# Vector 2

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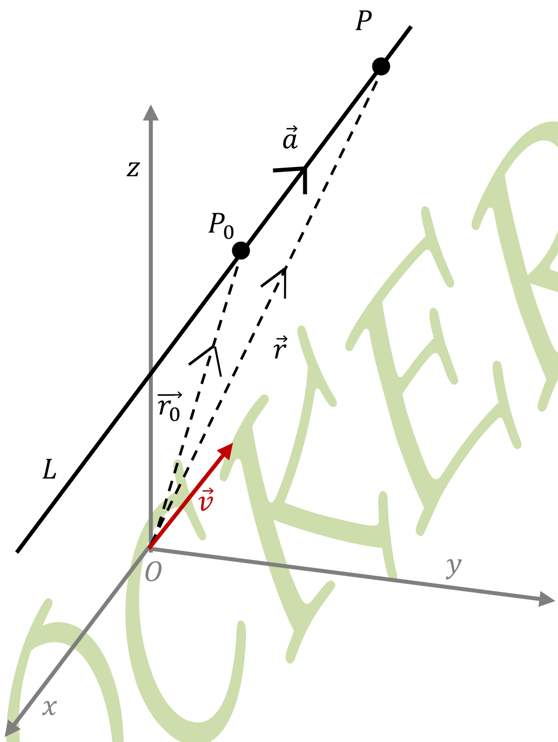
# SYLLABUS (6042) REQUIREMENTS

- find vector and Cartesian equations of a straight line in 3D
- find cross product of given vectors
- use cross product to find area of plane shapes
- find vector, parametric and Cartesian equations of a plane
- find the angle between a line and a plane, two lines and two planes
- determine whether two lines are parallel, intersecting or skewed
- determine whether two lines intersect
- find the perpendicular distance from a point to a straight line and the coordinates of the foot of the perpendicular
- determine whether a line lies in, is parallel to or intersects a plane
- find the point of intersection of a line and a plane
- find the line of intersection of two planes
- find the perpendicular distance from a point to a plane
- solve problems involving lines and planes

# VECTORS IN $\mathbb{R}^3$

## Vector equation of a line

- It is an equation that represents all possible points on the line.




## Derivation

- $P_0$ is a specific point on the line $L$ and it has coordinates $(x_0, y_0, z_0)$.
- $\vec{v}$ is a vector with direction $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and is parallel to the line $L$ . It is called the direction vector and $a$, $b$ and $c$ are direction numbers.
- $\vec{v}$ does not need to be in a standard position, it can have a completely different initial and terminal point.
- $P$ is a general or arbitrary point on the line $L$ and it has coordinates $(x, y, z)$.
- $\vec{r}$ is the position vector of $P$ and $\overrightarrow{r_0}$ is the position vector of $P_0$

- $\vec{a}$ is the position vector of $\overrightarrow{P_0P}$

  Now:

  $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$ $\quad$ $(Triangular\ law)$

  $\Rightarrow \vec{a} \ = \vec{r} - \vec{r_0}$

  $\Rightarrow \vec{r} \ = \vec{r_0} + \vec{a}$ $\quad$ (i)

  Note: Both vectors $\vec{a}$ and $\vec{v}$ are parallel

  $\Rightarrow \vec{a} = c\vec{v}$

  Replace the constant $c$ with $t$

  $\Rightarrow \vec{a} = t\vec{v}$ $\qquad$ (ii)

  Substituting equation (ii) into equation (i):

  $\Rightarrow \vec{r} \ = \vec{r_0} + t\vec{v}$ $\quad$ (iii)
- Equation (iii) is called the vector equation of a line
- $t$ is called the parameter and it plays the role of a scalar (a real number)
- Values assigned to $t$ will generate position vectors $\vec{r}$ with terminal points located on the line $L$.
- If $t$ is positive we move away from the original point in the direction of $\vec{v}$.
- If $t$ is positive we move away from the original point in the opposite direction of $\vec{v}$.

NB: *To write the vector equation of a line, we need a known fixed point* $P(x_0, y_0, z_0)$ *on the line and a vector* $\vec{v} = \ (a, b, c)$ *that is parallel to the line*

The vector equation given two points

Let the points be $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$

Steps

- Find the direction vector $\overrightarrow{AB}$:

NB: *For the direction vector* $\overrightarrow{AB}$

- $\overrightarrow{OB}$ *is the head and* $\overrightarrow{OA}$ *is the tail*
- $\overrightarrow{AB} = Head - Tail = \overrightarrow{OB} - \overrightarrow{OA}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

- Substitute this information into the general equation of a line:

$\vec{r} = \vec{r_0} + t\vec{a}$

NB: $\vec{r_0}$ is the position vector of any specific point on the line and is $\vec{a}$ is the direction vector of the line

$$\vec{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \quad \text{or} \quad \vec{r} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + t\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

∴ The general equation is given by:

$$\vec{r} = \overrightarrow{OA} + t\overrightarrow{AB} \quad \text{or} \quad \vec{r} = \overrightarrow{OB} + t\overrightarrow{AB}$$

## Solved Problems

### Question 1

Write the vector equation of a line which passes through $P(3;\ 5;\ 1)$ with direction vector $\vec{\boldsymbol{d}} = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$.

Solution

The equation is given by:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$$

### Question 2

Write the vector equation of a line which passes through $(2, -1, 3)$ and parallel to the vector $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

## Solution

The equation is given by:

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \text{ or } \boldsymbol{r} = (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) + t(3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$$

## Question 3

Find the vector equation for the line which passes through the points $A(3,6,2)$ and $B(6,2,-5)$.

## Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$$

Now choosing $A(3,6,2)$ to be the point on $\overrightarrow{AB}$.

NB: $B(6,2,-5)$ *can also be used to represent the specific point on the line.*

$$\boldsymbol{r} = \overrightarrow{\boldsymbol{OA}} + \boldsymbol{t}\overrightarrow{\boldsymbol{AB}}$$

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} + \boldsymbol{t}\begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$$

The equation $\boldsymbol{r}$ can also be written as $\boldsymbol{r} = (3\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}) + t(3\boldsymbol{i} - 4\boldsymbol{j} - 7\boldsymbol{k})$.

## Parametric equations

- The parametric equations of a line are derived from the vector equation of a line.

## Definition

- The parametric equations of a line that contains the point $P(x_0, y_0, z_0)$ and is parallel to the vector $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are:

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

## Derivation

- Rewrite the vector equation of a line in component form:

  $\vec{r} = \vec{r_0} + t\vec{a}$

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- Multiply the direction vector by $t$:

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} ta \\ tb \\ tc \end{pmatrix}$$

- Add the two vectors

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + ta \\ y_0 + tb \\ z_0 + tc \end{pmatrix}$$

- Equate corresponding components

Recall that two vectors are equal if and only if their corresponding components are equal

$x = x_0 + ta, y = y_0 + tb$ and $z = z_0 + ta$

This implies that:

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

- The value of the parameter will generate the coordinates of the points on the line

## Solved Problems

### Question 1

Find a parametric equation for the line which contains the point $(1, 2, 0)$ and has direction vector $(1, 2, 1)$.

### Solution

Obtaining the parametric equations:

$$\boldsymbol{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + t\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix}1+\mathrm{t}\\2+2\mathrm{t}\\\mathrm{t}\end{pmatrix}$$

Thus, the parametric equations are given by:

$$x = 1 + t$$

$$y = 2 + 2t$$

$$z = t$$

## Question 2

Let $L$ be a line which passes through the points $\mathrm{P}(1,1,1)$ and $\mathrm{Q}(3,2,1)$.

Find a vector which is parallel to the line. Find the vector-equation and parametric equations of the line.

## Solution

Obtaining the vector equation:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{PQ} = \begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}$$

Now choosing $P(1,1,1)$ to be the point on $\overrightarrow{PQ}$.

$$\boldsymbol{r} = \overrightarrow{\boldsymbol{OP}} + t\overrightarrow{\boldsymbol{PQ}}$$

The vector-equation of the line is given by:

$$\boldsymbol{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}$$

Obtaining the parametric equations:

$$\boldsymbol{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}$$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix}1+2\mathrm{t}\\1+\mathrm{t}\\1\end{pmatrix}$$

Thus, the parametric equations are:

$$x = 1 + 2t$$
$$y = 1 + t$$
$$z = 1$$

## Cartesian/Symmetric equation

- The Cartesian form of the vector equation is obtained by eliminating the parameter from the parametric equations.

  **Steps**

  - Make $t$ the subject of each parametric equation:

  $$t = \frac{x - x_0}{a}, \qquad t = \frac{y - y_0}{b}, \qquad t = \frac{z - z_0}{c}$$

  - Solve for $t$:

  In most cases $a$, $b$ and $c$ are non-zero numbers

  $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- $a, b$ and $c$ appearing in the denominators are direction numbers of the line $L$ i.e. they give the direction vector parallel to $L$ i.e. $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

## Solved Problem

**Question**

Determine the vector equation of the straight line passing through the point with position vector $i - 3j + k$ and parallel to the vector $2i + 3j - 4k$. Express the vector equation of the straight line in standard Cartesian form.

**Suggested Solution**

The vector parallel to the line is given by:

$$\boldsymbol{r} = \overrightarrow{\boldsymbol{OP}} + t\overrightarrow{\boldsymbol{AP}}$$

The vector-equation of the line is given by:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

Obtaining the parametric equations:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 + 2\mathrm{t} \\ -3 + 3\mathrm{t} \\ 1 - 4\mathrm{t} \end{pmatrix}$$

Thus the parametric equations are:

$$x = 1 + 2t$$

$$y = -3 + 3\mathrm{t}$$

$$z = 1 - 4\mathrm{t}$$

Now making t the subject of the formula in all equations:

$$x = 1 + 2t$$

$$x - 1 = 2t$$

$$\frac{x-1}{2} = t \qquad \text{(i)}$$

$$y = -3 + 3\mathrm{t}$$

$$y + 3 = 3t$$

$$\frac{y+3}{3} = t \qquad \text{(ii)}$$

$$z = 1 - 4\mathrm{t}$$

$$z - 1 = -4t$$

$$\frac{1-z}{4} = t \qquad \text{(iii)}$$

All three equations are equal to $t$ and so equal to one another, therefore the Cartesian or Symmetric equations is given by:

$$\frac{x-1}{2} = \frac{y+3}{3} = \frac{1-z}{4}.$$

**Follow up questions**

## Question 1

Determine the vector equation of the straight line passing through the two points $P_1(3,-1,5)$ and $P_2(-1,-4,2)$.

Answer: $r = 3i - j + 5k + t(4i + 3j + 3k)$.

## Question 2

Determine the vector equation of the straight line passing through the two points $A(3,4,-7)$ and $B(1,-1,6)$.

Answer: $r = 3i + 4j - 7k + t(-2i - 5j + 13k)$.

## Question 3

Find the parametric and Cartesian equations for the line $\boldsymbol{r} = 5\boldsymbol{i} + 2\boldsymbol{k} + \lambda(4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$

## Answers

The parametric equations are: $x = 5 + 4\lambda, y = -\lambda, z = 2 + 4\lambda$

The Cartesian equation is:

$$\frac{x-5}{4} = -y = \frac{z-2}{4}.$$

## Question 4

A line passes through points $A(2,-1,5)$ and $B(3,6,-4)$.

a) Write a vector equation of the line.

b) Write parametric equations for the line.

c) Determine if the point $C(0,-15,9)$ lies on the line.

## Answers

a) A vector equation is $r = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 7 \\ -9 \end{pmatrix}$

b) The corresponding parametric equations are

$x = 2 + t$

$y = -1 + 7t$

$z = 5 - 9t$

c) Substitute the coordinates of $C(0, -15, 9)$ into the parametric equations and solve for $t$.

$0 = 2 + t \quad \Rightarrow t = -2$

$-15 = -1 + 7t \quad \Rightarrow t = -2$

$9 = 5 - 9t \quad \Rightarrow t = -4/9$

The $t$-values are not equal, so the point does not lie on the line

## Determining whether a point lies on a line

The point is said to lie on a line when it satisfies the equation of the line i.e.

- If all three sets of parametric equations yield the same value for the parameter $t$, then the point lies on the line
- If at least one of parametric equations has a distinct value for the parameter $t$, then the point does not lie on the line

### Solved Problems

### Question 1

Determine whether the point (11,15,3) lies on the line $r = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

### Solution

$$r = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Substitute the point (11,15,3) to the LHS of the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Now LHS:

$$\begin{pmatrix}11\\15\\3\end{pmatrix}=\begin{pmatrix}1+2t\\3t\\-2+t\end{pmatrix}.$$

$\Rightarrow 11 = 1 + 2t$ (i)

$\Rightarrow 10 = 2t \therefore t = 5$

Also:

$15 = 3t$ (ii)

$\Rightarrow t = 5$

Finally:

$3 = -2 + t$ (iii)

$\Rightarrow t = 5$

Since all points give $t = 5 \therefore$ the point $(11,15,3)$ lies on the line $r = \begin{pmatrix}1\\0\\-2\end{pmatrix} + t\begin{pmatrix}2\\3\\1\end{pmatrix}$.

