6x + sin^6x + 2cos^4xsin^2x + sin^2xcos^4x + cos^2xsin^4x + 2cos^2xsin^4x$

$= cos^6x + sin^6x + 3cos^4xsin^2x + 3cos^2xsin^4x$

$= cos^6x + sin^6x + 3cos^2xcos^2xsin^2x + 3cos^2xsin^2xsin^2x$

$= cos^6x + sin^6x + 3cos^2xsin^2x(cos^2x + sin^2x)$

$= cos^6x + sin^6x + 3cos^2xsin^2x(1)$

$= cos^6x + sin^6x + 3cos^2xsin^2x$

But $cos^2x + sin^2x = 1$

$\Rightarrow (cos^2x + sin^2x)^3 = 1$

$\Rightarrow cos^6x + sin^6x + 3cos^2xsin^2x = 1$

$\Rightarrow cos^6x + sin^6x = 1 - 3cos^2xsin^2x$

REMEMBER that $sin^2 2x = (2sinxcosx)^2$

$= 4sin^2xcos^2x$

$= 4sin^2xcos^2x$

$\Rightarrow cos^6x + sin^6x = 1 - \frac{3}{4}(4cos^2xsin^2x)$

$\Rightarrow cos^6x + sin^6x = 1 - \frac{3}{4}sin^2 2x\ (As\ required)$

(ii) $cos^6x + sin^6x = \frac{2}{3}$

$\Rightarrow 1 - \frac{3}{4}sin^2 2x = \frac{2}{3}$

$\Rightarrow \frac{3}{4}sin^2 2x = 1 - \frac{2}{3}$

$\Rightarrow \frac{3}{4}sin^2 2x = \frac{1}{3}$

$\Rightarrow sin^2 2x = \frac{4}{9}$

$\Rightarrow sin2x = \pm\sqrt{\frac{4}{9}}$

$\Rightarrow sin2x = \pm\frac{2}{3}$

$\Rightarrow sin2x = -\frac{2}{3}$ or $sin2x = \frac{2}{3}$

Aside:

$\Rightarrow sin2x = -\frac{2}{3}$

$\Rightarrow 2x = sin^{-1}\left(-\frac{2}{3}\right)$

$\Rightarrow 2x = -41.8103°; 180° - (-41.8103)°; 360° + (-41.8103°)$

$\Rightarrow 2x = -41.8103°;\ \ 221.8103°;\ \ 318.1897°;$

$\Rightarrow x = \frac{-41.8103°}{2};\ \frac{221.8103°}{2};\ \frac{318.1897°}{2};$

$\Rightarrow x = -20.90515°;\ \ 110.90515°;\ \ 159.09485°$

$\Rightarrow x = -20.9°;\ \ 110.9°;\ \ 159.1°$

Also:

$\Rightarrow sin2x = \frac{2}{3}$

$\Rightarrow 2x = sin^{-1}\left(\frac{2}{3}\right)$

$\Rightarrow 2x = 41.8103°; 180° - (41.8103)°$

$\Rightarrow 2x = 41.8103°;\ \ 138.1897°$

$$\Rightarrow x = \frac{41.8103°}{2};\ \frac{138.1897°}{2}$$

$\Rightarrow x = 20.90515°;\ \ 69.09485°$

$\Rightarrow x = 20.9°;\ \ \ 69.1°$

$\therefore x = 20.9°;\ \ 69.1°;\ \ 110.9°;\ \ 159.1°$

Question 10

CAMBRIDGE NOVEMBER 2019 PAPER 3

(i) By first expanding $tan(2x + x)$ show that the equation $tan3x = 3cotx$ can be written in the form $tan^4x - 12tan^2x + 3 = 0$. **[4]**

(ii) Hence solve the equation $tan3x = 3cotx$ for $0° < x < 90°$. **[3]**

Suggested Solution

(i) $tan(2x + x) = \frac{tan\,2x + tan\,x}{1 - tan2x\,tan\,x}$

$$tan3x = \frac{tan\,2x + tan\,x}{1 - tan\,2x\,tan\,x}$$

Remember that $tan\,2x = \frac{2\,tan\,x}{1 - tan^2x}$

$$\Rightarrow tan3x = \frac{\left(\frac{2\,tan\,x}{1 - tan^2x}\right) + tan\,x}{1 - tan\,x\left(\frac{2\,tan\,x}{1 - tan^2x}\right)}$$

$$\Rightarrow tan3x = \frac{\frac{2\,tan\,x + tanx(1 - tan^2x)}{1 - tan^2x}}{\frac{1(1 - tan^2x) - 2tan^2x}{1 - tan^2x}}$$

$$\Rightarrow tan3x = \frac{3\,tan\,x - tan^3x}{1 - 3tan^2x}$$

Now

$tan3x = 3cotx$

$$\Rightarrow \frac{3\tan x - tan^3x}{1 - 3tan^2x} = \frac{3}{tanx}$$

$\Rightarrow tanx(3\tan x - tan^3x) = 3(1 - 3tan^2x)$

$\Rightarrow 3tan^2x - tan^4x = 3 - 9tan^2x$

$\Rightarrow tan^4x - 12tan^2x + 3 = 0 \quad (As\ required)$

(ii) $tan3x = 3cotx$

$\Rightarrow tan^4x - 12tan^2x + 3 = 0$

Let $y = tan^2x$

$\Rightarrow y^2 - 12y + 3 = 0$

$\Rightarrow y^2 - 12y = -3$

$\Rightarrow y^2 - 12y + (-6)^2 = -3 + (-6)^2$

$\Rightarrow (y - 6)^2 = 33$

$\Rightarrow y - 6 = \pm\sqrt{33}$

$\Rightarrow y = 6 \pm \sqrt{33}$

$\Rightarrow y = 6 + \sqrt{33}$ or $y = 6 + \sqrt{33}$

But $y = tan^2x$

$\Rightarrow tan^2x \quad = 6 + \sqrt{33}$ or $tan^2x \quad = 6 - \sqrt{33}$

$\Rightarrow tanx \quad = \pm\sqrt{(6 + \sqrt{33})}$ or $tanx \quad = \pm\sqrt{(6 - \sqrt{33})}$

Aside:

$\Rightarrow tanx \quad = \sqrt{(6 + \sqrt{33})}\ or\ tanx = -\sqrt{(6 + \sqrt{33})}$

