Vector 1

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SYLLABUS (6042) REQUIREMENTS

- define position and free vector
- carry out addition, subtraction and scalar multiplication of vectors
- use unit, displacement and position vector to solve problems
- calculate the magnitude of a vector and the scalar product of two vectors
- use scalar product to find the angle between two vectors
- calculate the area of plane shapes using the dot product
- solve problems involving vectors

VECTORS

Definition: Any physical quantity that has magnitude as well as direction for example weight, velocity, force, angular momentum and wavelength.

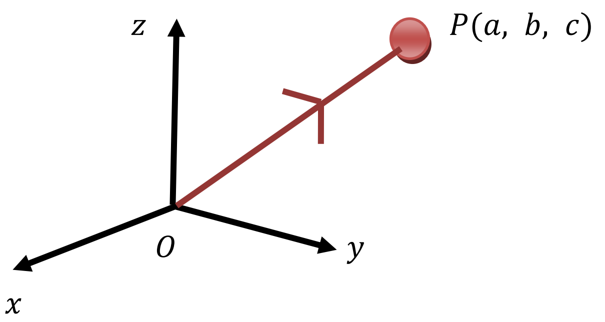
Notations

- Vectors are indicated by using a bold typeface e.g. $\boldsymbol{p}$.
- It is difficult when handwriting to reproduce the bold face and so it is conventional to underline vector quantities e.g. $\overrightarrow{OP}$ and $\vec{p}$
- The arrow indicates the direction of the vector

Types of Vectors

a) Position Vector

- Vectors used to describe position of a vector with respect to all coordinates of three dimensional systems
- It is usually denoted by an arrow e.g. $\overrightarrow{OP}$
- A position vector $\overrightarrow{OP}$ occurs when O is fixed
- The point O from where the vector $\overrightarrow{OP}$ starts is called its initial point
- The point P where it ends is called its terminal point.
- To locate the position of any point 'P' in a plane or space, generally a fixed point of reference called the origin 'O' is taken. The vector $\overrightarrow{OP}$ is called the position vector of P with respect to O as shown in the diagram below:



- Any point with coordinates $P(a, b, c)$ can be written as a vector with respect to the origin (position vector) as $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$ or $\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ or $\overrightarrow{OP} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

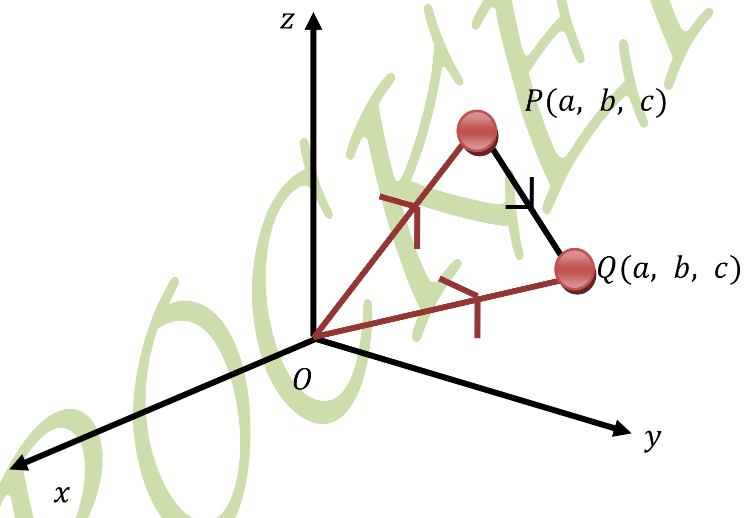
Note:

$\boldsymbol{i}$ = unit vector along x direction

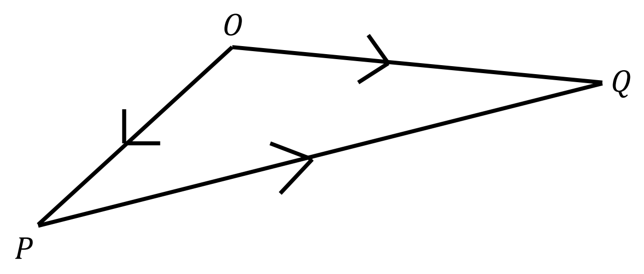
$\boldsymbol{j}$ = unit vector along y direction

$\boldsymbol{k}$ = unit vector along z direction

- $\overrightarrow{PQ}$ represents the displacement vector

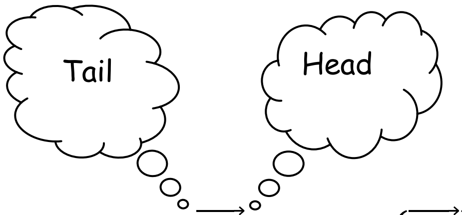


- It is a result of subtracting two position vectors
- The displacement of vector $\overrightarrow{PQ}$ is calculated as follows:



From the diagram above:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$
$$= \overrightarrow{OQ} - \overrightarrow{OP}$$



$$\overrightarrow{PQ} = Head\ (\overrightarrow{OQ}) - Tail\ (\overrightarrow{OP})$$

Note

(i) Given a point P, there is one and only one position vector for the point with respect to the origin 'O'.

(ii) Position vector of a point 'P' changes if the position of the origin 'O' is changed.

(iii) Subtracting two position vectors yields a displacement vector i.e. $\overrightarrow{PQ}$ is a displacement vector

Solved Problems

Example 1

Write down the following points as position vectors with respect to the origin

a) $P(1, -2,\ 3)$

b) $Q(4,\ 0,\ -1)$

Suggested Solution

a) $\overrightarrow{OP} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$ or $\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ or $\overrightarrow{OP} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

b) $\overrightarrow{OQ} = 4\vec{\imath} - \vec{k}$ or $\overrightarrow{OQ} = 4\boldsymbol{i} - \boldsymbol{k}$ or $\overrightarrow{OQ} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$

Example 2

If $P(1, -2,\ 3)$ and $Q(4,\ 0,\ -1)$ are points in a plane, write down the position vectors of P and Q with respect to the origin O.

Hence find the displacement $\overrightarrow{PQ}$

Suggested Solution

$\overrightarrow{OP} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$ or $\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ or $\overrightarrow{OP} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$\overrightarrow{OQ} = 4\vec{\imath} - \vec{k}$ or $\overrightarrow{OQ} = 4\boldsymbol{i} - \boldsymbol{k}$ or $\overrightarrow{OQ} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$

Now:

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$ or $3\vec{\imath} + 2\vec{\jmath} - 4\vec{k}$

Example 3

Write down the following position vectors as points

a) $\overrightarrow{OP} = 5\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

b) $\overrightarrow{OQ} = -2\vec{\imath} + 6\vec{\jmath}$

c) $\overrightarrow{OP} = \begin{pmatrix} 0 \\ 3 \\ -9 \end{pmatrix}$

Suggested Solution

a) $P(5, -3,\ 2)$

b) $Q(-2,\ 6,\ 0)$

c) $P(0,\ 3, -9)$

NB: Write down the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

Negative of a Vector

- A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{PQ}$), but direction is opposite to that of it, is called negative of the given vector.
- For example, vector $\overrightarrow{QP}$ is negative of the vector PQ
- It is written as $\overrightarrow{QP} = -\overrightarrow{PQ}$

Example

If $\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ then $\overrightarrow{OQ}$ is the negative of $\overrightarrow{OP}$.

Magnitude of a Vector

- The distance between initial and terminal points of a vector (say P and Q) is called the magnitude (or length or modulus) of the vector
- The magnitude of a vector is always positive
- The magnitude is denoted as $|\overrightarrow{PQ}|$ or $|\boldsymbol{p}|$ or $|\vec{p}|$
- If $\overrightarrow{PQ} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then the magnitude of $\overrightarrow{PQ}$ is calculated as follows:

$$|\overrightarrow{PQ}| = \sqrt{a^2 + b^2 + c^2}$$

Solved Problems

Question 1

Given that $\overrightarrow{OA} = (3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$ and $\overrightarrow{OB} = (7\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k})$, find size of the magnitude of vector $\overrightarrow{AB}$.

Suggested Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 8 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix}$$

$$= 4\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}$$

Now:

$$\left|\overrightarrow{AB}\right| = \sqrt{(4)^2 + (4)^2 + (-6)^2}$$

$$= \sqrt{16 + 16 + 36}$$

$$= \sqrt{68}$$

Question 2

$\overrightarrow{OA} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$. Calculate the size of magnitudes of vectors

(i) $\overrightarrow{OA}$

(ii) $\overrightarrow{BA}$

Suggested Solution

(i) $\overrightarrow{OA} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$

$$\left|\overrightarrow{OA}\right| = \sqrt{(0)^2 + (-3)^2 + (4)^2}$$

$$= \sqrt{0 + 9 + 16}$$

$$= \sqrt{25}$$

$$= 5$$

(ii) $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

$$\left|\overrightarrow{BA}\right| = \sqrt{(2)^2 + (-4)^2 + (1)^2}$$

$$= \sqrt{4 + 16 + 1}$$

$$= \sqrt{21}$$

b) Free Vector

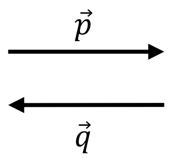
- A free vector is not restricted in any way, it can be placed to any point but parralel to itself
- It s completely defined by its magnitude and direction
- It can be drawn as any one set of equal length parallel lines
- All vectors are free vectors except position vectors
- Magnitude and direction remains constant

Types of Free Vectors

Zero Vector/ Null Vector

- A vector whose initial and terminal points coincide and is of length zero

- It is denoted as $\vec{0}$
- Zero vector cannot be assigned a definite direction as it has zero magnitude
- The vectors $\overrightarrow{PP}$ or $\overrightarrow{QQ}$ represent the zero vectors.
- Example: When adding two vectors which are equal in length and going in different directions



Note

a) Zero vectors have no specific direction.