## Question 2

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Relative to the origin $\boldsymbol{O}$, the position vectors of points $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $\begin{pmatrix}5\\3\\1\end{pmatrix}$ and $\begin{pmatrix}4\\c\\2\end{pmatrix}$ respectively

Determine whether the point $\boldsymbol{P}$ lies on $l$ whose equation is $\boldsymbol{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\5\end{pmatrix}$, where $\lambda$ is the parameter. **[2]**

### Solution

$$\boldsymbol{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\5\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\5\end{pmatrix}$$

Substitute the point $(5,3,1)$ to the LHS of the line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\5\end{pmatrix}$.

Now LHS:

$$\begin{pmatrix}5\\3\\1\end{pmatrix}=\begin{pmatrix}1+2\lambda\\-\lambda\\-2+5\lambda\end{pmatrix}.$$

$\Rightarrow 5 = 1 + 2\lambda$ (i)

$\Rightarrow 4 = 2\lambda \therefore \lambda = 2$

Also:

$3 = -\lambda$ (ii)

$\Rightarrow \lambda = -3$

Finally:

$1 = -2 + 5\lambda$ (iii)

$\Rightarrow \lambda = \frac{3}{5}$

Since all points give different values for $\lambda \therefore$ the point $(5,3,1)$ does not lie on the line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.

## Follow Up Questions

### Question 1

Show that the point $(5,2,-2)$ does not lie on the line $r = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

### Question 2

Determine whether the point $(2,0,-1)$ lies on the line $r = \begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

### Question 3

Determine whether the point $(1,1,3)$ lies on the line $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$

### Question 4

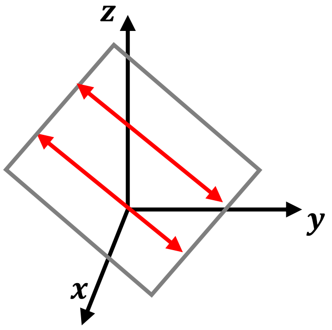
Show that the point $(4,13,-13)$ lies on the line which passes through points $A(2,-1,5)$ and $B(3,6,-4)$.

### Question 5

Determine whether the point $(1,3,2)$ lies on the line $r = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 7 \\ -9 \end{pmatrix}$

# Intersecting, Parallel and Skew Lines

## Parallel Lines

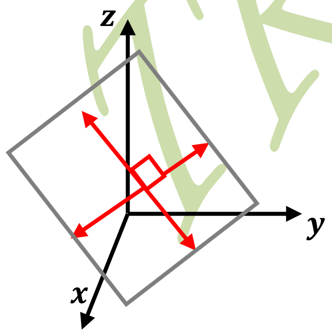



Let $l_1$ and $l_2$ be two lines in $\mathbb{R}^3$, with direction vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, and let $\theta$ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The lines $l_1$ and $l_2$ are in the same plane (coplanar)

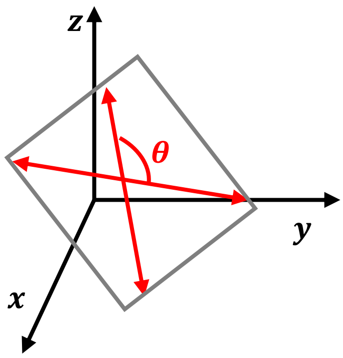
The lines $l_1$ and $l_2$ are parallel whenever:

- The direction vectors are scalar multiples of each other
- The direction vectors are in the same ratio
- The angle between the direction vectors is either $0°$ (same direction) or $180°$ (different directions)

## Intersecting Lines




(Perpendicular)



Let $l_1$ and $l_2$ be two lines in $\mathbb{R}^3$, with direction vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, and let $\theta$ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The lines $l_1$ and $l_2$ are in the same plane (coplanar). The lines are either perpendicular or are intersecting at any other angle in the range $0° < \theta < 180°$.

## Case 1

The lines $l_1$ and $l_2$ are intersect if:

- There are specific values of the parameters so that the lines share the same point

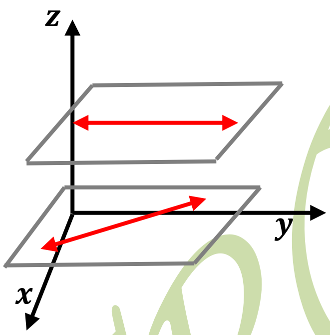
## Case 2

The lines $l_1$ and $l_2$ are perpendicular/orthogonal if:

- The dot product of their direction vectors is equal to zero

## Skew Lines

Two lines in $\mathbb{R}^3$ are skew if they are not parallel and do not intersect.




NB: *Skew lines lie in parallel planes.*



## Solved Problems

### Question 1

Show that the lines $l_1 : \boldsymbol{r} = (\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) + t(3\boldsymbol{i} - \boldsymbol{j})$ and $l_2 : \boldsymbol{r} = (4\boldsymbol{i} - \boldsymbol{k}) + u\,(2\boldsymbol{i} + 3\boldsymbol{k})$ intersect. Also, find their point of intersection.

**Solution**

Two lines intersect if $l_1 = l_2$.

$(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})+t(3\boldsymbol{i}-\boldsymbol{j})=(4\boldsymbol{i}-\boldsymbol{k}) \ + \ u\,(2\boldsymbol{i} \ + \ 3\boldsymbol{k})$

$$\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}3\\-1\\0\end{pmatrix}=\begin{pmatrix}4\\0\\-1\end{pmatrix}+u\begin{pmatrix}2\\0\\3\end{pmatrix}$$

$$\begin{pmatrix}1+3t\\1-t\\-1\end{pmatrix}=\begin{pmatrix}4+2u\\0\\-1+3u\end{pmatrix}$$

$1+3t=4+2u \quad (1)$

$1-t=0 \qquad\qquad (2)$

$\Rightarrow t=1$

$-1=-1+3u \qquad (3)$

$\Rightarrow u=0$

To check if the values of $s$ and $t$ are true, substitute them in (1)

$1+3t=4+2u \quad (1)$

LHS:

$1+3t=1+3(1)=4$

RHS:

$4+2u \ =4+0 \quad =4$

Since $LHS=RHS=4 \therefore$ for the values $t=1$ and $u=0$, line $l_1$ and line $l_2$ intersect.

To find the point of intersection, substitute $t$ in $l_1$ or $u$ in $l_2$:

$$l_1=\begin{pmatrix}1+3t\\1-t\\-1\end{pmatrix}=\begin{pmatrix}1+3(1)\\1-1\\-1\end{pmatrix}=\begin{pmatrix}4\\0\\-1\end{pmatrix} \text{ or}$$

$$l_2=\begin{pmatrix}4+2u\\0\\-1+3u\end{pmatrix}=\begin{pmatrix}4+0\\0\\-1+0\end{pmatrix}=\begin{pmatrix}4\\0\\-1\end{pmatrix}$$

$\therefore$ The point of intersection is $(4,0,-1)$.

## Question 2

Show that the lines $l_1: \frac{x-3}{2} = \frac{y-5}{6} = \frac{z+1}{4}$ and $l_2: \frac{x+1}{1} = \frac{y}{3} = \frac{z-8}{2}$ are parallel to each other.

## Solution

We write $l_1$ and $l_2$ in vector form.

$$r_1 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} -1 \\ 0 \\ 8 \end{pmatrix} + u\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Now the direction vector of $l_1$ is $\boldsymbol{a}$ and of $l_2$ is $\boldsymbol{b}$.

$\boldsymbol{a} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, then $l_1$ and $l_2$ are parallel since $\boldsymbol{a} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 2\boldsymbol{b}$ .

## Question 3

Show that lines $L_1: \frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3}$ and $L_2: \frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{2}$ are skew.

## Solution

Write the equation in parametric form.

$L_1 : x = 2t + 4; \; y = 4t - 5; \; z = -3t + 1$

$L_2 : x = s + 2; \; y = 3s - 1; \; z = 2s$

Checking if parallel:

The lines are not parallel since the vectors $\boldsymbol{v_1} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$ and $\boldsymbol{v_2} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ are not multiples of each other

Checking if they intersect:

Equate $x, y,$ and $z$ for both lines.

$$2t + 4 = s + 2 \qquad (1)$$

$$4t - 5 = 3s - 1 \qquad (2)$$

$$-3t + 1 = 2s \qquad (3)$$

From equation (1) $s = 2t + 2 \qquad (4)$

Substituting equation (4) into (2):

$4t - 5 = 3(2t + 2) - 1)$

$\Rightarrow t = -5.$

Now:

$s = 2t + 2 \qquad (4)$

$\Rightarrow s = 2(-5) + 2$

$\Rightarrow s = -10 + 2$

$\therefore s = -8.$

Substituting these numerical values into equation (3):

$-3t + 1 = 2s \qquad (3)$

LHS:

$-3t + 1 = -3(-5) + 1$

$= 15 + 1$

$= 16$

RHS:

$2s = 2(-8)$

$= -16$

Since $LHS = 16 \neq RHS = -16$ $\therefore$ for the values $t = -5$ and $s = -8$, line $l_1$ and line $l_2$ do not intersect.

$\therefore$ The two line s are skew because they are neither parallel nor intersecting

## Question 4

If the lines $l_1: \vec{r} = \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} a-1 \\ -a-1 \\ b \end{pmatrix}$ and $l_2: \vec{r} = \begin{pmatrix} 9 \\ 3 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 2a \\ 3-5a \\ 15 \end{pmatrix}$ are parallel, find the values of $a$ and $b$.

## Solution

Comparing $\boldsymbol{i}$ and $\boldsymbol{j}$ components:

$$\frac{a-1}{2a}=\frac{-a-1}{3-5a}$$

$\therefore (a-1)(3-5a)=2a(-a-1)$

$\Rightarrow 3a-5a^2-3+5a=-2a^2-2a$

$\therefore 3a^2-10a+3=0$

$\Rightarrow (3a-1)(a-3)=0$

$\therefore a=\dfrac{1}{3}$ or $3$

Now to find $b$ let's compare $\boldsymbol{i}$ components first to get the ratio:

Since direction vectors are multiples, let $\beta$ be the ratio:

$\beta(a-1)=2a$

When $a=1/3$:

$\Rightarrow \beta\left(\dfrac{1}{3}-1\right)=2\left(\dfrac{1}{3}\right)$

$\Rightarrow \beta\left(-\dfrac{1}{3}\right)=\dfrac{2}{3}$

$\Rightarrow \beta=-1$

Now comparing $\boldsymbol{k}$ components:

$\beta(b)=15$

$\Rightarrow -b=15$

$\therefore b=-15.$

When $a=3$:

$\Rightarrow \beta(3-1)=2(3)$

$\Rightarrow \beta(2)=6$

$\Rightarrow \beta=3$

Now comparing $\boldsymbol{k}$ components:

$\beta(b)=15$

$\Rightarrow 3b=15$

$\therefore b=5.$

$\therefore$ When $a=1/3$, $b=-15$ and when $a=3$, $b=5$.

## Follow up questions

### Question 1

Show that the lines $r = 2\boldsymbol{i} + 4\boldsymbol{k} + \lambda(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$;

$r = 5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \mu(\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k})$ intersect. Hence, find their point of intersection.

Answer: $(\mathbf{4},\ \mathbf{4},\ \mathbf{2})$

### Question 2

Determine whether the lines $\frac{x+5}{2} = \frac{2-y}{5} = \frac{1-z}{-1}$ and $\frac{x}{1} = \frac{2y+1}{4} = \frac{1-z}{-3}$

are parallel, intersecting or skewed.

Answer: **Skewed**

### Question 3

Show that the lines $r = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + t\,(3\boldsymbol{i} - \boldsymbol{j})$ and $r = 4\boldsymbol{i} - \boldsymbol{k} + u(2\boldsymbol{i} + \boldsymbol{k})$ intersect. Also find their point of intersection.

Answer: $(\mathbf{4},\ \mathbf{0},\ \mathbf{-1})$

### Question 4

Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also, find their point of intersection.

Answer: $\left(\frac{\mathbf{1}}{\mathbf{2}};\ -\frac{\mathbf{1}}{\mathbf{2}};\ -\frac{\mathbf{3}}{\mathbf{2}}\right)$

# Dot Product

- It is also called a scalar product. It is a result of multiplying one vector by a second vector so as to produce a scalar.
- The dot product is defined for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

## Algebraic Definition

The dot product 2 vectors $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbb{R}^3$ $\vec{a}.\vec{b}$ is defined to be the scalar

$$\vec{a}\,.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

## Geometric Definition

The dot product 2 vectors $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbb{R}^3$ $\vec{a}.\vec{b}$ is defined to be the scalar

$$\vec{a}\,.\vec{b} = |\vec{a}\,|.|\vec{b}|cos\theta,$$

where $\theta$ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

## Proposition

Let $\vec{p}$ and $\vec{q}$ be non-zero vectors.

- The vectors, $\vec{p}$ and $\vec{q}$, are perpendicular to each other if and only if $\vec{p}.\vec{q} = 0$.
- The angle between the vectors, $\vec{p}$ and $\vec{q}$, is obtuse if and only if $\vec{p}.\vec{q} < 0$
- The angle between the vectors, $\vec{p}$ and $\vec{q}$, is acute if and only if $\vec{p}.\vec{q} > 0$

## Solved Problems

### Question 1

Given that the vectors $\vec{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, find the dot product of $\vec{a}$ and $\vec{b}$.

**Solution**

$$\vec{a}.\vec{b} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = -3 + 2 + 12 = 11.$$

**Question 2**

Determine if the vectors $\vec{a} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ are perpendicular to each other.