$\Rightarrow x \quad = tan^{-1}\left[\sqrt{(6 + \sqrt{33})}\right]\ or\ x = tan^{-1}\left[-\sqrt{(6 + \sqrt{33})}\right]$

$\Rightarrow x \quad = 73.732886723° \quad or \quad x = -73.732886723°; 180° + (-73.732886723°)$

$\Rightarrow x \quad = 73.732886723° \quad or \quad x = -73.732886723°; 106.267113276°$

$\Rightarrow x \quad = 73.7° \quad or \quad x = -73.7°; 106.3°$

Also:

$\Rightarrow tanx \quad = \sqrt{(6-\sqrt{33})} \; or \; tanx = -\sqrt{(6-\sqrt{33})}$

$\Rightarrow x \quad = tan^{-1}\left[\sqrt{(6-\sqrt{33})}\right] \; or \; x = tan^{-1}\left[-\sqrt{(6-\sqrt{33})}\right]$

$\Rightarrow x \quad = 26.812403866° \quad or \quad x = -26.812403866°; 180° + (-26.812403866°)$

$\Rightarrow x \quad = 26.812403866° \quad or \quad x = -26.812403866°; 153.187596134°$

$\Rightarrow x \quad = 26.8° \quad or \quad x = -26.8°; 153.2°$

$\therefore x \quad = 26.8°; 73.7°$

Question 11

ZIMSEC JUNE 2020 PAPER 1

(a) Express $2cosx - 5sinx$ in the form $Rcos(x+\alpha)$, where $R > 0$ and $0° < x < 90°$. **[2]**

(b) Hence, or otherwise, solve the equation $2cos2x - 5sin2x = 2.5$ for $0° \leq x \leq 360°$. **[4]**

(c) Give values of x between $0°$ and $360°$ at which the maximum and the minimum values of $2cos2x - 5sin2x$ occur. **[4]**

Suggested Solution

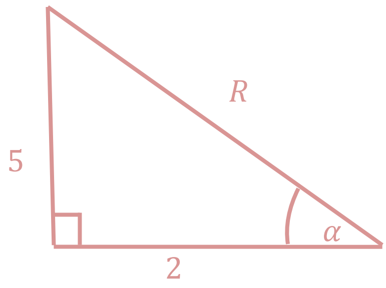
(a) $2cosx - 5sinx \equiv Rcos(x+\alpha)$

$= Rcosxcos\alpha - Rsinxsin\alpha$

Now:

$Rcos\alpha = 2 \Rightarrow cos\alpha = \frac{2}{R}$ (i)

$Rsin\alpha = 5 \Rightarrow sin\alpha = \frac{5}{R}$ (ii)



From the triangle:

$$Tan\alpha = \frac{5}{2}$$

$$\Rightarrow \alpha = Tan^{-1}\left(\frac{5}{2}\right)$$

$$\Rightarrow \alpha = 68.198590°$$

$$\therefore \alpha = 68°$$

Using the Pythagoras' theorem

$$\Rightarrow R^2 = (5)^2 + 2^2$$

$$\Rightarrow R = \sqrt{25+4}$$

$$\Rightarrow R = \sqrt{29}$$

$$\therefore 2cosx - 5sinx \equiv \sqrt{29}cos(x + 68°)$$

(b) $2cos2x - 5sin2x = 2.5$

$$\sqrt{29}cos(2x + 68°) = 2.5$$

$$cos(2x + 68°) = \frac{2.5}{\sqrt{29}}$$

$$2x + 68° = cos^{-1}\left(\frac{2.5}{\sqrt{29}}\right)$$

$$\Rightarrow 2x + 68° = 62.339060°;\ 360° - 62.339060°;\ 360° + 62.339060°;$$
$$360° + 297.66094°; 720° + 62.339060°$$

$$\Rightarrow 2x + 68° = 62.339060°;\ 297.66094°;\ 422.33906°;\ 657.66094°;\ 782.33906°$$

$$\Rightarrow 2x = 62.339060° - 68°;\ 297.66094° - 68°;\ 422.33906° - 68°;$$
$$657.66094° - 68°; 782.33906° - 68°$$

$\Rightarrow 2x = -5.66094°;\ 229.66094°;\ \ 354.33906°;\ 589.66094°;\ 714.33906°$

$\Rightarrow x = -2.83047°;\ 114.83047°;\ \ 177.16953°;\ 294.83047°;\ 357.16953°$

But $0° \leq x \leq 360°$

$\therefore x = 115°;\ \ 177°;\ 295°;\ 357°$

(c) $2cos2x - 5sin2x \equiv \sqrt{29}cos(2x + 68°)$

Maximum value $= \sqrt{29}$ and minimum value $= -\sqrt{29}$

Now at the maximum value:

$\sqrt{29}cos(2x + 68°) = \sqrt{29}$

$\Rightarrow cos(2x + 68°) = 1$

$\Rightarrow (2x + 68°) = cos^{-1}(1)$

$\Rightarrow (2x + 68°) = 0°;\ 360°;\ 720°$

$\Rightarrow 2x = 0° - 68°;\ 360° - 68°;\ 720° - 68°$

$\Rightarrow x = -34°;\ 146°;\ 326°$

$\therefore$The values of x for which the maximum value occurs in the range $0° \leq x \leq 360°$ are $146°;\ 326°$

Now at the minimum value:

$\sqrt{29}cos(2x + 68°) = -\sqrt{29}$

$\Rightarrow cos(2x + 68°) = -1$

$\Rightarrow (2x + 68°) = cos^{-1}(-1)$

$\Rightarrow (2x + 68°) = 180°;\ 360° + 180°$

$\Rightarrow 2x = 180° - 68°;\ 540° - 68°$

$\Rightarrow x = 56°;\ 236°$

$\therefore$The values of x for which the maximum value occurs in the range $0° \leq x \leq 360°$ are $56°;\ 236°$

Question 12

ZIMSEC JUNE 2020 PAPER 2

(a) Prove the identity

$$\sqrt{\frac{1-sin\theta}{1+sin\theta}} \equiv sec\theta - tan\theta \qquad [3]$$

(b) (i) Express $sinx - \sqrt{3}sinx$ in the form $Rsin(x-\alpha)$, where $R > 0$ and $0 < x < \frac{\pi}{2}$. **[2]**

(ii) Hence, or otherwise, solve the equation

$$sinx - \sqrt{3}sinx = -1 \text{ for } 0 \leq x \leq 2\pi. \qquad \textbf{[4]}$$