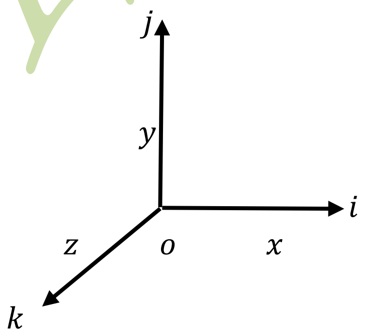
b) The position vector of origin is a zero vector.

c) The sum of any vector with a zero vector will give the same vector i.e. $\vec{0} + \vec{p} = \vec{p}$

d) Zero vectors are only of mathematical importance since a zero vector is the additive identity of the additive group of vectors

Unit Vector

- A vector whose magnitude is unity (i.e. 1 $unit$) is called a unit vector.
- A unit vector in the direction of $\boldsymbol{a}$ is given the 'hat' symbol $\hat{\boldsymbol{a}}$
- A unit vector in the direction of $\boldsymbol{OX}$ is represented by $\boldsymbol{i}$, $\boldsymbol{OY}$ is represented by $\boldsymbol{j}$ and $\boldsymbol{OZ}$ is represented by $\boldsymbol{k}$



- Illustration: Any point with coordinates $P(a, b, c)$ can be written a s a vector with respect to the origin as $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$

- A unit vector in the direction of a given vector is found by dividing the given vector by its magnitude i.e.

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Solved Problems

Question 1

Given that $\overrightarrow{OA} = (3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$ and $\overrightarrow{OB} = (7\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k})$, find the unit vectors in the direction of $\overrightarrow{AB}$.

Suggested Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 8 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix}$$

$= 4\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}$ and

$$|\overrightarrow{AB}| = \sqrt{(4)^2 + (4)^2 + (-6)^2}$$

$$= \sqrt{16 + 16 + 36}$$

$$= \sqrt{68}$$

Now

$$\widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{(4\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k})}{\sqrt{68}}$$

Question 2

$\overrightarrow{OA} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$. Find unit vectors in the direction of

(i) $\overrightarrow{OA}$

(ii) $\overrightarrow{BA}$

Suggested Solution

(i) $\overrightarrow{OA} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$

$$|\overrightarrow{OA}| = \sqrt{(0)^2 + (-3)^2 + (4)^2}$$

$$= \sqrt{0 + 9 + 16}$$

$$= \sqrt{25}$$

$$= 5$$

Now

$$\widehat{OA} = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} = \frac{\begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}}{5}$$

$$= \frac{1}{5}\begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} \text{ or } \frac{\mathbf{-3}\boldsymbol{j} + \mathbf{4}\boldsymbol{k}}{\mathbf{5}}$$

(ii) $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

$$|\overrightarrow{BA}| = \sqrt{(2)^2 + (-4)^2 + (1)^2}$$

$$= \sqrt{4 + 16 + 1}$$

$$= \sqrt{21}$$

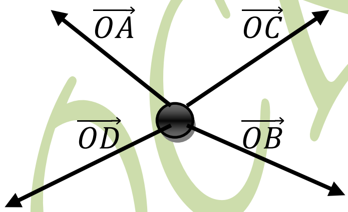
Now

$$\widehat{BA} = \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \frac{\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}}{\sqrt{21}}$$

$$= \frac{1}{\sqrt{21}}\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \text{ or } \frac{2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}}{\sqrt{21}}$$

Co-initial Vectors

Two or more vectors having the same initial point are called co-initial vectors.



Equal Vectors

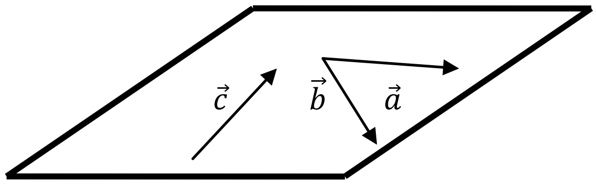
Two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\boldsymbol{p} = \boldsymbol{q}$

Example

$\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ are equal vectors

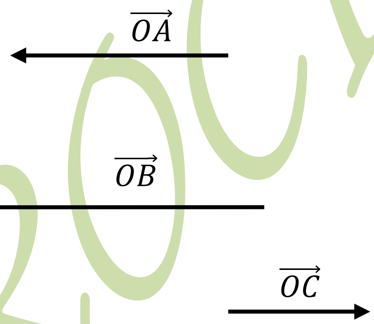
Coplanar vectors

The vectors in the same plane are called coplanar vectors.



Parallel Vectors

- These are vectors which have the same parallel support
- They can have equal or unequal magnitudes
- Their directions may be the same (like vectors) or opposite (unlike vectors)



- Two vectors are said to be parallel if an only if they are scalar multiples of one another

Solved Problems

Example 1

Show that vectors $\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\overrightarrow{OQ} = -5\boldsymbol{i} + 10\boldsymbol{j} - 15\boldsymbol{k}$ are parallel.

Suggested Solution

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and

$\overrightarrow{OQ} = \begin{pmatrix} -5 \\ 10 \\ -15 \end{pmatrix}$

$= -5\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$= -5\overrightarrow{OP}$

Since $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are scalar multiples ∴ they are parallel

Example 2

If $\overrightarrow{OA} = \begin{pmatrix} a-1 \\ -a-1 \\ b \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2a \\ 3-5a \\ 15 \end{pmatrix}$ are parallel, find the values of a and b.

Solution

Comparing $\boldsymbol{i}$ and $\boldsymbol{j}$ components:

$\frac{a-1}{2a} = \frac{-a-1}{3-5a}$

$\therefore (a-1)(3-5a) = 2a(-a-1)$

$\Rightarrow 3a - 5a^2 - 3 + 5a = -2a^2 - 2a$

$\therefore 3a^2 - 10a + 3 = 0$

$\Rightarrow (3a-1)(a-3) = 0$

$\therefore a = \frac{1}{3}$ or 3

Now to find b let's compare $\boldsymbol{i}$ components first to get the ratio:

Since direction vectors are multiples, let β be the ratio:

$\beta(a-1) = 2a$

When $a = 1/3$:

$\Rightarrow \beta\left(\frac{1}{3} - 1\right) = 2\left(\frac{1}{3}\right)$

$\Rightarrow \beta\left(-\frac{1}{3}\right) = \frac{2}{3}$

$\Rightarrow \beta = -1$

Now comparing $\boldsymbol{k}$ components:

$\beta(b) = 15$

$\Rightarrow -b = 15$

$\therefore b = -15.$

When $a = 3$:

$\Rightarrow \beta(3 - 1) = 2(3)$

$\Rightarrow \beta(2) = 6$

$\Rightarrow \beta = 3$

Now comparing $\boldsymbol{k}$ components:

$\beta(b) = 15$

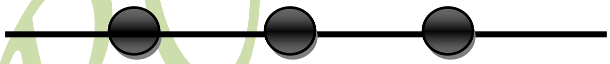
$\Rightarrow 3b = 15$

$\therefore b = 5.$

$\therefore$ When $a = 1/3$, $b = -15$ and when $a = 3$, $b = 5$.

Collinear Vectors

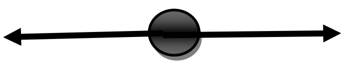
- Two or more points are said to be collinear if they lie in the same line



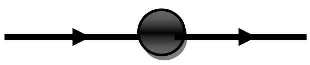
- Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.
- Direction vectors of collinear vectors are multiples i.e.

$$\overrightarrow{OA} = \lambda\overrightarrow{OB};\ \lambda \in \Re$$

- The angle between these vectors is 0° or 180°.



180°.



0°.

Solved Problems

Question 1

Given that $\overrightarrow{OA} = (3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$, $\overrightarrow{OB} = (7\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k})$ and $\overrightarrow{OC} = (13\boldsymbol{i} + 14\boldsymbol{j} - 17\boldsymbol{k})$, show that A, B and C are collinear.

Suggested Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 8 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 13 \\ 14 \\ -17 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ -15 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 13 \\ 14 \\ -17 \end{pmatrix} - \begin{pmatrix} 7 \\ 8 \\ -8 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -9 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

Since $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are multiples $\therefore A, B$ and C are collinear

NB: $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are sufficient to prove for co-linearity

Question 2

Given that $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -6 \\ 10 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 25 \\ 30 \\ -39 \end{pmatrix}$ show that A, B and C are not collinear.

Suggested Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ -6 \\ 10 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 14 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 25 \\ 30 \\ -39 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 21 \\ 27 \\ -37 \end{pmatrix}$$

Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not multiples $\therefore A, B$ and C are not collinear

Question 3

$P(1,3,-2)$, $Q(2,1,1)$ and $R(6,-7,13)$. Show that P, Q and R are collinear.

Suggested Solution

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} 6 \\ -7 \\ 13 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \\ 15 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are multiples $\therefore$ P, Q and R are collinear

Question 4

Find b and c such that $\begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ 9 \\ 6 \end{pmatrix}$ are collinear.

Suggested Solution

Let $\overrightarrow{OA} = \begin{pmatrix} c \\ 9 \\ 6 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}$

Now two vectors are collinear if $\overrightarrow{OA} = \lambda\overrightarrow{OB}$

$\overrightarrow{OA} = \lambda\overrightarrow{OB}$

$\begin{pmatrix} c \\ 9 \\ 6 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}$

$\Rightarrow 9 = 3\lambda$

$\therefore \lambda = 3$

Now:

$c = \lambda$

$\therefore c = 3$ and

$6 = b\lambda$

$\Rightarrow 6 = 3b$

$\therefore b = 2$

Follow Up Questions

Question 1

$P(1,2,3)$, $Q(-2,3,5)$ and $R(7,0,-1)$. Show that P, Q and R are collinear.

Question 2

Find a such that $\begin{pmatrix} -2 \\ 6 \\ a \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ are collinear

Question 3

If vectors $2\boldsymbol{i} - q\boldsymbol{j} + 3\boldsymbol{k}$ and $4\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$ are collinear, find the value of q.

Question 4

Given that $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 7 \\ 8 \\ -8 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 13 \\ 14 \\ -17 \end{pmatrix}$ show that A, B and C are collinear

Question 5

$P(4,5,2)$, $Q(3,2,4)$ and $R(5,8,0)$. Show that P, Q and R are collinear.