**Solution**

$$\vec{a}.\vec{b} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}.\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 10 - 10 + 0 = 0$$

Since $\vec{a}.\vec{b} = 0$ therefore $\vec{a}$ and $\vec{b}$, are perpendicular to each.

# Determinants of Matrices

## 2×2 Matrices

Suppose $A$ is any $(2 \times 2)$ Matrix such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of $A$ can be written as:

$|A|$ or $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

## Solved Problem

### Question

Given that $A = \begin{pmatrix} 6 & -1 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$ ;find the determinant of $A$ and the determinant of $B$.

### Solution

(i) $\begin{vmatrix} 6 & -1 \\ 4 & 1 \end{vmatrix} = ad - bc = 6(1) - (-1 \times 4) = 6 + 4 = 10.$

(ii) $\begin{vmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{vmatrix} = ad - bc$

$$= cos\theta(cos\theta) - (sin\theta \times -sin\theta)$$

$$= cos^2\theta + sin^2\theta \equiv 1$$

## 3×3 Matrices

Suppose $A$ is any $(3 \times 3)$ Matrix such that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then the determinant of $A$ can be written as $|A|$ or $\det(A)$ or $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\therefore \det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$

## Solved Problem

### Question

Given that $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}$; find the determinant of $A$ .

### Solution

$$\det(A) = 3\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 3(2-2) - 1(-4-2) + 1(2--1)$$

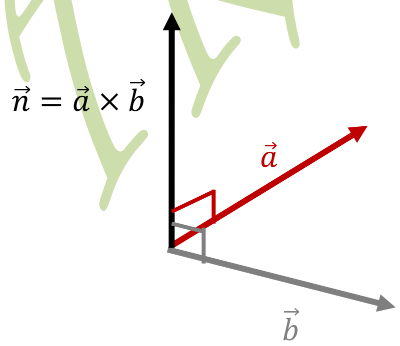
$$= 3(0) - 1(-6) + 1(3)$$

$$= 0 + 6 + 3$$

$$= 9$$

## Cross Product

- It is also called a vector product. It is a result of multiplying one vector by a second vector so as to produce another vector.
- This resultant vector is perpendicular to both vectors. This vector is called the normal vector ( $\vec{n}$ ).
- The cross product is only defined for vectors in $\mathbb{R}^3$



## Geometric definition

Given 2 vectors $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbb{R}^3$the cross product $\vec{a} \times \vec{b}$ can also be defined as follows

$$\vec{a} \times \vec{b} = (|\vec{a}|.\left|\vec{b}\right| \sin \theta)$$

## Algebraic definition

Given 2 vectors $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbb{R}^3$ the cross product $\vec{a} \times \vec{b}$ can also be defined as follows:

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are unit vectors along the $x$, $y$ and $z$ axes respectively. However while calculating the determinant they only have symbolic value.

## Note

- The cross product of parallel vectors is always equal to a zero vector
- The cross product of any vector with itself is equal to a zero vector i.e.
  - $\vec{a} \times \vec{a} = 0$
  - $\vec{b} \times \vec{b} = 0$

## Solved Problem

### Question

Consider the two vectors $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$.

Find

(a) the cross-product of the two vectors,

(b) verify that it is indeed perpendicular to both $\vec{a}$ and $\vec{b}$,

(c) the magnitude of $\vec{a} \times \vec{b}$.'

## Suggested Solution

(a) $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix}$

$= i\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} - j\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$

$= i(0-0) - j(0-0) + k[2-(-1)]$

$= 3k$

(b) $\vec{n} = \vec{a} \times \vec{b} = (0, 0, 3)$

$\vec{n}.\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 0 + 0 + 0 = 0.$

$\vec{n}.\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 0 + 0 + 0 = 0.$

Since both the dot products are zero $\therefore$ $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

(c) The magnitude of $\vec{a} \times \vec{b}$ is given b:

$\vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|$

$= \sqrt{0^2 + 0^2 + 3^2}$

$= \sqrt{9}$

$= 3$

## Follow up questions

### Question 1

Consider the two vectors $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ in $\mathbb{R}^3$. Find the cross-product of the two vectors.

**Answer:** $(-3, -22, -7)$

## Question 2

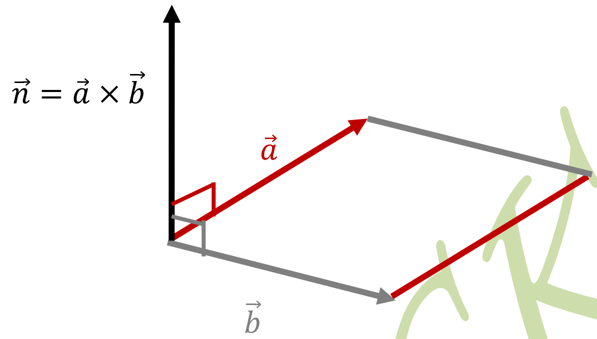
Find the cross-product of the two vectors $\vec{v} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\vec{w} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

Answer: $11\boldsymbol{i} - 7\boldsymbol{j} + 5\boldsymbol{k}$

## Using cross product to find area of plane shapes

### Parallelogram

#### Algebraic Approach

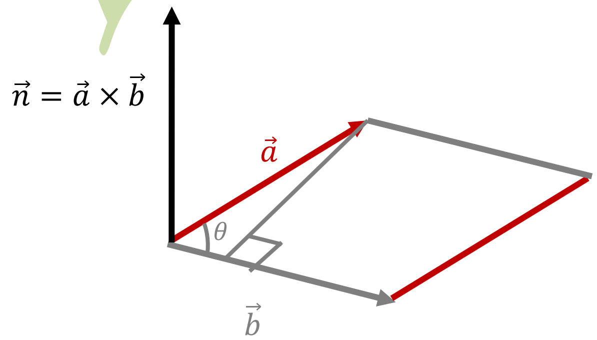



The area of a parallelogram with edges $\vec{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\vec{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is:

$Area = |\vec{a} \times \vec{b}|$ where $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

NB: $\vec{a}$ and $\vec{b}$ are non-zero vectors they intersect

#### Geometric Approach



The area of a parallelogram with edges $\vec{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\vec{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is:

$Area = Base \times Altitude$

$\quad = |\vec{a}| \times |\vec{b}|Sin\theta$

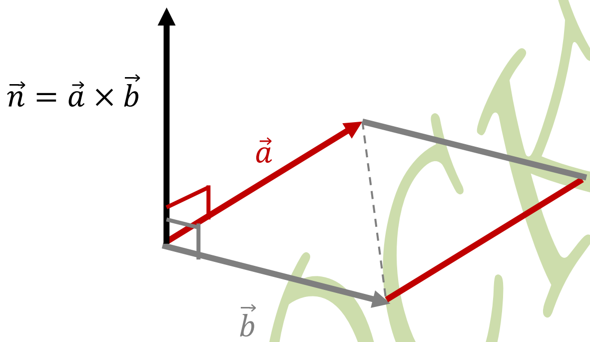
$\quad = |\vec{a}||\vec{b}|Sin\theta$

NB: $\theta$ is the angle between the two vectors and it is the smaller angle from the two possible angles in the range $0° < \theta < 180°$

## Triangle

The area of triangle is twice the area of parallelogram.

### Algebraic Approach

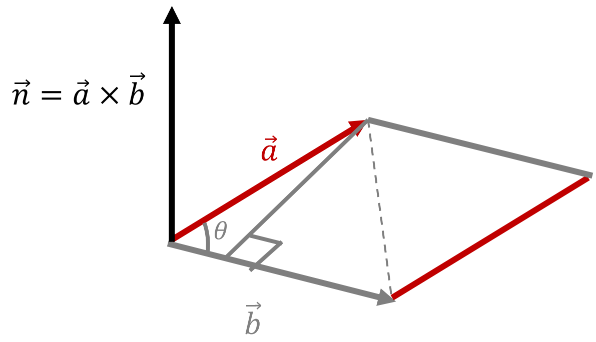



The area of a parallelogram with edges $\vec{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\vec{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is:

$Area = \frac{1}{2}|\vec{a} \times \vec{b}|$

NB: $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

## Geometric Approach




The area of a parallelogram with edges $\vec{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\vec{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is:

$$Area = \frac{1}{2} Base \times Altitude$$

$$= \frac{1}{2}|\vec{a}| \times |\vec{b}|Sin\theta$$

$$= \frac{1}{2}|\vec{a}||\vec{b}|Sin\theta$$

NB: $\theta$ is the angle between the two vectors and it is the smaller angle from the two possible angles in the range $0° < \theta < 180°$

### Solved Problem

### Question

Find the area of the parallelogram with edges $\vec{v} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\vec{w} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$.

### Solution

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix} = (2+9)\boldsymbol{i} - (4+3)\boldsymbol{j} + (6-1)\boldsymbol{k}$$

$$= 11\boldsymbol{i} - 7\boldsymbol{j} + 5\boldsymbol{k}$$

The area of the parallelogram with edges $\vec{v}$ and $\vec{w}$ is the magnitude of the vector $\vec{v} \times \vec{w}$.

$$Area = ||\vec{v} \times \vec{w}|| = \sqrt{11^2 + (-7)^2 + 5^2} = \sqrt{195}$$

## Follow up questions

### Question 1

Find the area of the parallelogram with edges $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ in $\mathbb{R}^3$.

Answer: $\sqrt{542}$

### Question 2

Find the area of the parallelogram with edges $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$.

Answer: 3

### Question 3

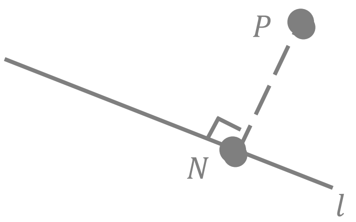
Show that the area of the triangle with edges $A(2,1,5)$, $B(4,2,-1)$ and $C(3,-2,1)$ is $\frac{\sqrt{537}}{2}$.

### Question 4

Show that the area of the parallelogram with edges $A(-1,2,1)$, $B(3,-2,4)$, $C(-2,1,5)$ and $D(-6,5,2)$ is $3\sqrt{66}$.

## The perpendicular distance from a point to a straight line

The scalar product plays a crucial role when determining the distance between a point and a line.

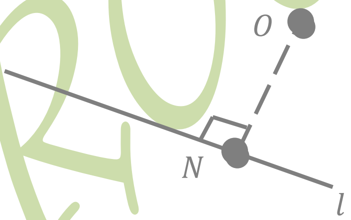


## Steps

- First we need to find the coordinates of $N$
- Find vector $\overrightarrow{NP}$
- Since $N$ lies on the line $l$, let the value of the scalar $= s$ or any other letter which different from the original scalar in the equation.
- In relation to the origin find $\overrightarrow{NP}$
- Find the dot product between $\overrightarrow{NP}$ and the direction vector of the line $l$ to find the value of $s$.
- Substitute that value into $\overrightarrow{ON}$ and $\overrightarrow{NP}$.
- Calculate the required modulus.

NB: *To find the coordinates of the foot of the perpendicular we simply substitute the value of $t$ into $\overrightarrow{ON}$*

## The perpendicular distance from the origin to a straight line




## Steps

- Let point $N$ be the foot of the perpendicular from $O$.
- Rewrite the vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{pmatrix}$
- State that vector $\boldsymbol{ON} = \begin{pmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{pmatrix}$

- Since $\boldsymbol{ON}$ is perpendicular to $l$, find the value of the scalar $t$ by equating the dot product between $\boldsymbol{ON}$ and the direction vector of the line $l$ to zero as follows:

$$\boldsymbol{ON}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

- Substitute this numerical value of $t$ in $\boldsymbol{ON} = \begin{pmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{pmatrix}$ to find the coordinates of the foot of the perpendicular
- To find the required distance then find the magnitude of $\boldsymbol{ON}$ as follows:

$$|\boldsymbol{ON}| = \sqrt{(x_0 + at)^2 + (y_0 + bt)^2 + (z_0 + ct)^2}$$

NB: *(i) Remember that to evaluate $|ON|$ replace $t$ by its numerical value.*

*(ii) The perpendicular distance is also called the shortest distance.*

## Solved Problems

### Question 1

Find

a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(7, -1, 6)$ to the line $l$ with equation

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

### Solution

a) Finding the coordinates of $N$:

$$\Rightarrow \overrightarrow{ON} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 + t \\ 1 - 2t \\ 3 + 4t \end{pmatrix}$$

Finding vector $\overrightarrow{NP}$

$$\Rightarrow \overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON}$$

$$= \begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 + t \\ 1 - 2t \\ 3 + 4t \end{pmatrix}$$

$$= \begin{pmatrix} 5 - t \\ -2 + 2t \\ 3 - 4t \end{pmatrix}$$

Since $\overrightarrow{NP}$ is perpendicular to line $l$ then:

$$\Rightarrow \overrightarrow{NP}.\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = 0$$

$$\begin{pmatrix} 5-t \\ -2+2t \\ 3-4t \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = 0$$

$$5 - t + 4 - 4t + 12 - 16t = 0$$
$$21 - 21t = 0$$
$$21 = 21t$$
$$\therefore t = 1$$

To find the coordinates of the foot of the perpendicular substitute the value of $t$ into $\overrightarrow{ON}$:

$$\Rightarrow \overrightarrow{ON} = \begin{pmatrix} 2+t \\ 1-2t \\ 3+4t \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix}$$

$\therefore$ the coordinates are $(3, -1, 7)$.

b) Substituting $t = 1$ into $\overrightarrow{NP}$.

$$\overrightarrow{NP} = \begin{pmatrix} 5-t \\ -2+2t \\ 3-4t \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$$

$$\therefore |\overrightarrow{NP}| = \sqrt{4^2 + 0^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17}$$

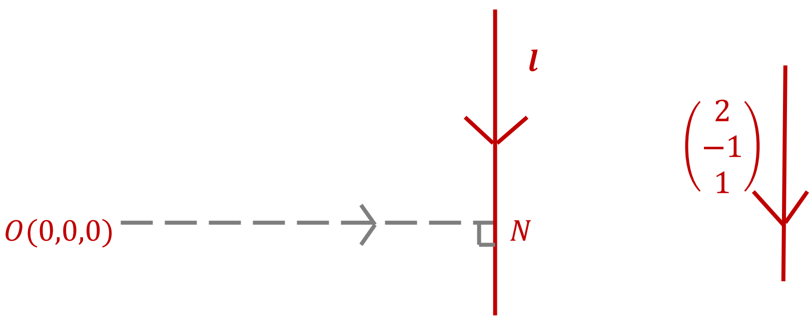
## Question 2

The line $l$ has the equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, where $t$ is a parameter.