Suggested Solution

a) Proving

$$\sqrt{\frac{1-sin\theta}{1+sin\theta}} \equiv sec\theta - tan\theta$$

LHS:

$$\sqrt{\frac{1-sin\theta}{1+sin\theta}} = \sqrt{\frac{(1-sin\theta)(1-sin\theta)}{(1+sin\theta)(1-sin\theta)}}$$

$$= \sqrt{\frac{(1-sin\theta)^2}{1-sin^2\theta}}$$

$$= \sqrt{\frac{(1-sin\theta)^2}{cos^2\theta}}$$

$$= \frac{1-sin\theta}{cos\theta}$$

$$= \frac{1}{cos\theta} - \frac{sin\theta}{cos\theta}$$

$$= sec\theta - tan\theta \equiv RHS\ (Proven)$$

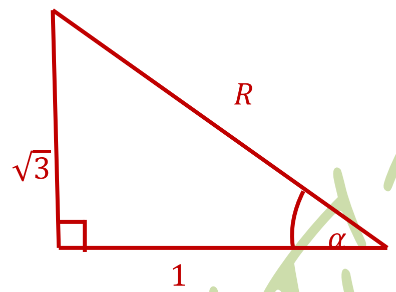
(b) (i) $sinx - \sqrt{3}cosx \equiv Rsin(x-\alpha)$

$$= Rsinxcos\alpha - Rcosxsin\alpha$$

Now:

$Rcos\alpha = 1 \Rightarrow cos\alpha = \frac{1}{R}$ (i)

$Rsin\alpha = \sqrt{3} \Rightarrow sin\alpha = \frac{\sqrt{3}}{R}$ (ii)



From the triangle:

$Tan\alpha = \frac{\sqrt{3}}{1}$

$\Rightarrow \alpha = Tan^{-1}(\sqrt{3})$

$\therefore \alpha = \frac{\pi}{3}$

Using the Pythagoras' theorem

$\Rightarrow R^2 = (\sqrt{3})^2 + 1^2$

$\Rightarrow R = \sqrt{3+1}$

$\Rightarrow R = \sqrt{4}$

$\Rightarrow R = 2$

$\therefore sinx - \sqrt{3}cosx \equiv 2sin\left(x - \frac{\pi}{3}\right)$

(b) $sinx - \sqrt{3}cosx = -1$

$2sin\left(x - \frac{\pi}{3}\right) = -1$

$$sin\left(x-\frac{\pi}{3}\right)=-\frac{1}{2}$$

$$x-\frac{\pi}{3}=sin^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow x-\frac{\pi}{3}=-\frac{\pi}{6};\ \pi-\left(-\frac{\pi}{6}\right);\ \ 2\pi+\left(-\frac{\pi}{6}\right)$$

$$\Rightarrow x-\frac{\pi}{3}=-\frac{\pi}{6};\ \frac{7\pi}{6};\ \ \frac{11\pi}{6}$$

$$\Rightarrow x=-\frac{\pi}{6}+\frac{\pi}{3};\ \frac{7\pi}{6}+\frac{\pi}{3};\ \ \frac{11\pi}{6}+\frac{\pi}{3}$$

$$\Rightarrow x=x=\frac{\pi}{6};\ \frac{9\pi}{6};\ \ \frac{13\pi}{6}$$

But $0\leq x\leq 2\pi$

$$\therefore x=\frac{\pi}{6};\ \frac{3\pi}{2}$$

TRIGONOMETIC IDENTITIES PRACTICE QUESTIONS

Prove the following identities:

Question 1

$$1 - 2sin^2\theta \equiv 2cos^2\theta - 1$$

Question 2

$$tan\theta \equiv cosec2\theta - cot2\theta$$

Question 3

$$cosec^2\theta - cos^2\theta sec^2\theta \equiv 1$$

Question 4

$$1 - \frac{sin^2\theta}{tan^2\theta} \equiv sin^2\theta$$

Question 5

$$(sin\theta + cos\theta)^2 \equiv 1 + sin2\theta$$

Question 6

$$tan^2\theta + 1 + tan\theta sec\theta \equiv \frac{1 + sin\theta}{cos^2\theta}$$

Question 7

$$\frac{sec\theta - cos\theta}{sec\theta + cos\theta} \equiv \frac{sin^2\theta}{1 + cos2\theta}$$

Question 8

$$sin\theta(cot\theta + tan\theta) \equiv sec\theta$$

Question 9

$$\sin 2\theta \cot\theta \equiv 2 - 2\sin^2\theta$$

Question 10

$$\sec\theta - \tan\theta \sin\theta \equiv \frac{1}{\sin\theta}$$

Question 11

$$\frac{\sec\theta}{\cos\theta} - \frac{\tan\theta}{\cot\theta} \equiv 1$$

Question 12

$$\tan^2\theta \sin^2\theta \equiv \tan^2\theta - \sin^2\theta$$

Question 13

$$\frac{1+\cos\theta}{\sin\theta} \equiv cosec\theta \cot\theta$$

Question 14

$$\frac{1-\cos\theta}{\sin\theta} \equiv \frac{\sin\theta}{1+\cos\theta}$$

Question 15

$$\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \equiv 2cosec^2\theta$$

Question 16

$$\cot^2\theta - \cos^2\theta \equiv \cos^2\theta \cot^2\theta$$

Question 17

$$\tan\theta + \sec\theta \equiv \frac{\cos\theta}{1-\sin\theta}$$

Question 18

$$\frac{sec\theta + 1}{tan\theta} \equiv \frac{sin\theta}{1 - cos\theta}$$

Question 19

$$\frac{sec\theta sin\theta}{tan\theta + cot\theta} \equiv sin^2\theta$$

Question 20

$$\frac{tan\theta}{sec\theta - 1} \equiv \frac{sec\theta + 1}{tan\theta}$$

Question 21

$$\frac{1 - tan^2\theta}{cot^2\theta - 1} \equiv tan^2\theta$$

Question 22

$$\frac{sin\theta cosec\theta}{cot\theta} \equiv tan\theta$$

Question 23

$$\frac{tan^2\theta}{tan^2\theta + 1} \equiv sin^2\theta$$

Question 24

$$\frac{cos\theta}{1 + sin\theta} + \frac{1 + sin\theta}{cos\theta} \equiv 2sec\theta$$