Question 6

Given that $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 11 \\ 4 \\ 2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix}$ show that A, B and C are not collinear

Question 7

If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are not coplanar points, test for co-linearity of the points whose position vectors are given by:

$\boldsymbol{a} - 2\boldsymbol{b} + 3\boldsymbol{c}$, $2\boldsymbol{a} + 3\boldsymbol{b} - 4\boldsymbol{c}$ and $-7\boldsymbol{b} + 10\boldsymbol{c}$.

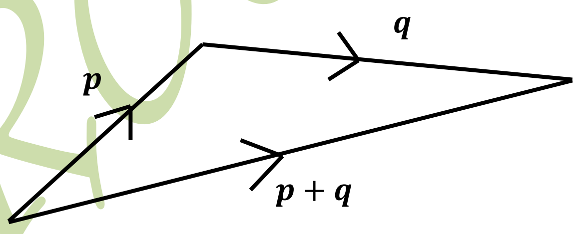
Question 8

$S(-1,0,2)$, $T(1,2,-1)$ and $U(0,-3,5)$. Show that S, T and U are not collinear.

VECTOR ALGEBRA

A. Addition of vectors

- The addition of scalars involves only the addition of their magnitudes.
- When a vector is added with another vector we have to consider their direction also.
- A vector can be added with another vector provided both the vectors represent the same physical quantity.
- Vectors are added in a particular way known as the triangle law.
- Triangle law of vector addition states that if two vectors can be represented in magnitude and direction by two sides of a triangle taken in the same order, then their resultant is represented completely by the third side of the triangle taken in opposite order



Note

- Vector addition is commutative i.e. $\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{q}+\boldsymbol{p}$
- Vector addition is associative i.e. $\boldsymbol{p}+(\boldsymbol{q}+\boldsymbol{r})=(\boldsymbol{p}+\boldsymbol{q})+\boldsymbol{r}$

Solved Problems

Question 1

Given that $\overrightarrow{OA} = (3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$, $\overrightarrow{OB} = (7\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k})$ and $\overrightarrow{OC} = (3\boldsymbol{i} - 2\boldsymbol{j} - 7\boldsymbol{k})$, find

(i) $\overrightarrow{AB} + \overrightarrow{OC}$

(ii) $\overrightarrow{BA} + \overrightarrow{CB}$

Suggested Solution

(i) $\overrightarrow{AB} + \overrightarrow{OC} = \left[\begin{pmatrix}7\\8\\-8\end{pmatrix} - \begin{pmatrix}3\\4\\-2\end{pmatrix}\right] + \begin{pmatrix}3\\-2\\-7\end{pmatrix}$

$= \begin{pmatrix}4\\4\\-6\end{pmatrix} + \begin{pmatrix}3\\-2\\-7\end{pmatrix}$

$= \begin{pmatrix}7\\2\\-13\end{pmatrix}$

(ii) $\overrightarrow{BA} + \overrightarrow{CB} = \left[\begin{pmatrix}3\\4\\-2\end{pmatrix} - \begin{pmatrix}7\\8\\-8\end{pmatrix}\right] + \left[\begin{pmatrix}7\\8\\-8\end{pmatrix} - \begin{pmatrix}3\\-2\\-7\end{pmatrix}\right]$

$= \begin{pmatrix}-4\\-4\\6\end{pmatrix} + \begin{pmatrix}4\\10\\-1\end{pmatrix}$

$= \begin{pmatrix}0\\6\\5\end{pmatrix}$

Question 2

Given that $\overrightarrow{OA} = \begin{pmatrix}1\\-3\\-1\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}3\\-2\\0\end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix}1\\3\\-3\end{pmatrix}$ find $\overrightarrow{AB} + \overrightarrow{OC} + \overrightarrow{CB}$

Suggested Solution

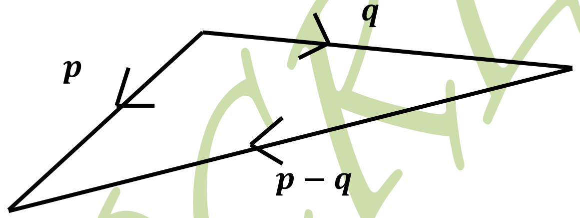
$\overrightarrow{AB} + \overrightarrow{OC} + \overrightarrow{CB} = \left[\begin{pmatrix}3\\-2\\0\end{pmatrix} - \begin{pmatrix}1\\-3\\-1\end{pmatrix}\right] + \begin{pmatrix}1\\3\\-3\end{pmatrix} + \left[\begin{pmatrix}3\\-2\\0\end{pmatrix} - \begin{pmatrix}1\\3\\-3\end{pmatrix}\right]$

$= \begin{pmatrix}2\\1\\1\end{pmatrix} + \begin{pmatrix}1\\3\\-3\end{pmatrix} + \begin{pmatrix}2\\-5\\3\end{pmatrix}$

$$= \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

B. Subtraction of vectors

- Subtraction of one vector from another is performed by adding the corresponding negative vector.
- That is, if we seek $\boldsymbol{p} - \boldsymbol{q}$ we form $\boldsymbol{p} + (-\boldsymbol{q})$.
- This is shown geometrically in the diagram below
- Subtraction of a vector is performed by adding a negative vector



Solved Problems

Question 1

Given that $\overrightarrow{OA} = (3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$, $\overrightarrow{OB} = (7\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k})$ and $\overrightarrow{OC} = (3\boldsymbol{i} - 2\boldsymbol{j} - 7\boldsymbol{k})$, find

(i) $\overrightarrow{AB} - \overrightarrow{OC}$

(ii) $\overrightarrow{BA} - \overrightarrow{CB}$

Suggested Solution

(i) $\overrightarrow{AB} - \overrightarrow{OC} = \left[\begin{pmatrix} 7 \\ 8 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}\right] - \begin{pmatrix} 3 \\ -2 \\ -7 \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ -7 \end{pmatrix}$$

$$=\begin{pmatrix}1\\6\\1\end{pmatrix}$$

$$\text{(ii) } \overrightarrow{BA}-\overrightarrow{CB}=\left[\begin{pmatrix}3\\4\\-2\end{pmatrix}-\begin{pmatrix}7\\8\\-8\end{pmatrix}\right]-\left[\begin{pmatrix}7\\8\\-8\end{pmatrix}-\begin{pmatrix}3\\-2\\-7\end{pmatrix}\right]$$

$$=\begin{pmatrix}-4\\-4\\6\end{pmatrix}-\begin{pmatrix}4\\10\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-8\\-14\\7\end{pmatrix}$$

Question 2

Given that $\overrightarrow{OA}=\begin{pmatrix}1\\-3\\-1\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}3\\-2\\0\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}1\\3\\-3\end{pmatrix}$ find $\overrightarrow{AB}+\overrightarrow{OC}-\overrightarrow{CB}$

Suggested Solution

$$\overrightarrow{AB}+\overrightarrow{OC}+\overrightarrow{CB}=\left[\begin{pmatrix}3\\-2\\0\end{pmatrix}-\begin{pmatrix}1\\-3\\-1\end{pmatrix}\right]+\begin{pmatrix}1\\3\\-3\end{pmatrix}-\left[\begin{pmatrix}3\\-2\\0\end{pmatrix}-\begin{pmatrix}1\\3\\-3\end{pmatrix}\right]$$

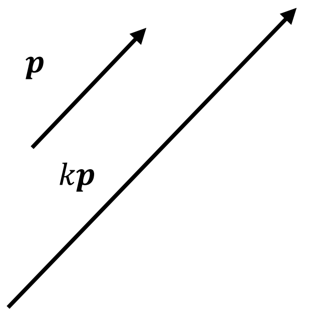
$$=\begin{pmatrix}2\\1\\1\end{pmatrix}+\begin{pmatrix}1\\3\\-3\end{pmatrix}-\begin{pmatrix}2\\-5\\3\end{pmatrix}$$

$$=\begin{pmatrix}1\\9\\-5\end{pmatrix}$$

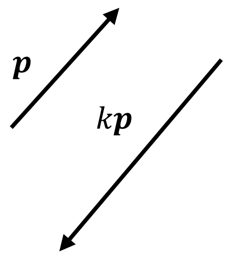
C. Multiplying a vector by a scalar

- If k is any positive scalar and $\boldsymbol{p}$ is a vector then $k\boldsymbol{p}$ is a vector in the same direction as $\boldsymbol{p}$ but k times as long.
- If k is negative, $k\boldsymbol{p}$ is a vector in the opposite direction to $\boldsymbol{p}$ and k times as long.

When k is positive



When k is negative



Note

For any scalars k and l, and any vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ the following rules hold:

(i) $k(\boldsymbol{p} + \boldsymbol{q}) = k\boldsymbol{p} + k\boldsymbol{q}$

(ii) $(k + l)\boldsymbol{p} = k\boldsymbol{p} + l\boldsymbol{p}$

(iii) $k(l)\boldsymbol{p} = (kl)\boldsymbol{p}$

Solved Problems

Question 1

Given that $\overrightarrow{OA} = (3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$, $\overrightarrow{OB} = (7\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k})$ and $\overrightarrow{OC} = (3\boldsymbol{i} - 2\boldsymbol{j} - 7\boldsymbol{k})$, find

(i) $\overrightarrow{AB} - 2\overrightarrow{OC}$

(ii) $3\overrightarrow{BA} + 5\overrightarrow{CB}$

Suggested Solution

(i) $\overrightarrow{AB} - 2\overrightarrow{OC} = \left[\begin{pmatrix}7\\8\\-8\end{pmatrix} - \begin{pmatrix}3\\4\\-2\end{pmatrix}\right] - 2\begin{pmatrix}3\\-2\\-7\end{pmatrix}$

$= \begin{pmatrix}4\\4\\-6\end{pmatrix} - \begin{pmatrix}6\\-4\\-14\end{pmatrix}$

$= \begin{pmatrix}-2\\8\\8\end{pmatrix}$

(ii) $3\overrightarrow{BA} + 5\overrightarrow{CB} = 3\left[\begin{pmatrix}3\\4\\-2\end{pmatrix} - \begin{pmatrix}7\\8\\-8\end{pmatrix}\right] + 5\left[\begin{pmatrix}7\\8\\-8\end{pmatrix} - \begin{pmatrix}3\\-2\\-7\end{pmatrix}\right]$

$$= 3\begin{pmatrix} -4 \\ -4 \\ 6 \end{pmatrix} + 5\begin{pmatrix} 4 \\ 10 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 38 \\ 13 \end{pmatrix}$$

Question 2

Given that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ find $2\overrightarrow{AB} - \overrightarrow{OC} + 4\overrightarrow{CB}$

Suggested Solution

$$2\overrightarrow{AB} - \overrightarrow{OC} + 4\overrightarrow{CB} = 2\left[\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}\right] - \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + 4\left[\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}\right]$$

$$= 2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + 4\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ -21 \\ 17 \end{pmatrix}$$

Dot Product

It is also called a scalar product. It is a result of multiplying one vector by a second vector so as to produce a scalar.