Find the perpendicular distance from the origin to $l$.

## Suggested Solution

Let point $N$ be the foot of the perpendicular from $O$.




The coordinates of $N$ are $(1 + 2t;\ -t;\ 1 + t)$

Now

$$\boldsymbol{ON} = \begin{pmatrix} 1+2t \\ -t \\ 1+t \end{pmatrix},$$

Now

$$\boldsymbol{ON}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1+2t \\ -t \\ 1+t \end{pmatrix}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$2 + 4t + t + 1 + t = 0$$

$$6t = -3$$

$$\therefore t = \frac{-3}{6} = \frac{-1}{2}$$

Now substituting $t = \frac{-1}{2}$ into $\boldsymbol{ON}$

$$\boldsymbol{ON} = \begin{pmatrix} 1+2\left(\frac{-1}{2}\right) \\ -\left(\frac{-1}{2}\right) \\ 1+\left(\frac{-1}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Now

$$|\boldsymbol{ON}| = \sqrt{(0)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}.$$

## Follow Up Questions

### Question 1

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(11,-5,-3)$ to the line $l$ with equation

$$r = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}.$$

Answers: a) $(7,3,-8)$. b) $\sqrt{105}$

## Question 2

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(-2,11,5)$ to the line $l$ with equation

$$r = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}.$$

Answers: a) $(-2,6,7)$. b) $\sqrt{29}$

## Question 3

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(8,4,-1)$ to the line $l$ with equation

$$r = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}.$$

Answers: a) $(2,7,-3)$. b) 7

## Question 4

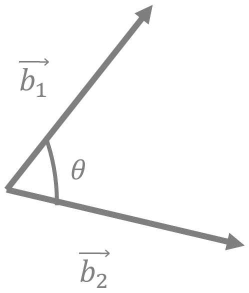
Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the origin to the line $l$ with equation

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix}$$

Answers: a) $\left(\frac{20}{13}; \frac{-24}{13}; \frac{57}{13}\right)$ b) 5

## The angle between two lines




- The angle between the two lines is calculated using the fact that the dot product of vectors $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ is equal the product of the magnitude of vector $\boldsymbol{b_1}$ and magnitude of vector $\boldsymbol{b_2}$ and cosine of angle $\theta$. This implies that $\vec{b_1}.\vec{b_2} = |\vec{b_1}|.|\vec{b_2}|\cos\theta$
- This implies that:

$$\cos\theta = \frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$$

- Therefore, if $\theta$ is the acute angle between the lines $\boldsymbol{r} = \boldsymbol{a}_1 + t\boldsymbol{b}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + t\boldsymbol{b}_2$ then $\theta$ is given by:

$$\theta = \cos^{-1}\left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$$

  - We use the direct method $\theta = \cos^{-1}\left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$ if $\theta$ is acute

NB: *The following formula will produce an angle which is either acute or obtuse:*

$$\theta = \cos^{-1}\left(\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right)$$

*When the calculated angle is obtuse and the required angle is acute then to find the acute angle subtract the obtuse angle from* $180°$.

$$Acute\ angle = 180° - Obtuse\ Angle$$

## Solved Problems

### Question 1

Find the angle between the skew lines $l_1: x = 2 + 3t, y = 4 - t, z = 3 + 2t$ and $l_2: x = 1 - t, y = 5 + 2t, z = 6 + 3t$.

### Solution

The vector $b_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ is parallel to line $l_1$ and $b_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ is parallel to line $l_1$.

The angle between the lines is equal to the angle between the vectors $b_1$ and $b_2$.

Let the angle be $\theta$:

$$cos\theta = \frac{\overrightarrow{b_1}.\overrightarrow{b_2}}{|\overrightarrow{b_1}||\overrightarrow{b_2}|}$$

$$cos\theta = \frac{\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{3^2 + (-1)^2 + 2^2}\sqrt{(-1)^2 + 2^2 + 3^2}}$$

$$cos\theta = \frac{-3 - 2 + 6}{\sqrt{14}\sqrt{14}}$$

$$cos\theta = \frac{1}{14}$$

$$\theta = cos^{-1}\left(\frac{1}{14}\right)$$

$$= 85.90395624°$$

$$= 86°$$

### Question 2

Find the acute angle between the line with equation $r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, where $t$ and $s$ are parameters.

Suggested Solution

$$cos\theta = \frac{\overrightarrow{b_1}.\overrightarrow{b_2}}{|\overrightarrow{b_1}||\overrightarrow{b_2}|}$$

$$cos\theta = \frac{\begin{pmatrix}-1\\-1\\3\end{pmatrix}\begin{pmatrix}1\\2\\-2\end{pmatrix}}{\sqrt{(-1)^2+(-1)^2+3^2}\sqrt{1^2+2^2+(-2)^2}}$$

$$cos\theta = \frac{-9}{3\sqrt{11}}$$

$$cos\theta = \frac{-3}{\sqrt{11}}$$

$$\theta = cos^{-1}\left(\frac{-3}{\sqrt{11}}\right)$$

$$= 154.7505982° \ (Obtuse\ angle).$$

$$\therefore Acute\ angle = 180° - 154.7605982°$$

$$= 25.239402°$$

$$= 25°$$

NB: To find the acute angle directly we use the following method:

$$cos\theta = \left|\frac{\overrightarrow{b_1}.\overrightarrow{b_2}}{|\overrightarrow{b_1}||\overrightarrow{b_2}|}\right|$$

$$cos\theta = \left|\frac{\begin{pmatrix}-1\\-1\\3\end{pmatrix}\begin{pmatrix}1\\2\\-2\end{pmatrix}}{\sqrt{(-1)^2+(-1)^2+3^2}\sqrt{1^2+2^2+(-2)^2}}\right|$$

$$cos\theta = \left|\frac{-9}{3\sqrt{11}}\right|$$

$$cos\theta = \frac{3}{\sqrt{11}}$$

$$\theta = cos^{-1}\left(\frac{3}{\sqrt{11}}\right)$$

$$= 25.239401821°$$

$$= 25°$$

## Follow Up Questions

### Question 1

Find the angle between the line with equation $r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where $t$ and $s$ are parameters.

**Answer**: $53.6°$

### Question 2

Find the angle between the line with equation $r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 12 \\ 3 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, where $t$ and $s$ are parameters.

**Answer**: $49.8°$

### Question 3

Find the angle between the line with equation $r = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$, where $t$ and $s$ are parameters.

**Answer**: $126°$

### Question 4

Find the angle between the line with equation $r = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + s\begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$, where $t$ and $s$ are parameters.

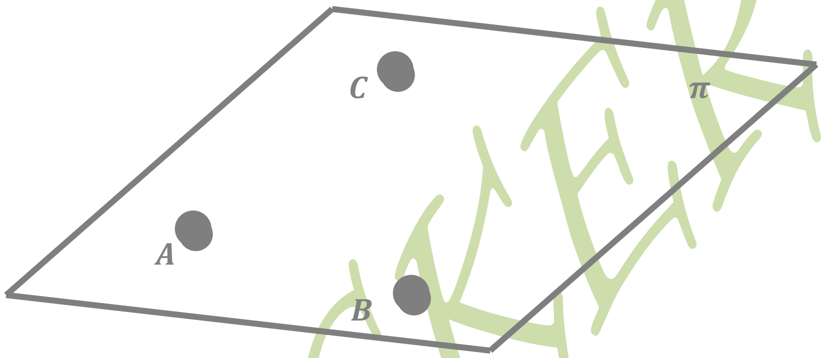
**Answer**: 8.7

# PLANES IN $\mathbb{R}^3$

## Definition

A plane can be thought of as the collection of all lines perpendicular/orthogonal to a given line. It is a flat surface without any thickness which extends without ends which contains infinitely many lines. A plane has infinite points on its surface

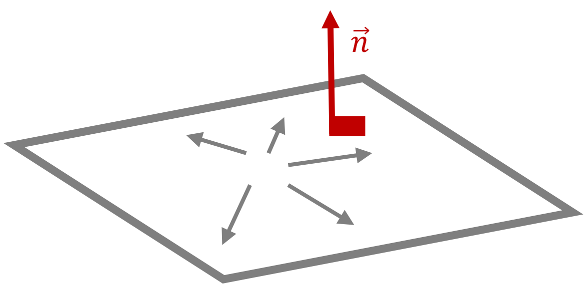
## Labelling a plane




- Planes can be named using any three non-collinear points which are called coplanar points. For example the plane above is called Plane $ABC$ (The order is not important).
- Planes can also be named using any letters or symbols. For example the plane above is called Plane $\pi$.

## Equation of a Plane

- There is infinite number of vectors in the plane (See diagram below).



- A non-zero vector $\vec{n}$ or $\boldsymbol{n}$, called the normal vector, and it is perpendicular/ orthogonal to all vectors in the plane or simply it perpendicular/ orthogonal to the plane.

NB: *Taking the dot product of $\vec{n}$ and any other vector yields a zero because they are perpendicular/ orthogonal to each other.*

- The equation of a plane is based on this dot- product

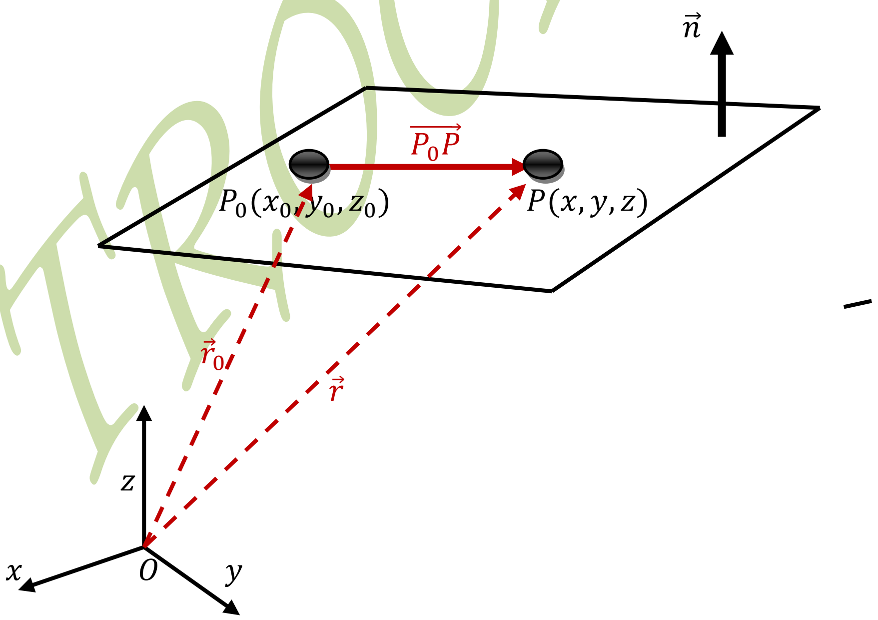
## The Plane which contains a point and the normal vector

- A plane in $\mathbb{R}^3$ is uniquely determined by a point and normal vector

### Derivation

Let:

- $P(x, y, z)$ be any general/arbitrary point on a plane
- $P_0(x_0;\ y_0;\ z_0)$ be any specific point on a plane
- $\vec{n} =< a, b, c >$ be a vector which is perpendicular/ orthogonal to the plane




- Let the position vectors of $P_0\ (x_0;\ y_0;\ z_0)$ and $P\ (x;\ y;\ z)$ be $\vec{r_0}$ and $\vec{r}$ respectively i.e.