Question 25

$$\frac{cos\theta}{1 - sin\theta} - tan\theta \equiv sec\theta$$

Question 26

$$\frac{sin\theta - cos\theta}{sin\theta + cos\theta} \equiv -\frac{2cos2\theta}{1 + sin2\theta}$$

Question 27

$$\frac{sin\theta}{cosec\theta} + \frac{cos\theta}{sec\theta} \equiv 1$$

Question 28

$$\frac{1}{sin^2\theta} - \frac{1}{tan^2\theta} \equiv 1$$

Question 29

$$\frac{1 - cos^2\theta}{sin2\theta} \equiv \frac{1}{2}tan\theta$$

Question 30

$$sec2\theta \equiv \frac{1}{1 - 2sin^2\theta}$$

Question 31

$$1 - \frac{sin^2\theta}{tan^2\theta} \equiv sin^2\theta$$

Question 32

$$cos\theta(sec\theta - cos\theta) \equiv sin^2\theta$$

Question 33

$$cos2\theta \equiv \frac{1 - tan^2\theta}{1 + tan^2\theta}$$

Question 34

$$\frac{sin^2\theta - 2sin\theta + 1}{sin\theta - 1} \equiv sin\theta - 1$$

Question 35

$$\frac{tan^2\theta}{sec\theta + 1} + 1 \equiv sec\theta$$

Question 36

$$sec2\theta \equiv \frac{sec^2\theta}{2 - sec^2\theta}$$

Question 37

$$\frac{sin\theta cos\theta}{1 - cos^2\theta} \equiv cot\theta$$

Question 38

$$cos3\theta \equiv 4cos^3\theta - 3cos\theta$$

Question 39

$$\frac{sin3\theta}{sin\theta} - \frac{cos3\theta}{cos\theta} \equiv 2$$

Question 40

$$1 + sin2\theta \equiv (sin\theta + cos\theta)^2$$

Question 41

$$\frac{sin^3\theta - cos^3\theta}{2 - sin2\theta} \equiv \frac{sin\theta + cos\theta}{2}$$

Question 42

$$\frac{1 + sec^2\theta}{sec^2\theta} \equiv 1 + cos^2\theta$$

Question 43

$$\frac{sec\theta - 1}{\theta sec\theta} \equiv \frac{1 - cos\theta}{\theta}$$

Question 44

$$\frac{sin\theta - sin3\theta}{sin^2\theta - cos^2\theta} \equiv 2sin\theta$$

Question 45

$$cosec\theta + cot\theta \equiv \frac{1}{cosec\theta - cot\theta}$$

Question 46

$$\sqrt{\frac{1 - cos\theta}{1 + cos\theta}} \equiv cosec\theta - cot\theta$$

Question 47

$$\sqrt{\frac{1 + sin\theta}{1 - sin\theta}} \equiv sec\theta + tan\theta$$

Question 48

$$\frac{cos\theta cot\theta}{1 - sin\theta} - 1 \equiv cosec\theta$$

Question 49

$$\frac{1 - 2cos^2\theta}{cos\theta sin\theta} \equiv tan\theta - cot\theta$$

Question 50

$$\frac{tan\theta}{1 - cot\theta} + \frac{cot\theta}{1 - tan\theta} \equiv 1 + sec\theta cosec\theta$$

Question 51

$$\frac{(2cos^2\theta - 1)^2}{cos^4\theta - sin^4\theta} \equiv 1 - 2sin^2\theta$$

Question 52

$$\frac{sin2\theta cos\theta - cos2\theta sin\theta}{cos2\theta cos\theta + sin2\theta sin\theta} \equiv tan\theta$$

Question 53

$$\frac{1}{sec\theta - tan\theta} \equiv \frac{1 + sin\theta}{cos\theta}$$

Question 54

$$\frac{sin\theta + sin2\theta}{1 + cos\theta + cos2\theta} \equiv tan\theta$$

Question 55

$$\sqrt{\frac{1 - sin\theta}{sin\theta + 1}} \equiv sec\theta - tan\theta$$

Question 56

$$\frac{sin^2\theta - cos^2\theta}{sin\theta + cos\theta} \equiv sin\theta - cos\theta$$

Question 57

$$\frac{sin\theta - 2sin^3\theta}{2cos^3\theta + cos\theta} \equiv tan\theta$$

Question 58

$$\frac{1 + tan^2\theta}{1 + cot^2\theta} \equiv sec^2\theta - 1$$

Question 59

$$2cos\theta - \frac{2cos\theta}{1 - tan\theta} \equiv \frac{sin2\theta}{sin\theta - cos\theta}$$

Question 60

$$\frac{1 + cos\theta}{1 - cos\theta} \equiv \frac{sec\theta + 1}{sec\theta - 1}$$

TRIGONOMETRIC EQUATIONS PRACTICE QUESTIONS

Solve the following equations:

Question 1

$$\sqrt{2}tan\theta sin\theta = tan\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0; \frac{\pi}{4}; \frac{3\pi}{4}; \pi$

Question 2

$$sec^2\theta - 5sec\theta = 2 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Question 3

$$2cos^2\theta + sin\theta = 1 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{\pi}{2}; \frac{7\pi}{6}; \frac{11\pi}{6}$

Question 4

$$cos2\theta = cos\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 2\pi; \frac{\pi}{4}; 2\pi$

Question 5

$$cos2\theta = 2 - 3sin\theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{\pi}{6}; \frac{\pi}{2}; \frac{5\pi}{6}$

Question 6

$$\frac{\sqrt{3}-1}{sin\theta}+\frac{\sqrt{3}+1}{cos\theta}=4\sqrt{2} \quad \text{for} \quad 0<\theta<\frac{\pi}{2}$$

Answer: $\theta=\frac{11\pi}{36};\ \frac{\pi}{12}$

Question 7

$$2tan^2\theta-6=0 \quad \text{for} \quad 0\leq\theta\leq 2\pi$$

Answer: $\theta=\frac{\pi}{3};\ \frac{2\pi}{3};\ \frac{4\pi}{3};\ \frac{5\pi}{3}$