Algebraic Definition

The dot product 2 vectors $\vec{p} = (p_1, p_2, p_3)$ and $\vec{q} = (q_1, q_2, q_3)$ in $\mathbb{R}^3$ $\vec{p}.\vec{q}$ is defined to be the scalar

$$\vec{p}.\vec{q} = p_1q_1 + p_2q_2 + p_3q_3.$$

Geometric Definition

The dot product 2 vectors $\vec{p} = (p_1, p_2, p_3)$and $\vec{q} = (q_1, q_2, q_3)$ in $\mathbb{R}^3$ $\vec{p}.\vec{q}$ is defined to be the scalar

$$\vec{p}.\vec{q} = |\vec{p}|.|\vec{q}|cos\theta,$$

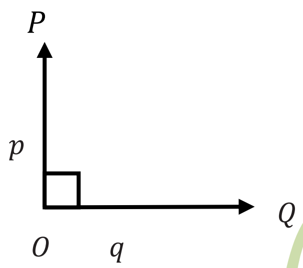
where θ is the angle between the vectors $\vec{p}$ and $\vec{q}$.

Proposition

Let $\vec{p}$ and $\vec{q}$ be non-zero vectors. The vectors, $\vec{p}$ and $\vec{q}$, are perpendicular to each other if and only if $\vec{p}.\vec{q} = 0$.

Orthogonal/Perpendicular vectors

- Two vectors are said to be orthogonal to one another if the angle between them is 90°.



- The dot product of perpendicular vectors is zero i.e.

$$\overrightarrow{OP}.\overrightarrow{OQ} = 0$$

Solved Problems

Question 1

Given that $\overrightarrow{OA} = (\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$, $\overrightarrow{OB} = (7\boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k})$ and $\overrightarrow{OC} = (3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$, show $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$.

Suggested Solution

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 7 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$ and

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Now:

$\overrightarrow{OA}.\overrightarrow{BC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.\begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$

$= -4 - 4 + 8$

$= 0$

Since $\overrightarrow{OA}.\overrightarrow{BC} = 0 \therefore \overrightarrow{OA}$ and $\overrightarrow{BC}$ are perpendicular.

Question 2

Find b such that $\begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 9 \\ 6 \end{pmatrix}$ are perpendicular.

Suggested Solution

Let $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 9 \\ 6 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}$

Two vectors are perpendicular if $\overrightarrow{OP}.\overrightarrow{OQ} = 0$

Now:

$\overrightarrow{OP}.\overrightarrow{OQ} = 0$

$\begin{pmatrix} -3 \\ 9 \\ 6 \end{pmatrix}.\begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix} = 0$

$\Rightarrow -3(1) + 9(3) + 6(b) = 0$

$\Rightarrow -3 + 27 + 6b = 0$

$\Rightarrow 24 + 6b = 0$

$\Rightarrow 6b = -24$

$\therefore b = -4$

Question 3

Given that the vectors $\vec{a} = (-1,2,3)$ and $\vec{b} = (3,1,4)$, find the dot product of $\vec{a}$ and $\vec{b}$.

Solution

$$\vec{a}.\vec{b} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$= -3 + 2 + 12$$

$$= 11.$$

Question 4

Determine if the vectors $\vec{a} = (5,-5,0)$ and $\vec{b} = (2,2,3)$ are perpendicular to each other.

Solution

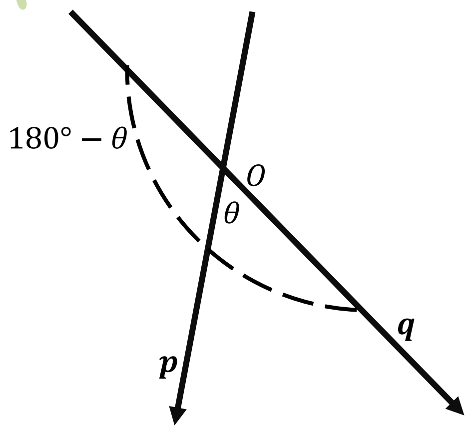
$$\vec{a}.\vec{b} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}.\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$= 10 - 10 + 0$$

$$= 0$$

Since $\vec{a}.\vec{b} = 0$ therefore $\vec{a}$ and $\vec{b}$, are perpendicular to each other

The angle between two vectors



- The angle between the two vectors is calculated using the fact that the dot product of vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ is equal the product of the magnitude of vector $\boldsymbol{p}$ and magnitude of vector $\boldsymbol{q}$ and cosine of angle θ. This implies that $\vec{p}.\vec{q} = |\vec{p}|.|\vec{q}| \cos \theta$
- This implies that:

$$\cos \theta = \frac{\vec{p}.\vec{q}}{|\vec{p}|.|\vec{q}|}$$

- Therefore, if θ is the acute angle between vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, then θ is given by:

$$\boldsymbol{\theta = cos^{-1}\left(\frac{p.q}{|p||q|}\right)}$$

NOTE

When the calculated angle is obtuse and the required angle is acute then to find the acute angle we simply subtract the obtuse angle from 180°.

$$\boldsymbol{Acute\ angle = 180° - Obtuse\ Angle}$$

WORKED EXAMPLES

Example 1

Find the angle between the vectors $b_1 = (3, -1, 2)$ and $b_2 = (-1, 2, 3)$.

Suggested Solution

$$\cos\theta = \frac{\boldsymbol{b_1.b_2}}{|\boldsymbol{b_1}||\boldsymbol{b_2}|}$$

$$\cos\theta = \frac{\begin{pmatrix}3\\-1\\2\end{pmatrix}\begin{pmatrix}-1\\2\\3\end{pmatrix}}{\sqrt{3^2+(-1)^2+2^2}\sqrt{(-1)^2+2^2+3^2}}$$

$$\cos\theta = \frac{-3-2+6}{\sqrt{14}\sqrt{14}}$$

$$\cos\theta = \frac{1}{14}$$

$$\theta = \cos^{-1}\left(\frac{1}{14}\right) = 85.90395624° = 86°$$

<u>Example 2</u>

Find the acute angle between vectors $\begin{pmatrix}-1\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\2\\-2\end{pmatrix}$.

<u>Suggested Solution</u>

Let $\boldsymbol{p} = \begin{pmatrix}-1\\-1\\3\end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$

$$\cos\theta = \frac{\boldsymbol{p}.\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$$

$$\cos\theta = \frac{\begin{pmatrix}-1\\-1\\3\end{pmatrix}\begin{pmatrix}1\\2\\-2\end{pmatrix}}{\sqrt{(-1)^2+(-1)^2+3^2}\sqrt{1^2+2^2+(-2)^2}}$$

$$\cos\theta = \frac{-9}{3\sqrt{11}}$$

$$\cos\theta = \frac{-3}{\sqrt{11}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{11}}\right) = 154.7505982°(Obtuse\ angle).$$

$$\therefore Acute\ angle = 180° - 154.7505982°$$
$$= 25.249402°$$
$$= 25°.$$

Follow Up Questions

Question 1

Find the angle between the vector $\begin{pmatrix}1\\4\\0\end{pmatrix}$ and vector $\begin{pmatrix}2\\1\\1\end{pmatrix}$ Answer: 53.6°

Question 2

Determine the angle between vectors $\begin{pmatrix}3\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\-1\end{pmatrix}$ Answer: 49.8°

Question 3

Find the acute angle between vectors $\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\3\\0\end{pmatrix}$. Answer: 54°

Question 4

Find the angle between vector $\begin{pmatrix}2\\8\\-5\end{pmatrix}$ and vector $\begin{pmatrix}3\\7\\-4\end{pmatrix}$. Answer: 8.7

Area of plane shapes using the dot product

The geometrical approach for the dot product of 2 vectors $\vec{p} = (p_1, p_2, p_3)$and $\vec{q} = (q_1, q_2, q_3)$ in $\mathbb{R}^3$ $\vec{p}.\vec{q}$ is defined to be the scalar

$$\vec{p}.\vec{q} = |\vec{p}\,|.|\vec{q}|cos\theta,$$

where θ is the angle between the vectors $\vec{p}$ and $\vec{q}$.

Area of triangle

The area of triangle is given by:

$$\text{Area of } PQR = \frac{1}{2}|\vec{p}\,|.|\vec{q}|sin\theta$$

or

$$\text{Area of } PQR = \frac{1}{2}|\vec{p}\,|.|\vec{q}| \text{ if } \vec{p}.\vec{q} = 0$$

Area of parallelogram

The area of parallelogram is given by:

Area of $PQRS = |\vec{p}\,|.|\vec{q}|sin\theta$

or

Area of $PQR = |\vec{p}\,|.|\vec{q}|$ if $\vec{p}.\vec{q} = 0$

Worked Problem

Question 1

The position vectors of the points A, B and C relative to the origin O are:

$\overrightarrow{OA} = 4\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$, $\overrightarrow{OB} = 8\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$ and $\overrightarrow{OC} = 2\boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k}$, respectively.