$\overrightarrow{OP_0} = \overrightarrow{r_0}$ and $\overrightarrow{OP} = \vec{r}$

Now:

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \vec{r} - \overrightarrow{r_0} = \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}.$$

Note

(i) $\overrightarrow{P_0P}$ is orthogonal/perpendicular to $\vec{n}$,

(ii) $\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

Now:

$\overrightarrow{P_0P} \equiv \vec{r} - \overrightarrow{r_0}$

$\Rightarrow (\vec{r} - \overrightarrow{r_0}).\vec{n} = 0$

$\Rightarrow \vec{r}.\vec{n} - \overrightarrow{r_0}.\vec{n} = 0$

$\Rightarrow \vec{r}.\vec{n} = \overrightarrow{r_0}.\vec{n}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\Rightarrow ax + by + cz = ax_0 + by_0 + cz_0$

$ax_0 + by_0 + cz_0 \equiv d$ (since it is just a constant)

$\Rightarrow \boldsymbol{ax + by + cz = d}$ (Cartesian equation) or $\boldsymbol{\vec{r}.\vec{n} = d}$ (Vector equation)

NB: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv \boldsymbol{r.n = r_0.n}$

## Solved Problems

### Question 1

Find the equation of the plane through the point $(2;\ 4; -1)$ with normal vector $\boldsymbol{n} = < 2;\ 3;\ 4 >$ .

Solution

The equation of the plane is

$$\begin{pmatrix} x-2 \\ y-4 \\ z+1 \end{pmatrix}.\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 0$$

$$2x - 4 \ + \ 3y - 12 \ + \ 4z + 4 \ = \ 0$$

$$2x + \ 3y \ + \ 4z \ = \ 0 + 12$$

$$2x \ + \ 3y \ + \ 4z \ = \ 12$$

Alternatively, we use the following method:

$$\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{r_0}.\boldsymbol{n}$$

$$r.\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}.\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$r.\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 4 + 12 - 4$$

$$r.\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 12\ (Vector\ equation)$$

$$2x \ + \ 3y \ + \ 4z \ = \ 12\ (Cartesian\ Equation)$$

Question 2

Find the equation of the plane with normal vector, $\boldsymbol{n} = < 1, 2, 3 >$ containing the point $(2, -1, 5)$

Solution

From the above, the equation of this plane is

$$\begin{pmatrix} x-2 \\ y+1 \\ z-5 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$$

$$x - 2 + 2y + 2 + 3z - 15 = 0$$

$$x + 2y + 3z = 0 + 15$$

$$x \ + \ 2y \ + \ 3z \ = \ 15$$

Alternatively we use the following method:

$$\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{r_0}.\boldsymbol{n}$$

$$r.\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\-1\\5\end{pmatrix}.\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$r.\begin{pmatrix}1\\2\\3\end{pmatrix} = 2 - 2 + 15$$

$$r.\begin{pmatrix}1\\2\\3\end{pmatrix} = 15\ (Vector\ equation)$$

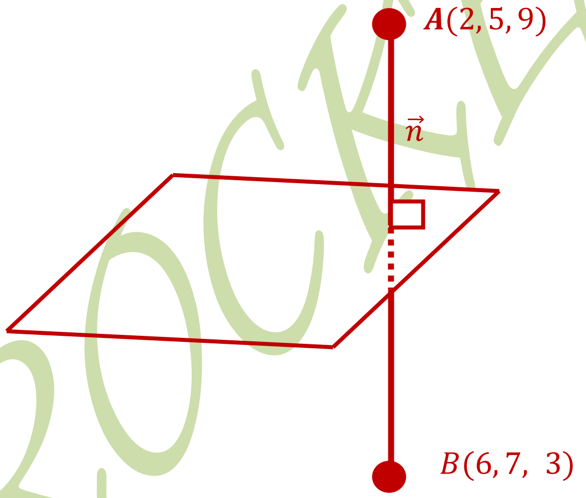
$$x + 2y + 3z = 15\ (Cartesian\ Equation)$$

## Question 3

Find the equation of the plane consisting of all points that are equidistant from $(2, 5, 9)$ and $(6, 7,\ 3)$.

## Solution

Let $a = (2, 5, 9)$ and $b = (6, 7,\ 3)$.




Point in the plane = Midpoint of $AB = \left[\left(\frac{2+6}{2}\right), \left(\frac{5+7}{2}\right), \left(\frac{9+3}{2}\right)\right]$

$= (4,6,6)$

$$\vec{n} = \overrightarrow{ab} = \vec{b} - \vec{a}$$

$$= \begin{pmatrix}6\\7\\3\end{pmatrix} - \begin{pmatrix}2\\5\\9\end{pmatrix}$$

$$= \begin{pmatrix}4\\2\\-6\end{pmatrix}$$

The equation of this plane is

$$\begin{pmatrix} x-4 \\ y-6 \\ z-6 \end{pmatrix} . \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = 0$$

$$4x - 16 + 2y - 12 - 6z + 36 = 0$$

$$4x + 2y - 6z = 0 + 28 - 36$$

$$4x + 2y - 6z = -8$$

$$2x + y - 3z = -4$$

Alternatively we use the following method:

$$\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{r_0}.\boldsymbol{n}$$

$$r.\begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} . \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix}$$

$$r.\begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = 16 + 12 - 36$$

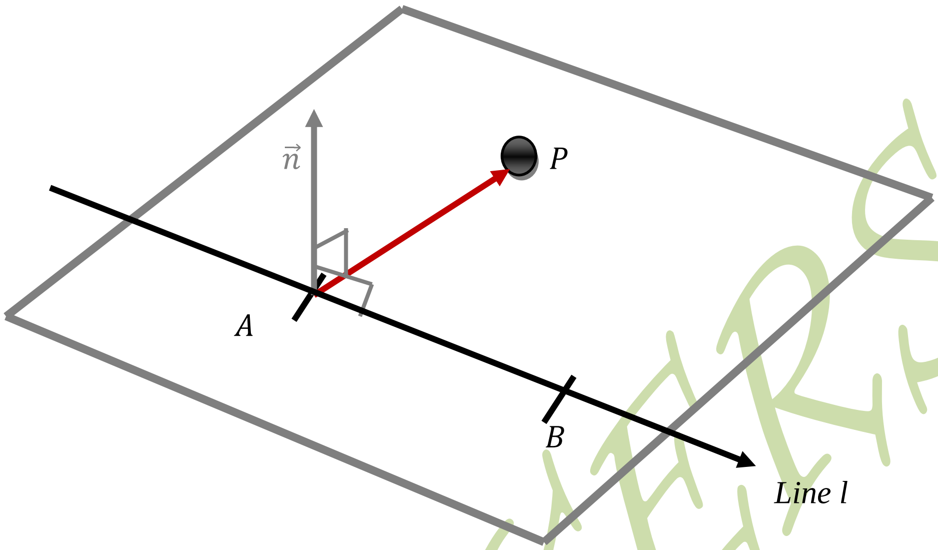
$$r.\begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = -8 \ (Vector\ equation)$$

$$r.\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = -4 \ (Vector\ equation)$$

$$2x + y - 3z = 4 \ (Cartesian\ Equation)$$

# The Plane which contains a point and a line

- A plane in $\mathbb{R}^3$ is uniquely determined a point and normal vector




If a point is in the plane and a line of the form $\boldsymbol{r} = \boldsymbol{OA} + t\boldsymbol{AB}$ we find the equation of the plane using the following method:

## STEPS

- Find the position vector $\boldsymbol{AP}$.

  ($A$ is the point on the line)
- Find the normal vector by simply taking the cross of $\boldsymbol{AB}$ and $\boldsymbol{AP}$.
- Find the position vector between the point $P$ and the arbitrary/general point $(x, y, z)$.
- Take the dot product of this vector and the normal vector and equate to zero

  (The dot product between perpendicular vectors is zero)

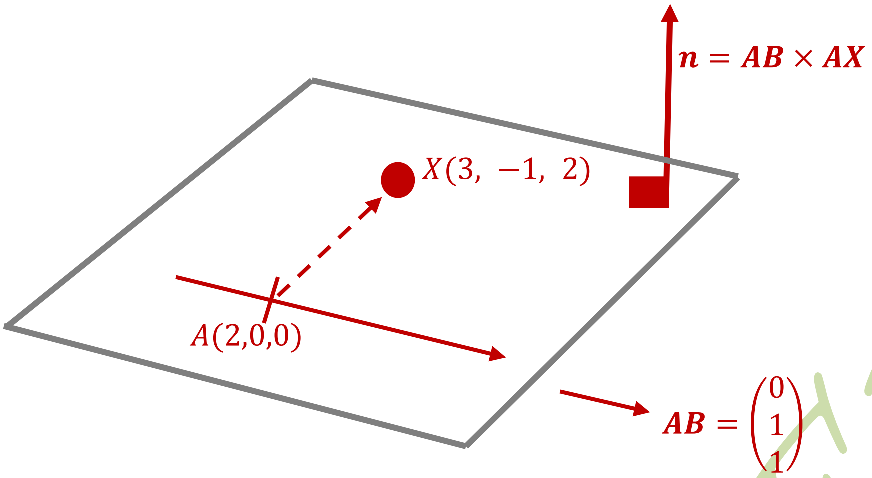
## Solved Problem

### Question

**ZIMSEC NOVEMBER 2018 SPECIMEN PAPER 2**

The plane $\rho$ contains the point $X$ with position vector $3\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and the line ,$l$, with vector equation $\boldsymbol{r}=2\boldsymbol{i}+\lambda(\boldsymbol{j}+\boldsymbol{k})$. Find the vector equation of $\rho$.

Suggested Solution




$\overrightarrow{AX}=\overrightarrow{OX}-\overrightarrow{OA}$

$$=\begin{pmatrix}3\\-1\\2\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AX}=\begin{vmatrix}i & j & k\\0 & 1 & 1\\1 & -1 & 2\end{vmatrix}$$

$$=i\begin{vmatrix}1 & 1\\-1 & 2\end{vmatrix}-j\begin{vmatrix}0 & 1\\1 & 2\end{vmatrix}+k\begin{vmatrix}0 & 1\\1 & -1\end{vmatrix}$$

$$=3\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$$

$$\therefore\ \vec{n}=\begin{pmatrix}3\\1\\-1\end{pmatrix}$$

Now the equation of the plane is

$r.\vec{n}=a.\vec{n}$

For '$a$' we choose any point in the plane but in this case let's choose $X(3,-1,2)$.

$$\Rightarrow \boldsymbol{r}.\begin{pmatrix}3\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\-1\\2\end{pmatrix}.\begin{pmatrix}3\\1\\-1\end{pmatrix}$$

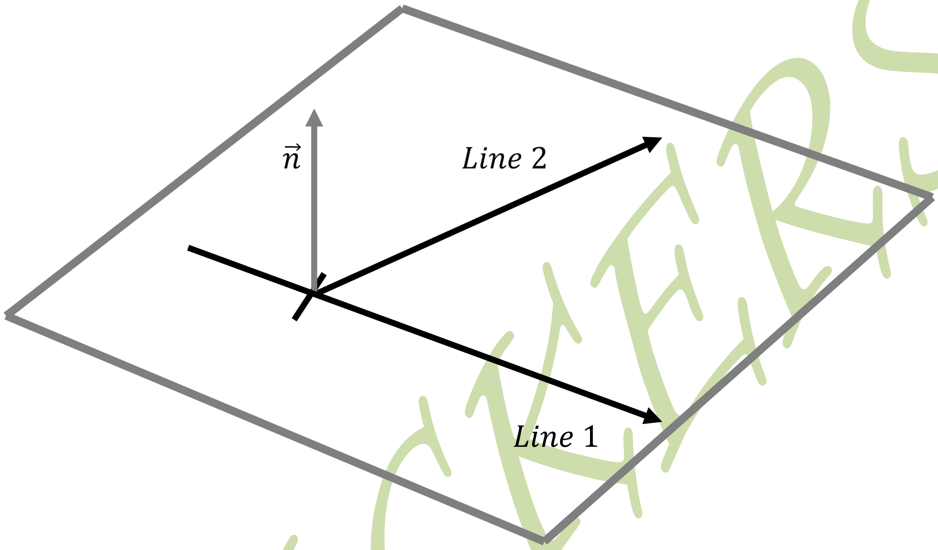
$$\boldsymbol{r}.\begin{pmatrix}3\\1\\-1\end{pmatrix}=9-1-2$$

∴The vector equation is $\boldsymbol{r}.\begin{pmatrix}3\\1\\-1\end{pmatrix} = 6.$

## The Plane which contains two lines

- A plane in $\mathbb{R}^3$ is uniquely determined by two lines

### Case 1: Intersecting Lines




The two lines are of the form $\boldsymbol{L_1}: \boldsymbol{r} = \boldsymbol{a_1} + t\boldsymbol{b_1}$ and $\boldsymbol{L_2}: \boldsymbol{r} = \boldsymbol{a_2} + t\boldsymbol{b_2}$

### STEPS

- Find the normal vector by simply taking the cross of the two vectors i.e. $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$
- Find the position vector between the known point and the arbitrary/general point $(x, y, z)$.

  (Choose the point from any of the equations of lines i.e. $\boldsymbol{a_1}$ or $\boldsymbol{a_2}$)
- Take the dot product of this vector and the normal vector and equate to zero

  (The dot product between perpendicular vectors is zero)

## Solved Problem

### Question

The plane $\pi$ contains the line $l_1$ with equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and the line $l_2$ with vector equation $\boldsymbol{r} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$. Find the cartesian equation of $\pi$.

### Suggested Solution

$$\vec{n} = \overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} i & j & k \\ 3 & 2 & 6 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= i\begin{vmatrix} 2 & 6 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= -8\boldsymbol{i} + 4\boldsymbol{k}$$

$$\therefore \vec{n} = \begin{pmatrix} -8 \\ 0 \\ 4 \end{pmatrix}$$

Now the equation of the plane is

$r.\vec{n} = a.\vec{n}$

Choose '$a$' from any of the equations of lines i.e. $\boldsymbol{a_1}$ or $\boldsymbol{a_2}$ $\Rightarrow A(5, -2, 0)$.