Question 8

$$3sin\theta=3cos\theta \quad \text{for} \quad 0°\leq\theta<360°$$

Answer: $\theta=53.1°;\ 233.1°$

Question 9

$$2sin^2\theta+3cos\theta=3 \quad \text{for} \quad 0\leq\theta\leq 2\pi$$

Answer: $\theta=0;\ \frac{\pi}{3};\ \frac{5\pi}{3};\ 2\pi$

Question 10

$$sin^2\theta-sin\theta=2 \quad \text{for} \quad 0\leq\theta\leq 2\pi$$

Answer: $\theta=\frac{\pi}{2}$

Question 11

$$3tan^2\theta=2sec^2\theta+1 \quad \text{for} \quad 0\leq\theta\leq 180°$$

Answer: $\theta=60°;\ 120°$

Question 12

$$2sin\,(\theta - 20°) = 1 \quad \text{for} \quad 0° \leq \theta \leq 360°$$

Answer: $\theta = 50°;\ 170°$

Question 13

$$3sin3\theta = 2cos3\theta \quad \text{for} \quad 0° \leq \theta \leq 180°$$

Answer: $\theta = 11.2°;\ 71.2°;\ 131.2°$

Question 14

$$2\,cot\left(\frac{\theta}{2}\right) = 2\sqrt{3} \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{\pi}{3}$

Question 15

$$sec\theta = -2 \quad \text{for} \quad -180° \leq \theta \leq 180°$$

Answer: $\theta = \pm 120°$

Question 16

$$sec^2\theta - sec\theta = 2 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0;\ \frac{\pi}{3};\ \frac{5\pi}{3}$

Question 17

$$3sin^2\theta - 2sin\theta = 0 \quad \text{for} \quad 0° \leq \theta \leq 360°$$

Answer: $\theta = 0°;\ 41.8°;\ 138.2°;\ 180°;\ 360°$

Question 18

$$\sin\theta\tan\theta = \sqrt{3}\sin\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0;\ \frac{\pi}{3};\ \frac{4\pi}{3}$

Question 19

$$\cos^2\theta = \sin\theta\cos\theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{\pi}{4};\ \frac{\pi}{2};\ \frac{3\pi}{2};\ \frac{5\pi}{4}$

Question 20

$$\mathrm{cosec}^4\theta - 4\,\mathrm{cosec}^2\theta = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0;\ \frac{\pi}{6};\ \frac{5\pi}{6};\ \pi;\ \frac{7\pi}{6};\ \frac{11\pi}{6}$

Question 21

$$2\cos^2\theta - 1 = \cos\theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = 0;\ \frac{2\pi}{3};\ \frac{4\pi}{3};\ 2\pi$

Question 22

$$4(\sin^2\theta - \cos\theta) = 3 - 2\cos\theta \quad \text{for} \quad 0° \leq \theta \leq 180°$$

Answer: $\theta = 72°;\ 144°$

Question 23

$$4\cos^2\left(\frac{\theta}{2}\right) - 4\cos\left(\frac{\theta}{2}\right) + 1 = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{2\pi}{3}$

Question 24

$$\cos\theta + \sin\theta = 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{3\pi}{4};\ \frac{7\pi}{4}$

Question 25

$$2\cos^2\theta = 7\sin\theta + 5 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{7\pi}{6};\ \frac{11\pi}{6}$

Question 26

$$\sin\theta - \sin 2\theta = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0;\ \frac{\pi}{3};\ \pi;\ \frac{5\pi}{3}$

Question 27

$$2\cos^2\theta - 3\sin^2\theta = 0 \quad \text{for} \quad 0° \leq \theta < 360°$$

Answer: $\theta = 30°;\ 150°;\ 210°;\ 330°$

Question 28

$$2\sin^2 2\theta = 1 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{8};\ \frac{3\pi}{8};\ \frac{5\pi}{8};\ \frac{7\pi}{8}$

Question 29

$$\sin\theta + \sin 2\theta = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0;\ \pi;\ \frac{2\pi}{3};\ \frac{4\pi}{3}$

Question 30

$$sec\theta = 2cos\theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{\pi}{4}; \frac{3\pi}{4}; \frac{5\pi}{4}; \frac{7\pi}{4}$

Question 31

$$2cos2\theta = cos2\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{2\pi}{3}; \frac{4\pi}{3}; \frac{\pi}{4}; \frac{3\pi}{4}; \frac{5\pi}{4}; \frac{7\pi}{4}$

Question 32

$$2\,sin(\theta - 20°) = 1 \quad \text{for} \quad 0° \leq \theta \leq 360°$$

Answer: $\theta = 50°; 170°$

Question 33

$$2sin^2\theta + 3sin\theta + 1 = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{7\pi}{6}; \frac{3\pi}{2}; \frac{11\pi}{6}$

Question 34

$$2sin^2\theta + 3sin\theta + 1 = 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{7\pi}{6}; \frac{3\pi}{2}; \frac{11\pi}{6}$

Question 35

$$2tan2\theta = 2 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{8}; \frac{5\pi}{8}; \frac{9\pi}{8}; \frac{13\pi}{8}$

Question 36

$$sin^2\theta = 3cos^2\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{3}; \frac{2\pi}{3}; \frac{4\pi}{3}; \frac{5\pi}{3}$

Question 37

$$cos3\theta + 1 = 0 \quad \text{for} \quad 0° \leq \theta \leq 360°$$

Answer: $\theta = 60°; 180° ; 300°$

Question 38

$$cos^3\theta = cos\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0; \frac{\pi}{2}; \frac{3\pi}{2} ; \pi$

Question 39

$$cos2\theta - cos\theta = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0; \frac{2\pi}{3}; \frac{4\pi}{3}$

Question 40

$$2cot\theta + cosec\theta = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{2\pi}{3}; \frac{4\pi}{3}$

Question 41

$$3tan^3\theta = tan\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0; \frac{\pi}{6}; \frac{5\pi}{6} ; \pi; \frac{7\pi}{6}; \frac{11\pi}{6}$

Question 42

$$sec\theta cosec\theta = 2cosec\theta \quad \text{for} \quad 0 \le \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Question 43

$$cos^2 2\theta - cos^4 2\theta = 0 \quad \text{for} \quad 0 \le \theta < 2\pi$$

Answer: $\theta = 0; \ \frac{\pi}{2}; \pi; \ \frac{3\pi}{2}; \frac{\pi}{4}; \frac{3\pi}{4}; \frac{5\pi}{4}; \frac{7\pi}{4}$