Find

(i) The scalar product $\overrightarrow{BA}.\overrightarrow{BC}$

(ii) The area of triangle ABC.

Suggested Solution

$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix}$; $\overrightarrow{OB} = \begin{pmatrix} 8 \\ 4 \\ -3 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix}$

(i) $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -2 \end{pmatrix}$

Now

$\overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}\,||\overrightarrow{BC}|(CosA\hat{B}C)$

$= 4\sqrt{56}\,(CosA\hat{B}C)$

Aside

$CosA\hat{B}C = \dfrac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}|.|\overrightarrow{BC}|}$

$$= \frac{24}{4\sqrt{56}}$$

$$\Rightarrow \overrightarrow{BA}.\overrightarrow{BC} = 4\sqrt{56}\left(\frac{24}{4\sqrt{56}}\right)$$

$$= 24$$

(ii) $Sin^2 A\hat{B}C = 1 - \left(Cos^2 A\hat{B}C\right)$

$$Sin^2 A\hat{B}C = 1 - \left(\frac{3}{\sqrt{14}}\right)^2$$

$$Sin^2 A\hat{B}C = 1 - \frac{9}{14}$$

$$Sin^2 A\hat{B}C = \frac{5}{14}$$

$$\therefore SinA\hat{B}C = \sqrt{\frac{5}{14}}$$

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right| SinA\hat{B}C$$

$$= \frac{1}{2} \times \sqrt{16} \times \sqrt{56} \times \sqrt{\frac{5}{14}}$$

$$= 4\sqrt{5} units^2.$$

Question 2

The points A, B and C have position vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}2\\2\\0\end{pmatrix}$ and $\begin{pmatrix}4\\0\\0\end{pmatrix}$ respectively.

The point O is the origin and the point M is the mid-point of AB.

a) Find the vectors $\overrightarrow{OM}$ and $\overrightarrow{CM}$.

b) Calculate $O\hat{M}C$. Hence find the area of triangle OMC.

Suggested Solution

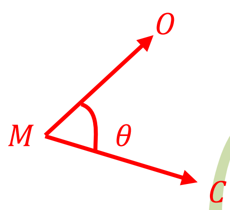
$\overrightarrow{OA} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}2\\2\\0\end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix}4\\0\\0\end{pmatrix}$

a) $$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}2\\2\\0\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\0\\-3\end{pmatrix}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-3\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\2\\\frac{3}{2}\end{pmatrix}$$

$$\overrightarrow{CM} = \overrightarrow{OM} - \overrightarrow{OC} = \begin{pmatrix}\frac{3}{2}\\2\\\frac{3}{2}\end{pmatrix} - \begin{pmatrix}4\\0\\0\end{pmatrix} = \begin{pmatrix}-\frac{5}{2}\\2\\\frac{3}{2}\end{pmatrix}$$

b)



$$Cos\left(O\widehat{M}C\right) = \frac{\overrightarrow{MO}.\overrightarrow{MC}}{|\overrightarrow{MO}|.|\overrightarrow{MC}|}$$

$$= \frac{\begin{pmatrix}-\frac{3}{2}\\-2\\-\frac{3}{2}\end{pmatrix}.\begin{pmatrix}\frac{5}{2}\\-2\\-\frac{3}{2}\end{pmatrix}}{\sqrt{\frac{17}{2}}\sqrt{\frac{25}{2}}}$$

$$\Rightarrow O\widehat{M}C = cos^{-1}\left(\frac{2.5}{10.30776406}\right)$$

$$= 75.96375653°$$

$$= 76°.$$

Hence the area of triangle $OMC = \frac{1}{2}|\overrightarrow{MO}|.|\overrightarrow{MC}|\text{Sin}(O\widehat{M}C)$

$$\Rightarrow \text{area } = \frac{1}{2}\sqrt{\frac{17}{2}}\sqrt{\frac{25}{2}}\ Sin75.96375653°$$

$$= 5\text{units}^2$$

FOLLOW UP QUESTION

ZIMSEC NOVEMBER 2003 PAPER 1

Given $P(3,2,1)$, $Q(0,4,2)$ and O the origin, find the scalar product of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ and

show that $cosP\hat{O}Q = \sqrt{\frac{5}{14}}$. **[3]**

Hence find the exact are of triangle OPQ. **[3]**

SOLVED EXAMINATION TYPE QUESTIONS

Question 1

Points A and B have positon vectors $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$, respectively.

a) Given that $\overrightarrow{OC} = \overrightarrow{AB}$ and $\overrightarrow{AD} = \overrightarrow{CB}$, find the position vectors of C and D.

b) Hence find the angle between $\overrightarrow{OC}$ and $\overrightarrow{AD}$.

Suggested Solution

$\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

a) $\overrightarrow{OC} = \overrightarrow{AB} \Rightarrow \overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$

Also

$\overrightarrow{AD} = \overrightarrow{CB} \Rightarrow \overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OC}$

$$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} - \overrightarrow{OC}$$

$$\therefore \overrightarrow{OD} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

b) $$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore Cos\ \theta = \frac{\overrightarrow{OC}.\overrightarrow{AD}}{|\overrightarrow{OC}|.|\overrightarrow{AD}|} = \frac{\begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{6}\sqrt{17}}$$

$$\Rightarrow \theta = cos^{-1}\left(\frac{-6}{\sqrt{6}\sqrt{17}}\right)$$

$$= 126.4476824°$$

$$= 126° \ \ (to\ the\ nearest\ degree)$$

Question 2

The position vectors of the points A, B and C relative to the origin O are:

$\boldsymbol{a} = 4\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = 8\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k}$ respectively.

Find

(i) $\overrightarrow{AB}$,

(ii) the exact value of $cos\ A\hat{B}C$,

(iii) the exact area of triangle ABC.

Suggested Solution

$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix}$; $\overrightarrow{OB} = \begin{pmatrix} 8 \\ 4 \\ -3 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix}$

(i) $$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

(ii) $Cos A\hat{B}C = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$

Now $\overrightarrow{BA} = \begin{pmatrix}-4\\0\\0\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}-6\\4\\-2\end{pmatrix}$

$$\therefore Cos A\hat{B}C = \frac{\begin{pmatrix}-4\\0\\0\end{pmatrix}.\begin{pmatrix}-6\\4\\-2\end{pmatrix}}{\sqrt{16} \quad \sqrt{56}}$$

$$= \frac{3\sqrt{14}}{14}$$

(iii) Using $Sin^2\theta + Cos^2\theta = 1$

Now

$$Sin^2 A\hat{B}C = 1 - \left(Cos^2 A\hat{B}C\right)$$

$$= 1 - \left(\frac{3\sqrt{14}}{14}\right)^2$$

$$= \frac{5}{14}$$

$$\therefore Sin A\hat{B}C = \sqrt{\frac{5}{14}}.$$

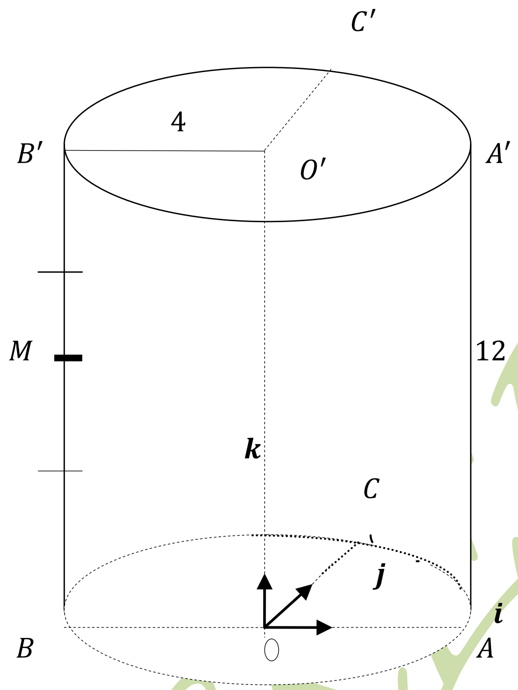
Thus area $= \frac{1}{2} ab Sin A\hat{B}C$

$$= \frac{1}{2} \times \sqrt{16} \times \sqrt{56} \times \sqrt{\frac{5}{14}}$$

$$= 4\sqrt{5} units^2.$$

Question 3

Cambridge June 2002 Paper 1



The diagram shows a solid cylinder standing on a horizontal circular base, centre O and radius 4 units. The line BA is a diameter and the radius OC is at 90° to OA. Points O', A', B' and C' lie on the upper surface of the cylinder such that OO', AA', BB' and CC' are all vertical and of the length 12 units. The mid-point of BB' is M.

Unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are parallel to OA, OC and OO' respectively.

i. Express the vectors $\overrightarrow{MO}$ and $\overrightarrow{MC'}$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

ii. Hence find the angle $O\hat{M}C'$

Suggested Solution

i. $\overrightarrow{MO} = \overrightarrow{MB} + \overrightarrow{BO}$

$= -6\boldsymbol{k} + 4\boldsymbol{i}$

$= 4\boldsymbol{i} - 6\boldsymbol{k}$

$$\overrightarrow{MC'} = \overrightarrow{MB'} + \overrightarrow{B'O'} + \overrightarrow{O'C'}$$

$$= 6\boldsymbol{k} + 4\boldsymbol{i} + 4\boldsymbol{j}$$

$$= 4\boldsymbol{i} + 4\boldsymbol{k} + 6\boldsymbol{k}$$

ii. $Cos O\hat{M}C' = \dfrac{\overrightarrow{MO}.\overrightarrow{MC'}}{|\overrightarrow{MO}||\overrightarrow{MC'}|}$

Now $\overrightarrow{MO} = \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix}$ and $\overrightarrow{MC'} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$

$$Cos O\hat{M}C' = \frac{\begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix}.\begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}}{\sqrt{52} \quad \sqrt{68}}$$

$$O\hat{M}C' = Cos^{-1}\left(\frac{-20}{\sqrt{52}\sqrt{68}}\right)$$

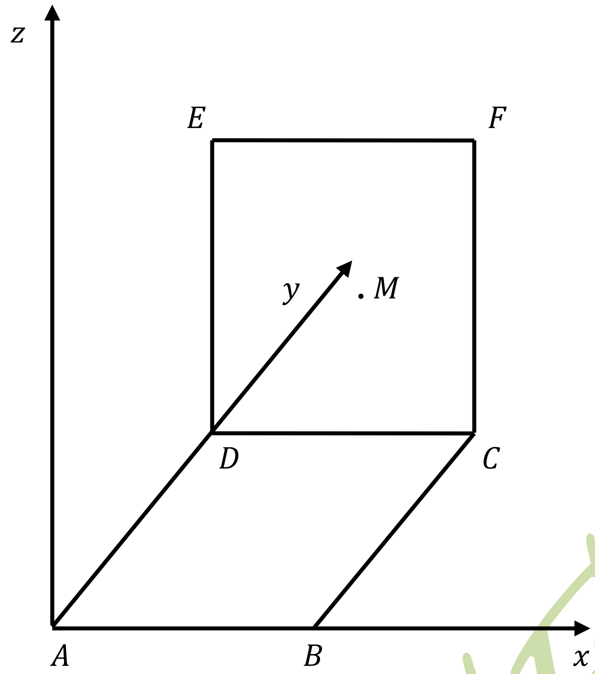
$$O\hat{M}C' = 109.6538241°$$

$$\therefore O\hat{M}C' = 109.7° \ (to\ the\ nearest\ degree)$$

Question 4

ZIMSEC NOVEMBER 2008 PAPER 1

The sides of a square $ABCD$ are each of length $4\,cm$. The rectangle $CDEF$ lies in a plane perpendicular to the plane $ABCD$ and $DE = CF = 8\,cm$. M is the centre of the rectangle $CDEF$. (See diagram).



Taking the point A as the origin and unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ in the directions $\overrightarrow{AB}, \overrightarrow{AD}$ and $\overrightarrow{DE}$, calculate the angle between the line AM and the line BE.

Suggested Solution

$$\overrightarrow{AM} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DE} + \frac{1}{2}\overrightarrow{DC}$$
$$= 4\boldsymbol{j} + 4\boldsymbol{k} + 2\boldsymbol{i}$$
$$= 2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$$

and

$$\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DE}$$
$$= -4\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}$$

Now:

$$Cos A\hat{B}C = \frac{\overrightarrow{AM}.\overrightarrow{BE}}{|\overrightarrow{AM}||\overrightarrow{BE}|}$$

Now $\overrightarrow{AM} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$ and $\overrightarrow{BE} = \begin{pmatrix} -4 \\ 4 \\ 8 \end{pmatrix}$

$$CosA\hat{B}C = \frac{\begin{pmatrix}2\\4\\4\end{pmatrix}.\begin{pmatrix}-4\\4\\8\end{pmatrix}}{\sqrt{48} \quad \sqrt{96}}$$

$$A\hat{B}C = Cos^{-1}\left(\frac{40}{6\sqrt{96}}\right)$$

$$A\hat{B}C = 47.12401133°$$

$$\therefore A\hat{B}C = 47° \ (to\ the\ nearest\ degree)$$

PAST EXAMINATION QUESTIONS

UCLES JUNE 1997 PAPER 1

The position vectors of A, B and C with respect to a fixed origin O are $2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ respectively. Find unit vectors in the directions of $\overrightarrow{CA}$ and $\overrightarrow{CB}$. **[4]**

Calculate angle ACB in degrees, correct to 1 decimal place. **[3]**

UCLES NOVEMBER 1997 PAPER1

The points A, B, C have position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ respectively, given by

$$\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(i) Express $\boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{c} - \boldsymbol{b}$ as column vectors, and hence describe precisely the position of $\boldsymbol{B}$ in relation to the points A and C. **[3]**

(ii) Calculate the angle between the directions of $\overrightarrow{OB}$ and $\overrightarrow{AB}$, where O is the origin, giving your answer correct to the nearest degree. **[3]**

UCLES JUNE 1998 PAPER 1

Two insects A and B are crawling on the walls of a room, with A starting from the ceiling. The floor is horizontal and forms the x-y plane, and the z-*axis* is vertically upwards. Relative to the origin O, the position vectors of the insects at time t seconds $(0 \leq t \leq 10)$ are

$$\overrightarrow{OA} = \boldsymbol{i} + 3\boldsymbol{j} + \left(4 - \frac{1}{10}t\right)\boldsymbol{k}, \qquad \overrightarrow{OB} = \left(\frac{1}{5}t + 1\right)\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k},$$

where the unit of distance is the metre.

(i) Write down the height of the room. **[1]**

(ii) Show that the insects move in such a way that the angle $B\hat{O}A = 90°$. **[3]**

(iii) For each insect, write down a vector to represent its displacement between $t = 0$ and $t = 10$, and show that these displacement are perpendicular to each other. **[3]**

(iv)Write down the expressions for the vector $\overrightarrow{AB}$ and for $|\overrightarrow{AB}|^2$ and hence find the minimum distance between the insects, correct to 3 significant figures. **[6]**

<u>UCLES NOVEMBER 1998 PAPER 1</u>

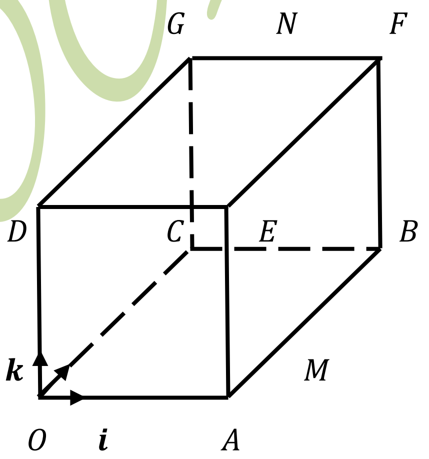
Points A, B and C have coordinates $(4, 0,4)$, $(0,6,6)$ and $(0,0, c)$ respectively. The point O is the origin, and the mid-point of AB is M.

(i) Find the vectors $\overrightarrow{OM}$ and $\overrightarrow{CM}$. **[2]**

(ii) Given that $c = 5$, calculate angle OMC. **[3]**

(iii)Find the value of c for which angle OMC is a right angle. **[2]**

<u>UCLES NOVEMBER 2000 PAPER 1</u>

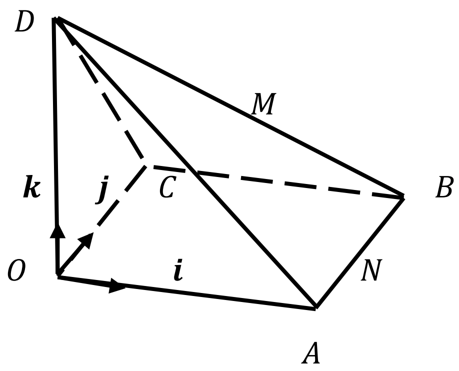


In the diagram $OABCDEFG$ is a cube in which the length of each edge is 2 units. Unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ respectively. The mid-points of AB and FG are M and N respectively.

(i) Express each of the following vectors $\overrightarrow{ON}$ and $\overrightarrow{MG}$ in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. **[3]**

(ii) Show that the acute angle between the directions of $\overrightarrow{ON}$ and $\overrightarrow{MG}$ is 63.6°, correct to the nearest $\overrightarrow{ON}$ and 0.1°. **[5]**

UCLES NOVEMBER 2001 PAPER 1



The diagram shows a pyramid $DOABC$. Taking unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as shown, the position vectors of A, B, C, D are given by

$$\overrightarrow{OA} = 4\boldsymbol{i}, \quad \overrightarrow{OB} = \boldsymbol{4i} + \boldsymbol{2j}, \quad \overrightarrow{OC} = 2\boldsymbol{j}, \quad \overrightarrow{OD} = \boldsymbol{6k}$$

The midpoints of BD and AB are M and N respectively.

(i) Find the vector $\overrightarrow{MN}$ and the angle between the directions of $\overrightarrow{MN}$ and $\overrightarrow{OB}$. **[4]**

(ii) The point P lying on OD has position vector $p\boldsymbol{k}$ and is such that angle PMN is a right angle. Find the value of p. **[3]**

ZIMSEC NOVEMBER 2002 PAPER 1

Points A, B, C and D have coordinates $(5, 7, 3)$; $(7, 8, 0)$; $(8,10,3)$ and $(6, 9, 6)$ respectively.

(a) Find

(i) $\overrightarrow{AB}$ and $\overrightarrow{CD}$, and hence state clearly two facts relating lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$, **[4]**

(ii) the length of BC. **[2]**

(b) Evaluate the scalar product $\overrightarrow{AC}.\overrightarrow{BD}$, and deduce a relationship between AC and CD. **[3]**

ZIMSEC NOVEMBER 2003 PAPER 1

Given $P(3, 2, 1)$, $Q(0, 4, 2)$ and O the origin, find the scalar product of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ and

show that $cosP\hat{O}Q = \sqrt{\frac{5}{14}}$. **[3]**

Hence find the exact are of triangle OPQ. **[3]**

ZIMSEC JUNE 2004 PAPER 1

The position vectors of the points A, B and C relative to a fixed origin O are

$\boldsymbol{a} = 4i + 4\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = 8\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 8\boldsymbol{j} - 5\boldsymbol{k}$ respectively. Find

(i) $\overrightarrow{AB}$, **[2]**

(ii) the exact value of cos $A\hat{B}C$, **[3]**

(iii)the exact area of triangle ABC. **[3]**

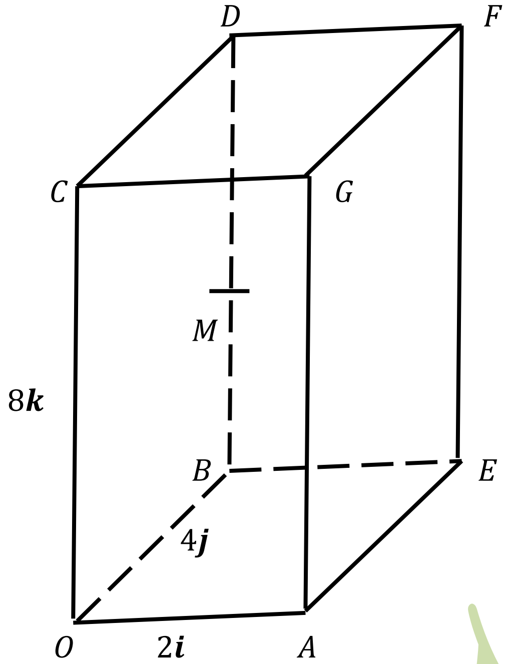
ZIMSEC NOVEMBER 2004 PAPER 1

The position vectors of the points A, B and C are $3\boldsymbol{i} - \boldsymbol{j}$, $2\boldsymbol{i} + 5\boldsymbol{j}$ and $8\boldsymbol{i} + 3\boldsymbol{j}$ respectively.