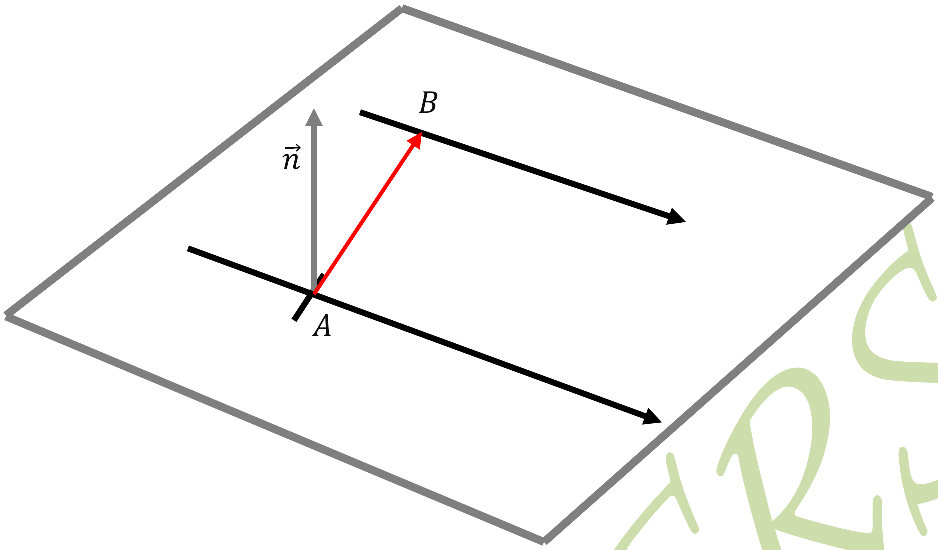
$$\Rightarrow \boldsymbol{r}.\begin{pmatrix} -8 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}.\begin{pmatrix} -8 \\ 0 \\ 4 \end{pmatrix}$$

$$\boldsymbol{r}.\begin{pmatrix} -8 \\ 0 \\ 4 \end{pmatrix} = -40$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} -8 \\ 0 \\ 4 \end{pmatrix} = -40$$

$\therefore$The vector equation is $-8x + 4z = -40$.

## Case 2: Parallel Lines




The two lines are of the form $\boldsymbol{L_1}: \boldsymbol{r} = \boldsymbol{a_1} + t\boldsymbol{b_1}$ and $\boldsymbol{L_2}: \boldsymbol{r} = \boldsymbol{a_2} + t\boldsymbol{b_2}$

## STEPS

- Find the third vector using $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$
- Find the normal vector by simply taking the cross of the third vector and $\boldsymbol{b_1}$ or $\boldsymbol{b_2}$
- Find the position vector between the known point and the arbitrary/general point $(x, y, z)$.

  (Choose the point from any of the equations of lines i.e. $\boldsymbol{a_1}$ or $\boldsymbol{a_2}$)
- Take the dot product of this vector and the normal vector and equate to zero

  (The dot product between perpendicular vectors is zero)

## Solved Problem

### Question

The plane $\pi$ contains the line $l_1$ with equation $\boldsymbol{r} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\1\\2\end{pmatrix}$ and the line $l_2$ with vector equation $\boldsymbol{r} = \begin{pmatrix}0\\-4\\-1\end{pmatrix} + t\begin{pmatrix}3\\3\\6\end{pmatrix}$. Find the cartesian equation of $\pi$.

## Suggested Solution

Let $\overrightarrow{d_1} = \begin{pmatrix}3\\3\\6\end{pmatrix}$, $\overrightarrow{d_2} = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $A_1 = (-1,1,1)$ and $A_2 = (0,-4,-1)$

$\Rightarrow \overrightarrow{A_1A_2} = \overrightarrow{A_2} - \overrightarrow{A_1}$

$= \begin{pmatrix}0\\-4\\-1\end{pmatrix} - \begin{pmatrix}-1\\1\\1\end{pmatrix}$

$= \begin{pmatrix}1\\-5\\-2\end{pmatrix}$

$\vec{n} = \overrightarrow{A_1A_2} \times \overrightarrow{d_1} = \begin{vmatrix} i & j & k \\ 1 & -5 & -2 \\ 3 & 3 & 6 \end{vmatrix}$

$= i\begin{vmatrix} -5 & -2 \\ 3 & 6 \end{vmatrix} - j\begin{vmatrix} 1 & -2 \\ 3 & 6 \end{vmatrix} + k\begin{vmatrix} 1 & -5 \\ 3 & 3 \end{vmatrix}$

$= -24\boldsymbol{i} - 12\boldsymbol{j} + 18\boldsymbol{k}$

$\therefore \vec{n} = \begin{pmatrix}-24\\-12\\18\end{pmatrix}$

Now the equation of the plane is

$r.\vec{n} = a.\vec{n}$

For '$a$' we choose any point from the two lines i.e. $A(-1,1,1)$.

$\Rightarrow \boldsymbol{r}.\begin{pmatrix}-24\\-12\\18\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}.\begin{pmatrix}-24\\-12\\18\end{pmatrix}$

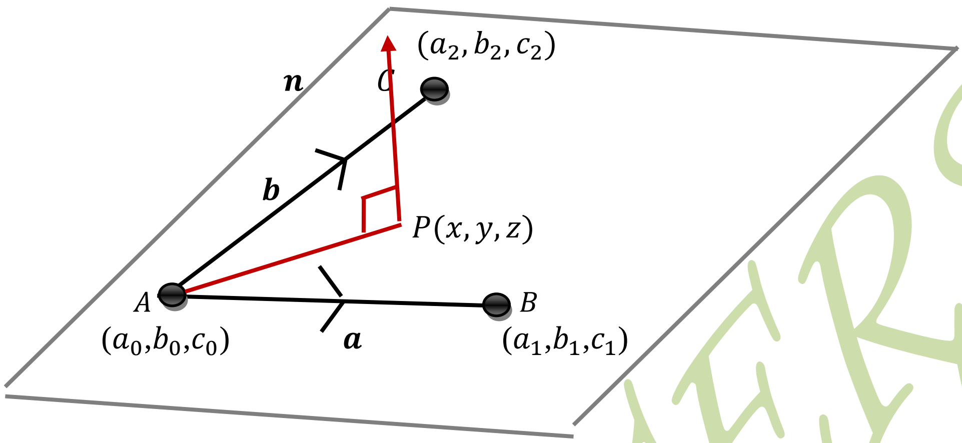
$\boldsymbol{r}.\begin{pmatrix}-24\\-12\\18\end{pmatrix} = 24 - 12 + 18$

$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}-24\\-12\\18\end{pmatrix} = 30$

$\therefore$The vector equation is $-24x - 12y + 18z = 30$ or $-4x - 2y + 3z = 5$

# The Plane which contains three non-collinear points

- A plane in $\mathbb{R}^3$ is uniquely determined by three non-collinear points
- If the points are collinear then they represent a line




## METHOD 1

### STEPS

- Let one of the points be the centre
- Find the position vectors of the two line segments produced.

  If $A$ is the centre then find $\boldsymbol{AB}$ and $\boldsymbol{AC}$.
- Find the normal $\boldsymbol{n}$ by taking the cross of the two vectors $(\boldsymbol{a} \times \boldsymbol{b})$

$$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = \boldsymbol{a} = \begin{pmatrix} a_1 - a_0 \\ b_1 - b_0 \\ c_1 - c_0 \end{pmatrix} \text{ and}$$

$$\boldsymbol{AC} = \boldsymbol{OC} - \boldsymbol{OA} = \boldsymbol{b} = \begin{pmatrix} a_2 - a_0 \\ b_2 - b_0 \\ c_2 - c_0 \end{pmatrix}.$$

NB: $\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- Determine the equation:

  $\boldsymbol{PA}.\boldsymbol{n} = 0$

$\boldsymbol{PA}$ is perpendicular to $\boldsymbol{n}$.

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} . \boldsymbol{n} = 0,$$ where the point $P(x;\ y;\ z)$ is the arbitrary point in the plane.

## METHOD 2

### STEPS

- Write the parametric equations i.e. $\boldsymbol{r} = \boldsymbol{OA} + s\boldsymbol{AB} + t\boldsymbol{AC}$
- Express the constants $s$ and $t$ in terms of $x$, $y$ or $z$ by attempting to solve the simultaneous equations
- Replace these values in any equation

## METHOD 3

The equation of the plane passing through the points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$ is given by:

$$\begin{vmatrix} x - a_1 & y - b_1 & z - c_1 \\ a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ a_3 - a_1 & b_3 - b_1 & c_3 - c_1 \end{vmatrix} = 0$$

## Solved Problem

### Question

Find the equation of the plane containing the points $P = (1;\ 3;\ 2)$, $Q = (3;\ -1;\ 6)$ and $R = (5;\ 2;\ 0)$.

### Suggested Solution

### Method 1

First we must find a vector orthogonal to the plane containing the three points.

Let $\boldsymbol{a} = \overrightarrow{PQ} = <2;\ -4;\ 4>$ and let $\boldsymbol{b} = \overrightarrow{PR} = <4;\ -1;\ -2>$, then using the cross product we have a vector perpendicular to the plane.

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= i\begin{vmatrix}-4 & 4\\ -1 & -2\end{vmatrix} - j\begin{vmatrix}2 & 4\\ 4 & -2\end{vmatrix} + k\begin{vmatrix}2 & -4\\ 4 & -1\end{vmatrix}$$

$$= 12\boldsymbol{i} + 20\boldsymbol{j} + 14\boldsymbol{k}$$

$$\therefore \vec{n} = \begin{pmatrix}12\\20\\14\end{pmatrix}$$

The equation of the plane is

$$\begin{pmatrix}x-1\\y-3\\z-2\end{pmatrix}.\begin{pmatrix}12\\20\\14\end{pmatrix} = 0$$

$$12x - 12 + 20y - 60 + 14z - 28 = 0$$

$$12x + 20y + 14z = 0 + 60 + 28 + 12$$

$$6x + 10y + 7z = 50$$

## Method 2

$P = (1;\ 3;\ 2)$, $Q = (3;\ -1;\ 6)$ and $R = (5;\ 2;\ 0)$.

$\overrightarrow{PQ} = \begin{pmatrix}2\\-4\\4\end{pmatrix}$ and let $\overrightarrow{PR} = \begin{pmatrix}4\\-1\\-2\end{pmatrix}$

Now using $\boldsymbol{r} = \overrightarrow{OP} + s\overrightarrow{PQ} + t\overrightarrow{PR}$

$$\boldsymbol{r} = \overrightarrow{OP} + s\overrightarrow{PQ} + t\overrightarrow{PR}$$

$$\boldsymbol{r} = \begin{pmatrix}1\\3\\2\end{pmatrix} + s\begin{pmatrix}2\\-4\\4\end{pmatrix} + t\begin{pmatrix}4\\-1\\-2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+2s+4t\\3-4s-t\\2+4s-2t\end{pmatrix}$$

$x = 1 + 2s + 4t$ (i)

$y = 3 - 4s - t$ (ii)

$z = 2 + 4s - 2t$ (iii)

Solving the above equations simultaneously:

$4 \times$ equation (ii) + equation (i)

$4y + x = 13 - 14s$

$$\Rightarrow s = \frac{13 - 4y - x}{14} \qquad \text{(iv)}$$

equation (ii) + equation (iii)

$y + z = 5 - 3t$

$$\Rightarrow t = \frac{5 - y - z}{3} \qquad \text{(v)}$$

Now substituting equation (iv) and equation (v) into equation (i)

$x = 1 + 2s + 4t \quad \text{(i)}$

$$\Rightarrow x = 1 + 2\left(\frac{13 - 4y - x}{14}\right) + 4\left(\frac{5 - y - z}{3}\right)$$

$$\Rightarrow x = 1 + \left(\frac{13 - 4y - x}{7}\right) + 4\left(\frac{5 - y - z}{3}\right)$$

$\Rightarrow 21x = 21 + (39 - 12y - 3x\ ) + 28(5 - y - z)$

$\Rightarrow 21x = 21 + 39 - 12y - 3x + 140 - 28y - 28z$

$\Rightarrow 24x + 40y + 28z = 200$

$\Rightarrow 6x\ +\ 10y\ +\ 7z\ =\ 50$

## Method 3

The equation of the plane passing through the points $P\ =\ (1;\ 3;\ 2)$, $Q\ =\ (3;\ -1;\ 6)$ and $R\ =\ (5;\ 2;\ 0)$ is given by:

$$\begin{vmatrix} x-1 & y-3 & z-2 \\ 3-1 & -1-3 & 6-2 \\ 5-1 & 2-3 & 0-2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y-3 & z-2 \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 0$$

$$(x-1)\begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} - (y-3)\begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} + (z-2)\begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix} = 0$$

$12(x-1) - (-20)(y-3) + 14(z-2) = 0$

$12x - 12 + 20y - 60 + 14z - 28 = 0$

$12x + 20y + 14z = 100$

$6x + 10y + 7z = 50$

## Follow Up Questions

### Question 1

Find the equation of the plane containing the points $P = (-2;\ -3;\ 5)$, $Q = (2;\ 5;\ -3)$ and $R = (5; 3;\ -3)$.