Question 44

$$3sin3\theta = 2cos3\theta \quad \text{for} \quad 0° \le \theta \le 180°$$

Answer: $\theta = 11.2°; \ 71.2°; \ 131.2°$

Question 45

$$tan\theta + 1 = \sqrt{3} + \sqrt{3}cot\theta \quad \text{for} \quad 0 \le \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{3}; \ \frac{3\pi}{4}; \frac{4\pi}{3}; \frac{7\pi}{4}$

Question 46

$$18cos^2\theta - 9 = 0 \quad \text{for} \quad 0 \le \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{4}; \frac{3\pi}{4}; \frac{5\pi}{4}; \frac{7\pi}{4}$

Question 47

$$sin\left(\frac{\theta}{2}\right) = \sqrt{2} - sin\left(\frac{\theta}{2}\right) \quad \text{for} \quad 0° \le \theta < 360°$$

Answer: $\theta = 90°; 270°$

Question 48

$$2sin^2\theta = 3cos\theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Question 49

$$cos2\theta - sin\theta tan\theta = 2 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Question 50

$$sin2\theta = cos\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{6}; \frac{\pi}{2}; \frac{5\pi}{6}; \frac{3\pi}{2}$

Question 51

$$2cos3\theta + \sqrt{3} = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{5\pi}{18}; \frac{7\pi}{18}; \frac{17\pi}{18}; \frac{19\pi}{18}; \frac{29\pi}{18}; \frac{31\pi}{18}$

Question 52

$$2sec^2\theta + tan^2\theta - 3 = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{6}; \frac{5\pi}{6}; \frac{7\pi}{6}; \frac{11\pi}{6}$

Question 53

$$\frac{4}{sec^2\theta} + 3cos\theta = 2cos\theta tan\theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

Answer: $\theta = 1.1;\ 5.2$

Question 54

$$3cos^2\theta - 5sin\theta = 1 \quad \text{for} \quad 0° \leq \theta < 360°$$

Answer: $\theta = 19.5°;\ 160.5°$

Question 55

$$2sin\theta cot\theta + cot\theta = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{2};\ \frac{3\pi}{2};\ \frac{7\pi}{6};\ \frac{11\pi}{6}$

Question 56

$$2cos2\theta = 1 \quad \text{for} \quad -180° \leq \theta \leq 180°$$

Answer: $\theta = \pm 30°;\ \pm 150°$

Question 57

$$tan\theta = 3sin\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = 0;\ \ 1.2;\ \ \pi;\ \ 5.1$

Question 58

$$sin^2(\theta - 30°) = \frac{1}{2} \quad \text{for} \quad 0° \leq \theta < 360°$$

Answer: $\theta = 75°;\ 165°;\ 255°;\ 345°$

Question 59

$$cos\theta + 1 = sin\theta \quad \text{for} \quad 0 \leq \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{2};\ \pi$

Question 60

$$4sin\theta = tan\theta \quad \text{for} \quad -180° \le \theta \le 180°$$

Answer: $\theta = 0°;\ \pm 75.5°;\ 180°$

Question 61

$$\cos(70° - x) = 0.6 \quad \text{for} \quad 0° \le \theta < 180°$$

Answer: $\theta = 16.9°;\ 123.1°$

Question 62

$$sin\theta = cos\theta \quad \text{for} \quad 0 \le \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{4};\ \frac{5\pi}{4}$

Question 63

$$sin2x - cosx = 0 \quad \text{for} \quad 0 \le \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{6};\ \frac{\pi}{2};\ \frac{5\pi}{6};\ \frac{3\pi}{2}$

Question 64

$$sin\theta - \sqrt{3}cos\theta = 1 \quad \text{for} \quad 0 \le \theta < 2\pi$$

Answer: $\theta = \frac{\pi}{2};\ \frac{7\pi}{6}$

Question 65

$$sin(x - 35°) = cosx \quad \text{for} \quad 0° \le \theta < 360°$$

Answer: $\theta = 62.5°;\ 242.5°$

PAST EXAMINATION QUESTIONS

CAMBRIDGE

Question 1

CAMBRIDGE NOVEMBER 2019 PAPER 3

(i) By first expanding $tan\ (2x + x)$, show that the equation $tan3x = 3cotx$ can be written in the form $tan^4x - 12tan^2x + 3 = 0$. **[4]**

(ii) Hence solve the equation $tan3x = 3cotx$ for $0° < x < 90°$. **[3]**

Answers: $x = 26.8°;\ \ 73.7°$

Question 2

CAMBRIDGE JUNE 2019 PAPER 3

Showing all necessary working, solve the equation $cot2\theta = 2tan\theta$ for $0° < \theta < 180°$. **[5]**

Answers: $x = 24.1°;\ \ 155.9°$

Question 3

CAMBRIDGE NOVEMBER 2017 PAPER 3

By expressing the equation $tan(\theta + 60°) + tan(\theta - 60°) = tan\theta$ in terms of $tan\theta$ only, solve the equation for $0 \le \theta < 90°$. [5]

Answer: $\theta = 16.8°$

Question 4

CAMBRIDGE JUNE 2016 PAPER 3

(i) Express $\sqrt{5}cosx + 2sinx$ in the form $Rcos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. **[3]**

(ii) Hence solve the equation

$$\sqrt{5}cos\frac{1}{2}x + 2sin\frac{1}{2}x = 1.2,$$

for $0° < x < 360°$. **[4]**

Answers: (i) $3cos(x - 41.81°)$

(ii) $x = 216.5°$

Question 5

CAMBRIDGE NOVEMBER 2015 PAPER 3

The angles A and B are such that

$$sin(A + 45°) = \left(2\sqrt{2}\right)\cos A \quad \text{and} \quad 4sec^2B + 5 = 12tanB\ 180°.$$

Without using a calculator, find the exact value of $tan(A - B)$. **[8]**

Answer: $\frac{3}{11}$

Question 6

CAMBRIDGE JUNE 2015 PAPER 3

Solve the equation $cot2x + cotx = 3$ for $0° < x < 180°$. **[6]**

Answers: $24.9°$; $98.8°$

Question 7

CAMBRIDGE NOVEMBER 2014 PAPER 3

(i) Show that $cos(\theta - 60°) + \cos(\theta + 60°) \equiv cos\theta$. [3]

(ii) Given that $\frac{cos(2x-60°)+\cos(2x+60°)}{cos(x-60°)+\cos(x+60°)} = 3$, find the exact values of $cosx$. **[4]**