(i) Calculate $\overrightarrow{AB}$, where M is the mid-point of BC **[2]**

(ii) Calculate the cosine of the angle between $\overrightarrow{AM}$ and $\overrightarrow{BC}$. **[2]**

ZIMSEC NOVEMBER 2005 PAPER 1



The diagram shows a rectangular box $OBDCGAEF$ in which the length OA is 2 units, the length OB is 4 units and the length OC is 8 units. Unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are taken along the edges OA, OB and OC respectively.

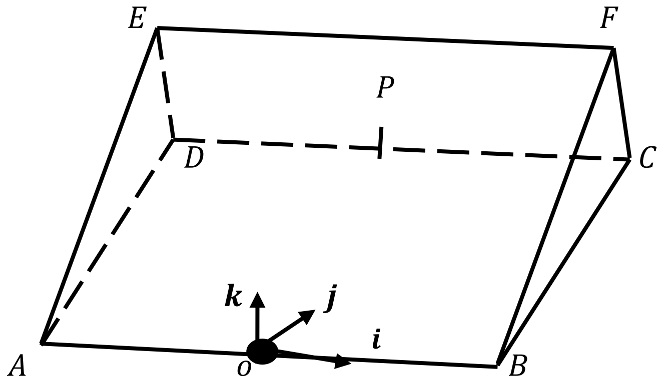
(i) Find the position vector of the mid-point M of BD. **[1]**

(ii) Find a unit vector in the direction of $\overrightarrow{AM}$. **[3]**

(iii) The point L inside the box has position vector $\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

Calculate the angle LOA in degrees, correct to 1 decimal place. **[4]**

ZIMSEC JUNE 2006 PAPER 1



The diagram shows a solid triangular prism standing on a horizontal rectangular base $ABCD$. The rectangular face $CDEF$ is vertical. The edge AB has mid-point O, and the edge DC has midpoint P. Unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are taken parallel to edges AB, BC and CF respectively. The rectangular base has length $AB = 10$ units and width $BC = 8$ units. $\overrightarrow{CF} = 6\boldsymbol{k}$. Calculate

(i) $\overrightarrow{OC}$, **[1]**

(ii) $\overrightarrow{EP}$, **[1]**

(iii)angle FAP. **[5]**

ZIMSEC NOVEMBER 2006 PAPER 1

Given that the position vectors of points A, B and C are $(6\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$; $(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$ and $(14\boldsymbol{i} + 16\boldsymbol{k})$ respectively,

(i) find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and state the exact value of $|\overrightarrow{AB}|$, **[4]**

(ii) state a precise relationship between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Hence draw a sketch to show the relative arrangement of points A, B and C in space. **[3]**

(iii) $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{AB}$. Given that the position vector of D is $(\boldsymbol{i} - d\boldsymbol{j} + 9\boldsymbol{k})$, find the value of d

Hence determine the exact area of triangle ABD. **[5]**

ZIMSEC NOVEMBER 2007 PAPER 1

Two birds, P and Q fly such that their position vectors with respect to an origin O is give by

$$\overrightarrow{OP} = (2t+3)\boldsymbol{i} + (t-1)\boldsymbol{j} + 3t\boldsymbol{k}$$

$$\overrightarrow{OQ} = (t-2)\boldsymbol{i} + (3t+1)\boldsymbol{j} + (t+2)\boldsymbol{k}$$

for $0 \leq t \leq 10$, where $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors of magnitude 1 metre in the x, y and z directions respectively.

(a) For the time $t = 0$,

  (i) calculate the distance between the two birds, **[3]**

  (ii) find the position vector of the point mid-way between the two birds. **[1]**

(b) Find the value of t for which $P\hat{O}Q = 90°$, giving your answer to 2 significant figures. **[3]**

ZIMSEC JUNE 2009 PAPER 1

The points A, B and C have position vectors $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and

$\boldsymbol{c} = 5\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ respectively relative to the origin O

(a) Evaluate the scalar product $(\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{c} - \boldsymbol{b})$. Hence calculate the size of the angle $A\hat{B}C$, giving your answer to the nearest $0.1°$ **[5]**

(b) Given that $ABCD$ is a parallelogram, determine

  (i) the position vector of D, **[2]**

  (ii) the area of $ABCD$ giving your answer in exact form. **[4]**

ZIMSEC NOVEMBER 2009 PAPER 1

The position vectors of points A, B and P are given by $\begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}; \begin{pmatrix} -1 \\ 11 \\ -1 \end{pmatrix}; \begin{pmatrix} 2p+1 \\ -5p+6 \\ 3p+2 \end{pmatrix}$

(a) Show that A, B and P are all collinear for all values of p. **[3]**

(b) Given that the point C has position vector $\begin{pmatrix} -2 \\ -5 \\ 11 \end{pmatrix}$, find in terms of p,

(i) an expression for $\overrightarrow{CP}$, **[1]**

(ii) the value of $\overrightarrow{CP}.\overrightarrow{AB}$. **[2]**

(c) (i) Given also that CP is perpendicular to AB, find the value of p. **[2]**

(ii) Hence, or otherwise, obtain the shortest distance from C to the line AB. **[2]**

ZIMSEC NOVEMBER 2010 PAPER 1

The position vectors of points A and B with respect to the origin O, are given by

$\overrightarrow{OA} = \boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$,

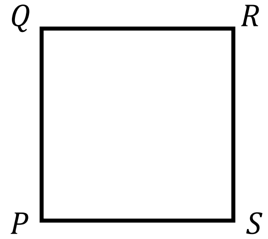
$\overrightarrow{OB} = -4\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$.

Show that $cos(AOB) = \frac{4}{\sqrt{38}}$. **[2]**

Hence, or otherwise, find the position vector of the point P on B such that

AP is perpendicular to OB. **[4]**

ZIMSEC JUNE 2011 PAPER 1



In the diagram above, $PQRS$ is a square. The position vector of Q relative to an origin, O, is given by $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and the displacement vectors $\overrightarrow{PR} = \begin{pmatrix} -3 \\ 3 \\ 12 \end{pmatrix}$ and $\overrightarrow{SQ} = \begin{pmatrix} 5 \\ -11 \\ 4 \end{pmatrix}$

(a) Find

(i) $\overrightarrow{PQ}$,

(ii) the position vectors of P, R and S. **[7]**

(b) Calculate the area of the square $PQRS$. **[2]**

ZIMSEC NOVEMBER 2011 PAPER 1

The points O, P, Q and R form a parallelogram, where O is the origin. The position vectors of P and R are $\boldsymbol{p} = \boldsymbol{i} + \lambda\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{r} = 4\boldsymbol{i} + 2\boldsymbol{k}$ respectively; where λ is a positive constant. The angle $O\hat{P}R$ is a right angle.

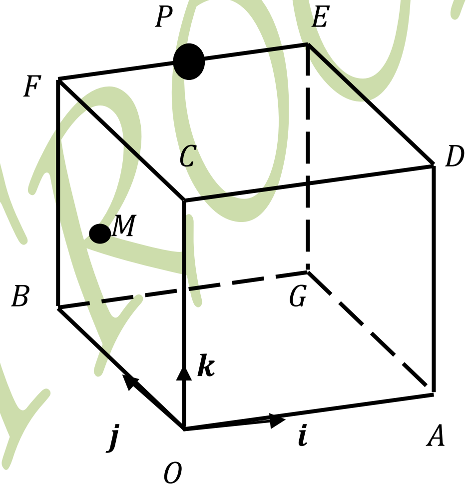
Find

(a) the value of λ, **[3]**

(b) the position vector of Q, **[1]**

(c) the exact area of the parallelogram $OPQR$. **[3]**

ZIMSEC JUNE 2012 PAPER 1



The diagram shows a cube of length 8 units. The unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are parallel to $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively. M is the point of intersection of $\overrightarrow{OF}$ and $\overrightarrow{BC}$. P is the midpoint of $\overrightarrow{EF}$.

Find

(i) a unit vector parallel to $\overrightarrow{MP}$, **[4]**

(ii) angle DMP. **[4]**

ZIMSEC NOVEMBER 2012 PAPER 1

The points A, B and C have position vectors $8\boldsymbol{j} - 4\boldsymbol{k}$; $p\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $-2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$ respectively.

(a) Calculate the

(i) unit vector parallel to $\overrightarrow{AC}$,

(ii) positive value of p such that BA is perpendicular to BC. **[7]**

(b) Hence or otherwise, find the area of triangle ABC for the value of p in (**a**)(**i**). **[3]**

ZIMSEC JUNE 2013 PAPER 1

The position vectors of points A, B and C relative to the origin O are $4\boldsymbol{i} - 9\boldsymbol{j} - \boldsymbol{k}$;

$\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$ and $\lambda\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ respectively.