Answer: $2x + 3y + 4z = 7$

### Question 2

Find the equation of the plane containing the points $P = (1;\ 0;\ 2)$, $Q = (-1;\ 1;\ 2)$ and $R = (5;\ 0;\ 3)$.

Answer: $x + 2y - 4z = -7$

### Question 3

Find the equation of the plane containing the points $P = (1; 2; 3)$, $Q = (-2;\ 4;\ -1)$ and $R = (3;\ -1;\ 5)$.

Answer: $x + 2y - 4z = -7$

### Question 4

Find the equation of the plane with normal vector, $\boldsymbol{n} = < 1, 2, 3 >$ containing the point $(1;\ 0;\ -1)$

Answer: $x + 2y + 3z = -2$

### Question 5

Find the equation of the plane with normal vector, $\boldsymbol{n} = < 5; 2;\ -3 >$ containing the point $(3;\ -2; 5)$

Answer: $5x + 2y - 3z = -4$

## Question 6

Find the equation of the plane with normal vector, $\boldsymbol{n} = < -2, 0, 1 >$ containing the point $(2, 2, 2)$

## Question 7

Find the equation of the plane that is equidistant from $(0, -3, 1)$ and $(2, -1, 5)$.

## Question 8

Find the equation of the plane that is equidistant from $(3, 4, 3)$ and $(3, 0, 7)$.

## Question 9

Find the equation of the plane that is equidistant from $(1, -1, 1)$ and $(5, 1, \ 3)$.

## Question 10

Find the equation of the plane which contains the point $(2, -1, 3)$ and the line

$\boldsymbol{r} = \boldsymbol{j} - 3\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$.

## Question 11

Find the equation of the plane which contains the point $(0, 2, 0)$ and the line

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

## Question 12

Find the equation of the plane which contains the point $(1, -2, 3)$ and the line $\frac{x-1}{4} = \frac{y}{-2} = \frac{z+2}{3}$.

# Parametric equation of a Plane

Let:

$(x_0;\ y_0;\ z_0)$ be an initial point on the plane;

$\vec{a} = < a_1, a_2, a_3 >$ and

$\vec{b} = < b_1, b_2, b_3 >$ are contained the plane

The vector equation is given by $< x, y, z > = < x_0, y_0, z_0 > + t\vec{a} + u\vec{b}$, where $t$ and $u$ are parameters. This implies that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + u\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Thus the parametric equations are:

$$x = x_0 + a_1 t + b_1 u$$
$$y = y_0 + a_2 t + b_2 u$$
$$z = z_0 + a_3 t + b_3 u$$

## Solved Problems

### Question 1

Find the parametric equations through the point $(3,2,1)$ and containing the vectors $\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$.

### Solution

The vector equation is given by $< x, y, z > = < 3,2,1 > + t < 1, -1, 2 > + u < 3, 4, -1 >$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + u\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$$

Thus the parametric equations are:

$$x = 3 + t + 3u$$
$$y = 2 - t + 4u$$
$$z = 1 + 2t - u$$

## Question 2

Consider the plane with direction vectors $\vec{a} = \begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ through $P(3,7,0)$.

Write the vector equations of the plane.

## Answer

The vector equation is given by $< x, y, z > = < 3,7,0 > +t < 8,-5,4 > +u < 1,-3,-2 >$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix} + u\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

Parametric equations for the plane are

$$x = 3 + 8t + u$$

$$y = 7 - 5t - 3u$$

$$z = 4t - 2u$$

## Follow Up Questions

## Question 1

Find the parametric equations through the point $(2, 2, -2)$ and containing the vectors $\vec{a} =< -3, -3,\ 1 >$ and $\vec{b} =< 1, -1,\ 1 >$ .

## Question 2

Find the parametric equations through the point $(1, 0, -2)$ and containing the vectors $\vec{a} =< 0, 8, -2 >$ and $\vec{b} =< -3, -4, 1 >$ .

## Question 3

Find the parametric equations through the point $(-1, -1, -1)$ and containing the vectors $\vec{a} = < 1, -1, 2 >$ and $\vec{b} =< 1, 4, -4 >$ .

## Question 4

Find the parametric equations through the point $(1,3,2)$ and containing the vectors $\vec{a} =< 3, -1, 0 >$ and $\vec{b} =< 1, -4,1 >$ .

## Question 5

Find the parametric equations through the point (8,0,2) and containing the vectors $\vec{a} =< -3,1,1 >$ and $\vec{b} =< 2,0,5 >$.

## Summary Notes

- The Vector equation of the plane is given by: $r\,.\,\boldsymbol{n} = d$
- The Cartesian equation of the plane is given by:

$$ax + by + cz = d$$

- The normal to the plane $ax + by + cz = d$ is $< a, b, c >$ or $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Just consider the coefficients of $x$, $y$ and $z$.

  Take note of the following:

$$ax + by + cz = d$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$$

$$r\,.\,\boldsymbol{n} = d$$

  Hence $\boldsymbol{n} =< a, b, c >$

- For a scalar equation, any point $(x, y, z)$ that satisfies the equation lies on the plane.
- For vector and parametric equations, any combination of values of the parameters $t$ and $s$ will produce a point on the plane.
- The $x$ −intercept of a plane is found by setting $y = z = 0$ and solving for $x$. Similarly, the $y -$ and $z -$intercepts are found by setting $x = z = 0$ and $x = y = 0$, respectively.
- The equation of a plane that cuts the co-ordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

# Determining whether a point lies in a plane

The point is said to lie in a plane if its coordinates satisfy the equation of a plane

## Solved Problems

### Question 1

Show that the point $(5,2,-2)$ lies in the plane $2x + 4y + 5z = 8$.

### Solution

Substitute the point $(5,2,-2)$ to the LHS of the plane $2x + 4y + 5z = 8$.

Now

LHS:

$$\begin{aligned} 2x + 4y + 5z &= 2(5) + 4(2) + 5(-2) \\ &= 10 + 8 - 10 \\ &= 8 \equiv RHS \end{aligned}$$

Since $LHS \equiv RHS$ $\therefore$ the point $(5,2,-2)$ lies in the plane $2x + 4y + 5z = 8$.

### Question 2

Consider the plane with direction vectors $\vec{a} = \begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ through $P(3,7,0)$.

(a) Write the vector and parametric equations of the plane.

(b) Determine if the point $Q(-10,8,-6)$ is on the plane.

(c) Find the coordinates of two other points on the plane.

(d) Find the $x$-intercept of the plane.

### Suggested Solution

(a) $x = 3 + 8t + u$

$y = 7 - 5t - 3u$

$z = 4t - 2u$

(b) If the point $(-10,8,-6)$ is on the plane, then there exists a single set of $t$ and $s$ values that satisfy the equations.

$-10 = 3 + 8t + s \ (1)$

$8 = 7 - 5t - 3s \ (2)$

$6 = 4t - 2s$ (3)

Solve (1) and (2) for $t$ and $s$ using elimination.

$-39 = 24t + 3s$ (4)

$\underline{1 = -5t - 3s \ (2)}$

$-38 = 19t \qquad (4) + (2)$

$\therefore t = -2$

Substitute $t = -2$ into either equation 1 or 2.

$-13 = 8(-2) + s$

$s = 3$

Now check if $t = -2$ and $s = 3$ satisfy (3).

RHS $= 4(-2) - 2(3)$

$= -14$

$\neq$ LHS

Since the values for $t$ and $s$ do not satisfy equation (3), the point $(-10, 8, -6)$ does not lie on the plane.

(c) To find other points on the plane, use the vector equation and choose arbitrary values for the parameters $t$ and $s$.

Let $t = 1$ and $s = -1$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} + \begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 5 \\ 6 \end{pmatrix}$$

Let $t = 2$ and $s = 1$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} + (2)\begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix} + (1)\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} + \begin{pmatrix} 16 \\ -10 \\ 8 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 20 \\ -6 \\ 6 \end{pmatrix}$$

The coordinates of two other points on the plane are $(10, 5, 6)$ and $(20, -6, 6)$.

(d) To find the $x$-intercept, set $y = z = 0$ and solve for $s$ and $t$.

$x = 3 + 8t + s \quad (1)$

$0 = 7 - 5t - 3s \quad (2)$

$0 = 4t - 2s \quad (3)$

From (3) $s = 2t \quad (4)$

Substituting (4) into (2):

$0 = 7 - 5t - 3(2t)$

$0 = 7 - 11t$

$\Rightarrow t = \dfrac{7}{11}$

$s = 2t \quad (4)$

$\Rightarrow s = \dfrac{14}{11}$

Now substitute $t = 7/11$ and $t = 14/11$ into (1)

$x = 3 + 8t + s \quad (1)$

$x = 3 + 8\left(\dfrac{7}{11}\right) + \dfrac{14}{11}$

$x = 3 + \dfrac{7}{11} + \dfrac{14}{11}$

$\therefore x = \dfrac{103}{11}$

## Follow Up Questions

### Question 1

Show that the point $(5,2,-2)$ lies in the plane $2x+6y-z=24$.

### Question 2

Show that the point $(2,0,-1)$ lies in the plane $3x+y+z=5$.

### Question 3

Show that the point $(1,1,3)$ lies in the plane $x-3y+2z=4$.

### Question 4

Show that the point $(-2,3,0)$ lies in the plane $-5x-3y-2z=1$.

### Question 5

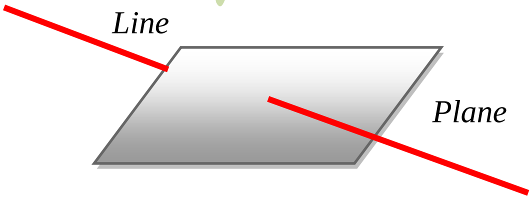
Show that the point $(1,3,2)$ lies in the plane $x+2y+z=9$.

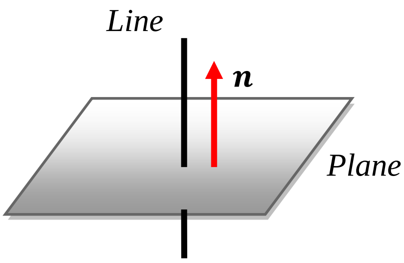


## Determining whether a line lies in a plane, is parallel to or intersects a plane

### Case 1

The line and the plane are not parallel so they intersect on one point



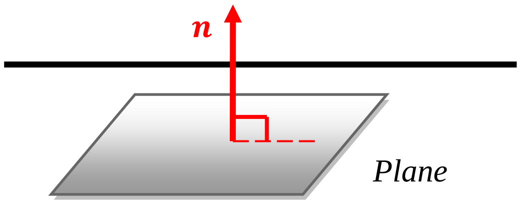

- A line intersects the plane when a value of the parameter can be estimated.

- Alternatively, when the direction vector of the line is parallel to the normal vector of the plane, then line is said to be perpendicular to the plane i.e. $\vec{d} = \phi\vec{n}$
- If the line and the plane intersect then the dot product of the normal and the direction vector is not equal to zero i.e.

$$\vec{d}.\vec{n} \neq 0$$

**Case 2**

The line and the plane are parallel so they do not intersect




- When a line is parallel to a plane, its direction vector is perpendicular to the plane's normal vector i.e.

$$\vec{d}.\vec{n} = 0$$

**Case 3**

The line and the plane are parallel and the line lies in the plane.


- To prove that a line lies in a plane, you need to show that the line and the plane are parallel and that any point on the line also lies in the plane i.e.
  - $\vec{d}.\vec{n} = 0$ and
  - The point '$a$' on the line should also be contained on the plane :
    - $r = a$ satisfies $r.n = D$ or
    - the coordinates of the position vector of A must satisfy $Ax + By + Cz = D$.

## Solved Problems

### Question 1

Does the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ lie in the plane $2x + 4y + 5z = 8$?

### Solution

The direction vector of the line, $\boldsymbol{d}$, is $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and normal vector of the plane, $\widehat{\boldsymbol{n}}$, is $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

$$\boldsymbol{d}.\widehat{\boldsymbol{n}} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}.\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = 6 + 4 - 10 = 0.$$

Since $\boldsymbol{d}.\widehat{\boldsymbol{n}} = 0$, the line is parallel to the plane.

Now find a point on the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ by setting $t = 1$ or any value of your choice.

So the point $(5,2,-2)$ lies on the line

Now check if the point satisfies the equation of the plane $2x + 4y + 5z = 8$.

LHS:

$$\begin{aligned} 2x + 4y + 5z &= 2(5) + 4(2) + 5(-2) \\ &= 10 + 8 - 10 \\ &= 8 \equiv RHS \end{aligned}$$

$LHS \equiv RHS \therefore$ the point $(5,2,-2)$ lies in the plane $2x + 4y + 5z = 8$.

Since the line and the plane are parallel and the point $(5,2,-2)$ lies both on the line and in the plane, therefore the line must lie in the plane.

### Alternative Method:

The direction vector of the line, $\boldsymbol{d}$, is $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and normal vector of the plane, $\widehat{\boldsymbol{n}}$, is $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

$$\boldsymbol{d}.\hat{\boldsymbol{n}} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = 6 + 4 - 10 = 0.$$

Since $\boldsymbol{d}.\hat{\boldsymbol{n}} = 0$, the line is parallel to the plane.