Answer: $\frac{1}{4}\left(3 \pm \sqrt{17}\right)$

Question 8

CAMBRIDGE NOVEMBER 2013 PAPER 33

(i) Given that $sec\theta + 2cosec\theta = 3cosec2\theta$, show that $2sin\theta + 4cos\theta = 3$. **[3]**

(ii) Express $2sin\theta + 4cos\theta$ in the form $Rsin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. **[3]**

(iii) Hence solve the equation $sec\theta + 2cosec\theta = 3cosec2\theta$ for $0° < \theta < 360°$. **[4]**

Answers: (ii) $\sqrt{20}sin(\theta + 63.44°)$

(iii) $74.4°$; $338.7°$

Question 9

CAMBRIDGE NOVEMBER 2013

Paper 32

(i) Prove that $cot\theta + tan\theta \equiv 2cosec2\theta$. **[3]**

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} cosec2\theta d\theta = \frac{1}{2}In3$$ [**4**]

Question 10

CAMBRIDGE NOVEMBER 2008 PAPER 2

(i) Show that the equation

$$sin(x + 30°) = 3\,cos(x + 60°)$$

can be written in the form

$$3\sqrt{3}sinx = cosx.$$ [**3**]

(ii) Hence solve the equation

$$sin(x + 30°) = 3\,cos(x + 60°)$$

for $-180° \leq x \leq 180°$. **[3]**

Answers: (ii) $x = -169.1°;\ \ 10.9°$

Question 11

CAMBRIDGE JUNE 2008 PAPER 3

(i) Show that the equation $tan(30° + \theta) = 2\,tan(60° - \theta)$ can be written in the form

$$tan^2\theta + (6\sqrt{3})tan\theta - 5 = 0.$$ [4]

(ii) Hence, or otherwise, solve the equation

$$tan(30° + \theta) = 2\,tan(60° - \theta),$$

for $0° \leq \theta \leq 180°$. **[3]**

Answers: (ii) $24.7°;\ \ 95.3°$

Question 12

CAMBRIDGE JUNE 2008 PAPER 2

(i) Express $5cos\theta - sin\theta$ in the form $Rcos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. **[3]**

(ii) Hence solve the equation

$$5cos\theta - sin\theta = 4,$$

giving all solutions in the interval $0° \leq \theta \leq 360°$. **[4]**

Answers: (i) $\sqrt{26}cos(\theta - 11.31°)$

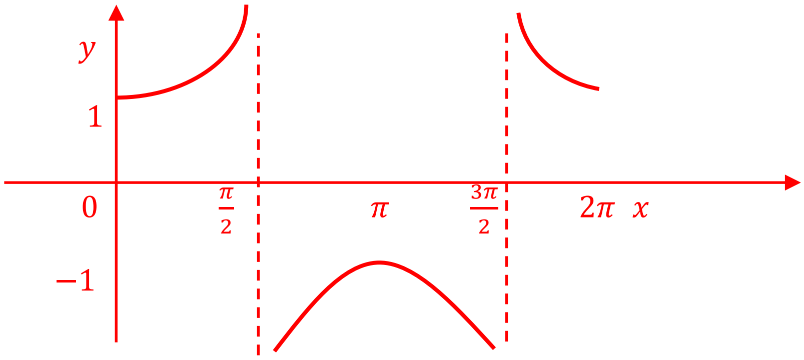
(ii) $27.0°;\ \ 310.4°$

Question 13

CAMBRIDGE JUNE 2004 PAPER 3

Sketch the graph of $y = secx$, for $0 \leq x \leq 2\pi$. **[3]**

Answer:



Question 14

CAMBRIDGE JUNE 2004 PAPER 3

(i) Prove that $sin^2\theta cos^2\theta \equiv \frac{1}{8}(1 - cos4\theta)$. **[3]**

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} sin^2\theta cos^2\theta d\theta$$ [3]

Answer: (ii)$\frac{1}{8}\left(\frac{1}{3}\pi + \frac{\sqrt{3}}{8}\right)$

Question 15

CAMBRIDGE NOVEMBER 2002 PAPER 2

The angle x, measured in degrees, satisfies the equation

$$\cos(x - 30°) = 3\sin(x - 60°).$$

(i) By expanding each side, show that the equation may be simplified to

$$2\sqrt{3}cosx = sinx.$$ [3]

(ii) Find two possible values of x lying between 0° and 360° . **[3]**

(iii)Find the exact value of $cos2x$, giving your answer as a fraction. **[3]**

Answers: (ii) 73.9°; 253.9°

(iii) $-\frac{11}{13}$

Question 16

CAMBRIDGE NOVEMBER 2002 PAPER 3

(i) Express $4sin\theta - 3cos\theta$ in the form $Rsin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the value of α correct to 2 decimal places. **[3]**

Hence

(ii) solve the equation

$$4sin\theta - 3cos\theta = 2,$$

giving all values of θ such that $0° < \theta < 360°$, **[4]**

(iii)Write down the greatest value of $\frac{1}{4sin\theta - 3cos\theta + 6}$. **[1]**

Answers: (i) $5sin(\theta - 36.87°)$

(ii) $60.4°;\ 193.3°$

(iii) 1

Question 17

CAMBRIDGE JUNE 2002 PAPER 2

(i) Express $3cos\theta + 2sin\theta$ in the form $Rcos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the exact value of R and the value of α correct to 1 decimal place. **[3]**

(ii) Solve the equation

$$3cos\theta + 2sin\theta = 3.5,$$

giving all solutions in the interval $0° \leq \theta \leq 180°$. **[4]**

(iii)The graph of $y = 3cos\theta + 2sin\theta$ for $0° \leq \theta \leq 180°$, has one stationary point. State the coordinates of this point. **[1]**

Answers: (i) $\sqrt{13}cos(\theta - 33.7°)$

(ii) $19.8°;\ 47.6°$

(iii) $(33.7;\ \sqrt{13})$

Question 18

CAMBRIDGE NOVEMBER 2001 PAPER 2

Solve the equation $2sec^2x - tanx = 5$ for $0° \leq x \leq 360$. **[5]**

Answers: $56.3°$; $135°$; $236.3°$; $315°$

ZIMSEC

Question 1

ZIMSEC NOVEMBER 2007 PAPER 1

Show that the equation

$$\frac{4sin^2\theta}{cosec\theta} + \frac{3}{cosec^2\theta sec\theta} = 2sin^2\theta$$

may be written as

$$sin^2\theta(4sin\theta + 3cos\theta - 2) = 0$$

and hence solve the equation for $0° \leq \theta \leq 360°$. **[7]**

Answer: $\theta = 120°; 167°$

Question 2

ZIMSEC NOVEMBER 2009 PAPER 1

(i) Express $3cosx - 5sinx$ in the form $Rsin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 360°$ **[3]**