(a) Find the unit vector parallel to $\overrightarrow{AB}$,

Find the value of λ such that A, B and C are collinear. **[5]**

(b) Calculate the angle AOB . **[3]**

ZIMSEC NOVEMBER 2013 PAPER 1

The position vectors of M and N relative to the origin O are

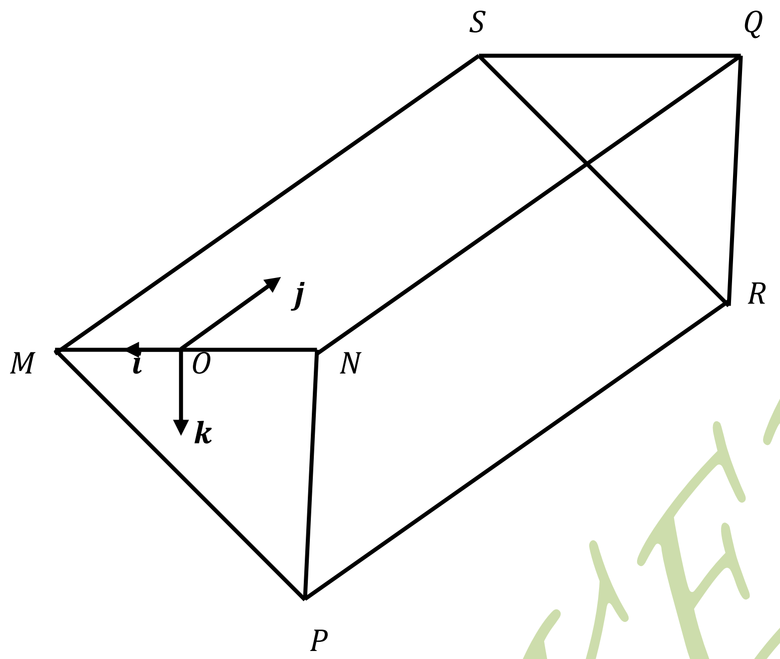
$(2p - 2)\boldsymbol{i} + (1 - p)\boldsymbol{j} + (p - 2)\boldsymbol{k}$ and $(p + 2)\boldsymbol{i} + p\boldsymbol{j} + 2p\boldsymbol{k}$ respectively.

Find the values of p when

i. $|\overrightarrow{OM}| = |\overrightarrow{ON}|$, **[3]**

ii. $M\hat{O}N = 90°$. **[3]**

ZIMSEC NOVEMBER 2014 PAPER 1



The diagram shows a triangular prism $MNPRQS$ with $MP = NP = 5$ units. O is the midpoint of MN.

The unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$ are taken along $\overrightarrow{OM}$ and $\overrightarrow{OP}$ respectively, and $\boldsymbol{j}$ is taken parallel to MS.

Given that $\overrightarrow{OR} = 9\boldsymbol{j} + 4\boldsymbol{k}$,

find

(i) the unit vector in the direction $\overrightarrow{PS}$, **[3]**

(ii) the angle SPQ. **[3]**

ZIMSEC JUNE 2015 PAPER 1

The points A, B and C have position vectors $\boldsymbol{a} = 3\boldsymbol{j} - p\boldsymbol{j} - \boldsymbol{k}$; $\boldsymbol{b} = 5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and

$\boldsymbol{c} = 7\boldsymbol{i} + (2 + \sqrt{5})\boldsymbol{j} - \boldsymbol{k}$ respectively.

(i) Find the exact value of p if

1. $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$,
2. $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$,

. **[5]**

(ii) 1. Find the unit vector in the direction of $\overrightarrow{BC}$,

2. Hence write down a vector parallel to $\overrightarrow{BC}$ with modulus 15.

[3]

<u>ZIMSEC JUNE 2016 PAPER 1</u>

(a) Relative to the origin O, the position vectors of P, Q and R are $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$ respectively, Find $P\hat{Q}R$. **[4]**

(b) The vector $W = \begin{pmatrix} m - \frac{1}{2} \\ m + \frac{1}{2} \\ m\sqrt{6} \end{pmatrix}$

(i) If $\boldsymbol{W}$ is a unit vector, find the possible values of m. **[2]**

(ii) The vector $\boldsymbol{V} = \begin{pmatrix} m - \frac{1}{2} \\ m + \frac{1}{2} \\ m\sqrt{6} \end{pmatrix}$.

If $\boldsymbol{W}$ is normal to $\boldsymbol{V}$, find possible values of m. **[4]**

(iii)The vector $\boldsymbol{U} = \begin{pmatrix} m - \frac{1}{2} \\ n \\ 2\sqrt{6} \end{pmatrix}$.

Given that $\boldsymbol{W}$ is parallel to $\boldsymbol{U}$, find the values of m and n. **[4]**

ZIMSEC NOVEMBER 2016 PAPER 1

Relative to the origin O the position vectors of A and B are

$\overrightarrow{OA} = 3\boldsymbol{i} + 4\boldsymbol{j} + (2-b)\boldsymbol{k}$

$\overrightarrow{OB} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

(i) Find the unit vector in the direction of $\overrightarrow{OB}$ **[2]**

(ii) Find the value of b, if $B\hat{O}A = 90°$. **[2]**

(iii) Write down an expression for $\overrightarrow{AB}$. **[3]**

(iv)Hence find the value of b if $|\overrightarrow{AB}|^2$ and is minimum. **[3]**

ZIMSEC 2017 JUNE PAPER 1

The coordinates of P, Q and R are $(1, 2, 1)$, $(4, 7, 8)$ and $(6, 4, 12)$ respectively.

(a) S is a point such that $PQRS$ is a parallelogram.

Find the coordinates of S. **[3]**

(b) The points M ad N are the midpoints of P, Q and QR respectively.

Find the unit vector in the direction of $\overrightarrow{MN}$. **[3]**

ZIMSEC NOVEMBER 2017 PAPER 1

The position vectors of points A, B and C relative to the origin O are $\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$;

$3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$ and $p\boldsymbol{i} - p\boldsymbol{j} + (p-1)\boldsymbol{k}$ respectively.

Find

(a) a unit vector parallel to $\overrightarrow{AB}$, **[3]**

(b) the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, **[3]**

(c) the value of p for which $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{OC}$. **[2]**

ZIMSEC JUNE 2018 PAPER 1

Relative to the origin O, the position vectors of points A, B and C are

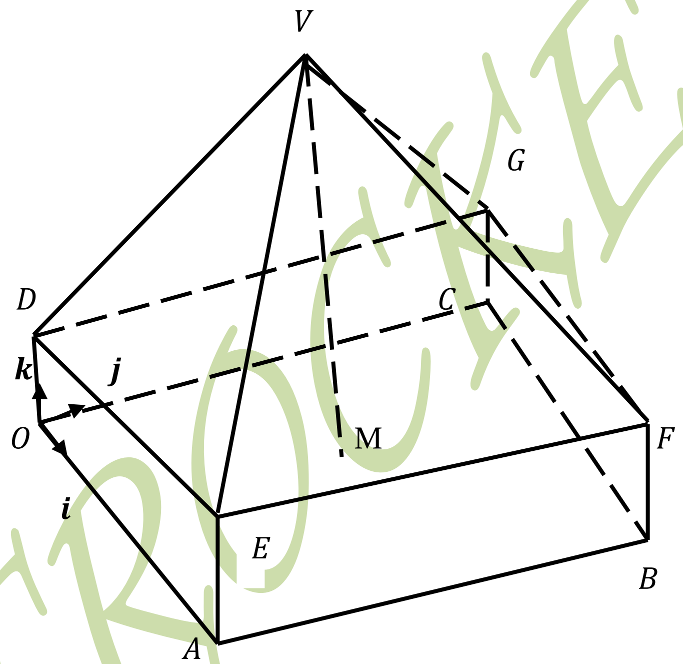
$6\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$; $\mathbf{2}\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $4\boldsymbol{i} + 5\boldsymbol{k}$, respectively.

Find

(a) Calculate angle ABC. **[3]**

(b) Determine the exact value of the area of triangle ABC. **[3]**

ZIMSEC JUNE 2019 PAPER 1

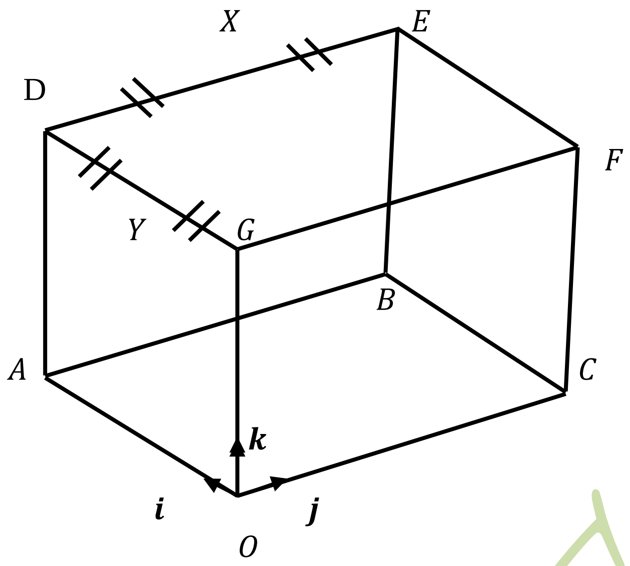


The diagram shows a plan of a building whose floor is a rectangle $OABC$ and whose roof is in the form of a pyramid $DEFGV$, with $OA = 4m$, $OC = 8m$, $OD = 2m$ and $MV = 5m$. M is the midpoint where the diagonals OB and AC intersect. Taking O as the origin and $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the directions $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively,

(i) write position vectors of B and V, **[2]**

(ii) find the exact value of the angle between $\overrightarrow{OV}$ and $\overrightarrow{OB}$. **[4]**

ZIMSEC NOVEMBER 2019 PAPER 1



$OABCDEFG$ is a perfect cube of edge 10 units. The unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are along OA, OC and OG, respectively. X and Y are the midpoints on ED and GD respectively.

Find

(i) $\overrightarrow{OX}$ in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, **[1]**

(ii) a unit vector parallel to $\overrightarrow{OE}$. **[2]**

(iii) $X\hat{O}Y$. **[4]**

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

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