(ii) Hence, or otherwise, solve the equation

$$3cosx - 5sinx = 2,$$

for $0° < x < 360°$. **[3]**

Answers: (i) $\sqrt{34}cos(\theta - 149.03°)$

(ii) $11°$; $231°$

Question 3

ZIMSEC NOVEMBER 2009 PAPER 1

(i) Solve the equation

$|2sinx - 3| = 1$

for $0° \leq x \leq 360°$. **[3]**

(ii) Solve the equation $\frac{3}{2}sin2\theta = 2sin^2\theta$ for $0° < \theta < 360°$.

$$5cos\theta - sin\theta = 4,$$

giving all solutions in the interval $0° \leq \theta \leq 360°$, giving your answers correct to the nearest $0.1°$. **[5]**

Answers: (i) $90°$

(ii) $0°$; $56.3°$; $180°$; $236.3°$; $360°$

Question 4

ZIMSEC NOVEMBER 2012 PAPER 1

(a) (i) Write down the function $3\sqrt{2}(cosx + sinx)$ in the form $Rcos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

(ii) Hence, or otherwise, solve the equation $\sqrt{2}(cosx + sinx) = \sqrt{3}$ for $0° \leq \theta \leq 360°$. **[6]**

(b) (i) State the maximum and the minimum values of the function

$$y = 6 + 3\sqrt{2}(cosx + sinx).$$

(ii) Sketch the graph of y against x for $0° \leq \theta \leq 360°$.

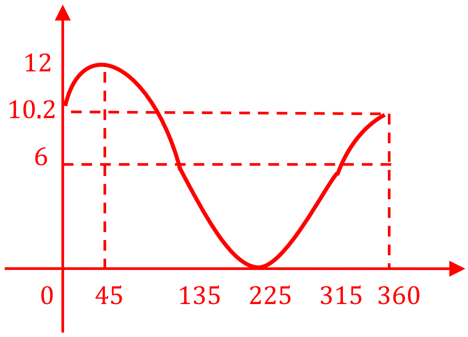
(iii) Describe the basic transformations which the graph of $y = cosx$ undergoes to obtain the graph of $y = 6 + 3\sqrt{2}(cosx + sinx)$. **[8]**

Answers: a) (i) $6cos(\theta - 45°)$

(ii) $15°;\ 75°$

b) (i) maximum value $= 12$ and minimum value$= 0$

(ii)



(iii) (1) Translation along the $x - axis$ with translation vector $\binom{45°}{0}$ or moving $45°\ units$ rightwards

(2) Stretch along the $y - axis$ with factor 6 or multiply every y −coordinate by 6

(3) Translation along the $y - axis$ with translation vector $\binom{0}{6}$ or moving $6\ units$ upwards

Question 5

ZIMSEC NOVEMBER 2013 PAPER 1

(i) Prove the identity

$$\frac{1 + sin2\theta}{1 - sin2\theta} \equiv \frac{(tan\theta + 1)^2}{(tan\theta - 1)^2}$$ **[4]**

(ii) Hence solve the equation

$$tanxcos2x = sinx \text{ for } 0° \leq x \leq 360°$$ **[6]**

Answers: (i) $0°;\ 120°;\ 180°;\ 240°;\ 360°$

Question 6

ZIMSEC NOVEMBER 2015 PAPER 1

(i) Express $6cos\theta - 8sin\theta$ in the form $Rcos(\theta + \alpha)$. **[3]**

(ii) Hence or otherwise, solve the equation $6cos\theta - 8sin\theta = 2.5$ for $-180° \leq \theta \leq 180°$. **[3]**

(iii)State the minimum and the maximum values of

$$\frac{1}{6cos\theta - 8sin\theta + 13}$$ [**2**]

Answers: (i) $10cos(\theta + 53.1°)$

(ii) $-157.6°;\ \ 51.4°$

(iii) Maximum $= \frac{1}{3}$

Minimum $= \frac{1}{23}$

Question 7

ZIMSEC NOVEMBER 2019 PAPER 1

(a) Express $3cos\theta - 2sin\theta$ in the form $Rcos(\theta + \alpha)$ where $R > 0$ and $0° < \alpha < 90°$. **[3]**

(b) Hence or otherwise, solve the equation $3cos\theta - 2sin\theta = \frac{2}{3}$ for $-180° \leq \theta \leq 180°$. **[3]**

(c) Determine the maximum and the minimum values of $\frac{1}{3cos\theta - 2sin\theta}$. **[2]**

Answers: (a) $\sqrt{13}cos(\theta - 33.7°)$

(b) $-113°;\ \ 46°$

(b) Maximum $= -\frac{1}{\sqrt{13}}$

Minimum $= \frac{1}{\sqrt{13}}$

Question 8

ZIMSEC NOVEMBER 2019 PAPER 1

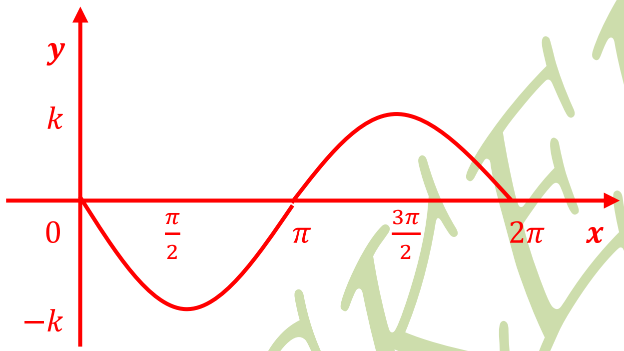
(a) Given that $h(\theta) = ksin\theta; k < 0$

Sketch the graph of $y = h(\theta)$ for $0 < \theta < 2\pi$. **[3]**

(b) Describe the geometrical transformation which maps $f(x) = sinx$ onto

$y = 6\,sin(x - \pi)$. **[3]**

Answers: (a)



(b) Translation along the $x - axis$ with translation vector $\begin{pmatrix}\pi\\0\end{pmatrix}$ or moving π *units* rightwards followed by a stretch along the $y - axis$ with factor 6 or multiply every y −coordinate by 6

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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