Now substituting $\boldsymbol{r} = \begin{pmatrix} 2+3t \\ 1+t \\ -2t \end{pmatrix}$ into $2x + 4y + 5z = 8$:

LHS:

$$2x + 4y + 5z = 2(2+3t) + 4(1+t) + 5(-2t)$$

$$= 4 + 6t + 4 + 4t - 10t$$

$$= 8 \equiv RHS$$

$LHS \equiv RHS$ $\therefore$ the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ lies in the plane $2x + 4y + 5z = 8$.

<u>Question 2</u>

Show that the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ is parallel to the plane $2x + 4y + 5z = 8$.

<u>Solution</u>

The direction vector of the line, $\boldsymbol{d}$, is $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and normal vector of the plane, $\hat{\boldsymbol{n}}$, is $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

$$\boldsymbol{d}.\hat{\boldsymbol{n}} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = 6 + 4 - 10 = 0.$$

Since $\boldsymbol{d}.\hat{\boldsymbol{n}} = 0$, the line is parallel to the plane.

## Question 3

Given a line $L$ and a plane $P$: $L: x = 15 - 3t, y = 6 - 5t, z = 2 + 3t$

$P: 2x + 3y + 4z = 20.$ Determine if $L$ and $P$ intersect or are parallel.

## Solution

The normal vector of $P$ is $\boldsymbol{n} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and the line $L$ is parallel $to\ \boldsymbol{v} = \begin{pmatrix} -3 \\ -5 \\ 3 \end{pmatrix}$

Now:

$$\boldsymbol{n} \cdot \boldsymbol{v} = (2)(-3) + (3)(-5) + 4(3)$$

$$= -6 - 15 + 12$$

$$= -9 \neq 0,$$

Since $\boldsymbol{n} \cdot \boldsymbol{v} \neq 0$ $\therefore$ $L$ and $P$ are not parallel to each other.

Checking for intersection:

- Substitute the parametric equations of $L$ into the equation of $P$
- Solve the resulting equation for $t$:

$$2(15 - 3t) + 3(6 - 5t) + 4(2 + 3t) = 20$$

$$\Rightarrow -9t = -36$$

$$\Rightarrow t = 4.$$

Substitute $t = 4$ in the parametric equations of $L$

$\therefore$ The point of intersection is $(3, -14, 14)$

## Question 4

Given a line $L$ and a plane $P$:

$$L: x = 15 + 7t, y = 10 + 12t, z = 5 - 4t$$

$$P: 12x - 5y + 6z = 10.$$

Determine if $L$ and $P$ intersect or are parallel.

## Solution

The normal vector of $P$ is $\boldsymbol{n} = \begin{pmatrix} 12 \\ -5 \\ 6 \end{pmatrix}$ and the line $L$ is parallel to $\boldsymbol{v} = \begin{pmatrix} 7 \\ 12 \\ -4 \end{pmatrix}$.

Now:

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{v} &= (7)(12) + (-5)(12) + 6(-4) \\ &= 84 - 60 - 24 \\ &= 0 \end{aligned}$$

Since $\boldsymbol{n} \cdot \boldsymbol{v}$ $\therefore$ $L$ and $P$ are parallel to each other.

Checking for intersection:

Since $L$ is parallel to $P$, if $L$ intersects $P$, then any point of $L$ should be in $P$.

When $t = 0$, the point $(15, 10, 5)$ is on $P$.

Now:

$$\begin{aligned} 12x - 5y + 6z &= 12(15) - 5(10) + 6(6) \\ &= 180 - 50 + 36 \\ &= 166 \\ &\neq 10 \therefore L \text{ and } P \text{ do not intersect.} \end{aligned}$$

## Question 5

Show that the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ intersects the plane $5x + y - z = 1$.

## Solution

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and so for any point on the line

$$x = 2 + t, \quad y = 3 + 2t \quad \text{and} \quad z = 4 - t$$

Substituting these into the equation of the plane $5x + y - z = 1$ gives

$$\begin{aligned} 5(2 + t) + (3 + 2t) - (4 - t) &= 1 \\ 8t &= -8 \\ \therefore t &= -1 \end{aligned}$$

Since there is a constant value of the parameter $t$, thus the line intersects and the plane.

## Question 6

Show that the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 6 \\ -3 \end{pmatrix}$ is perpendicular to the plane $3x + 2y - z = 1$.

## Solution

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 6 \\ -3 \end{pmatrix}$$

Now:

$$\boldsymbol{d} = \begin{pmatrix} 9 \\ 6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

Also

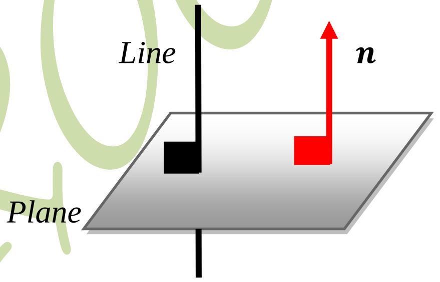
$3x + 2y - z = 1$

Now:

$$\boldsymbol{n} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

Since $\boldsymbol{d} = 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 3\boldsymbol{n}$, $\therefore$ they are parallel hence the plane is perpendicular to the plane.

NB: Consider the following diagram for clarity






## Follow Up Questions

## Question 1

Does the line $\frac{x-1}{2} = \frac{y+5}{3} = \frac{z-3}{-1}$ lie in the plane $5x - y + z = 11$?

## Question 2

Given a line $M$ and a plane $P$, $M: x = -2t, y = 3 - t, z = 2 + 2t$ and $P: x - 3y + z = 14$. Determine if $M$ and $P$ intersect or are parallel.

## Question 3

Does the line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ lie in the plane $\boldsymbol{r}.\begin{pmatrix} 4 \\ -8 \\ -8 \end{pmatrix} = 4$?

## Question 4

Discuss the line $l: \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k} + \lambda(5\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$ and the plane $\pi$, where $\pi: \boldsymbol{r}.(2\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k}) = -8$ .

## Question 5

Determine whether the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is perpendicular to the plane $x - 2y + z = 2$.

## Question 6

Does the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ lie in the plane $\boldsymbol{r}.\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2$?

## Question 7

Given a line $L$ and a plane $P$:

$$L: x = 10 + 2t, \; y = -3 - 10t, \; z = -5 + 6t$$

$$P: x - 5y + 3z = 10.$$

Determine if $L$ and $P$ intersect or are parallel.

### Question 8

Show that the line $\frac{x+11}{5} = \frac{y-25}{-10} = \frac{z}{15}$ is parallel to plane $-x + 2y - 3z = 11$.

### Question 9

Discuss the line $l: 3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k} + \lambda(\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k})$ and the plane $\pi$, where

$\pi: 2x + 3y - 2z = 2$.

## Finding the point of intersection of a line and a plane

- The vector equation of a straight line passing through the fixed point, with position vector $\boldsymbol{r}_1$, and parallel to the fixed vector $\boldsymbol{a}$, is

$$r = \boldsymbol{r}_1 + t\boldsymbol{a}.$$

- For the point of intersection of this line with the plane, whose vector equation is

$$\boldsymbol{r} \,.\, \boldsymbol{n} = d,$$

the value of $t$ must be such that

$$(\boldsymbol{r}_1 + t\boldsymbol{a}).\boldsymbol{n} = d,$$

which is an equation from which the appropriate value of t and, hence, the point of intersection may be found.

## Solved Problems

### Question 1

Find the intersection of the line $x = t, y = 2t, z = 3t$, and the plane $x + y + z = 1$.

### Solution

Substitute the line into the plane:

$t + 2t + 3t = 1 \Rightarrow t = \frac{1}{6}$

Put $t$ back to the line:

$x = \frac{1}{6}, y = \frac{2}{6} = \frac{1}{3}, z = \frac{3}{6} = \frac{1}{2}$

Hence the intersection point is $\left(\frac{1}{6};\frac{1}{3};\frac{1}{2}\right)$.

Question 2

Determine the point of intersection of the plane, whose vector equation is $r\,.(\boldsymbol{i}-3\boldsymbol{j}-\boldsymbol{k})=7$, and the straight line passing through the point $(4,-1,3)$ which is parallel to the vector $2\boldsymbol{i}-2\boldsymbol{j}+5\boldsymbol{k}$.

Solution

The equation of the line is given by:

$r=(4\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k})+t(2\boldsymbol{i}-2\boldsymbol{j}+5\boldsymbol{k}.)$

We need to obtain the value of the parameter:

$r\,.(\boldsymbol{i}-3\boldsymbol{j}-\boldsymbol{k})=7$

$(4\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}+t[2\boldsymbol{i}-2\boldsymbol{j}+5\boldsymbol{k}]).(\boldsymbol{i}-3\boldsymbol{j}-\boldsymbol{k})=7.$

$\Rightarrow (4+2t)(1)+(-1-2t)(-3)+(3+5t)(-1)=7$

$\Rightarrow 4+3t=7$

$\Rightarrow t=1$

Put $t$ back to the line:

$(4+2,-1-2,3+5)=(6,-3,8).$

Hence the intersection point is $(6,-3,8)$.

Follow Up Questions

Question 1

Find the intersection of the line $r=\begin{pmatrix}-5\\1\\3\end{pmatrix}+t\begin{pmatrix}4\\-1\\3\end{pmatrix}$ and the plane $r.\begin{pmatrix}1\\1\\1\end{pmatrix}=1$

Question 2

Find the intersection of the line $r=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$ and the plane $x-2y-z=3$.

Question 3

Find the intersection of the line $\frac{x-1}{2}=\frac{y+5}{3}=\frac{z-3}{-1}$ and the plane $.\begin{pmatrix}-1\\-1\\2\end{pmatrix}=7$ .

## Question 4

Determine the point of intersection of the plane, whose vector equation is $r\,.(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 4$, and the straight line passing through the point, $(1, -2, 1)$, which is parallel to the vector $3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$.

## Question 5

Find the point intersection of the line $\frac{x-1}{2} = \frac{y+5}{3} = \frac{z-3}{-1}$ and the plane $x - 2y - z = 3$.

## Question 6

Find the intersection of the line $x = 1 - t, y = 2 + 2t, z = 3 - 3t$, and the plane $r.\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = 1$ .

## Question 7

Find the intersection of the line $r = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ and the plane $r.\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$

# Coplanarity of two lines

- Two lines are coplanar if they lie on the same plane
- The lines are of the form $\boldsymbol{r} = \overrightarrow{OA} + s\overrightarrow{d_1}$ and $\boldsymbol{r} = \overrightarrow{OB} + t\overrightarrow{d_2}$

## Method 1

### Steps

- Find the cross product of the two vectors to determine the normal

$$\vec{n} = \overrightarrow{d_1} \times \overrightarrow{d_2}$$

- Use the two points on the lines to determine the 3rd vector

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

- Find the dot product between the 3rd vector and the normal.

$$\overrightarrow{AB}.\vec{n} = 0$$

NB: (i) If the dot product is zero it means the lines are coplanar

(ii) Do not take the dot product between $\vec{n}$ and either $\overrightarrow{d_1}$ or $\overrightarrow{d_2}$

(ii) Intersecting lines are coplanar lines

## Method 2: Cartesian Form

- Let $\overrightarrow{d_1} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\overrightarrow{d_2} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\overrightarrow{OA} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.
- Two lines are coplanar if and only if $\overrightarrow{AB}.(\overrightarrow{d_1} \times \overrightarrow{d_2}) = 0$.
- This can be expressed in cartesian form as:

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Solved Problem

Question

Show that

(i) the lines $\boldsymbol{r} = \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix} + s\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$ are coplanar,

(ii) $x - 2y + z = 0$ is the equation of the plane which contains the two lines.

Suggested Problem

Method 1

(i) $\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix}$

$$= i\begin{vmatrix} 5 & 7 \\ 4 & 7 \end{vmatrix} - j\begin{vmatrix} 3 & 7 \\ 1 & 7 \end{vmatrix} + k\begin{vmatrix} 3 & 5 \\ 1 & 4 \end{vmatrix}$$

$$= i(35 - 28) - j(21 - 7) + k(12 - 5)$$

$$= 7i - 14j + 7k$$

$$= \begin{pmatrix} 7 \\ -14 \\ 7 \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}$$

Now:

$$\overrightarrow{AB}.\vec{n} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}.\begin{pmatrix} 7 \\ -14 \\ 7 \end{pmatrix}$$

$$= 21 - 98 + 77$$

$$= 0$$

Since $\overrightarrow{AB}.\vec{n} = 0$ $\therefore$ the lines are coplanar

(ii) $\boldsymbol{r}.\vec{n} = \vec{a}.\vec{n}$

$$\boldsymbol{r}.\begin{pmatrix}7\\-14\\7\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix}.\begin{pmatrix}7\\-14\\7\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}7\\-14\\7\end{pmatrix} = 14 - 56 + 42$$

$$\Rightarrow 7x - 14y + 7z = 0$$

$$\therefore x - 2y + z = 0\ (As\ required).$$

## Method 2

$$\begin{vmatrix}x_2 - x_1 & y_2 - y_1 & z_2 - z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix} = \begin{vmatrix}3 & 7 & 11\\ 3 & 5 & 7\\ 1 & 4 & 7\end{vmatrix}$$

$$= 3\begin{vmatrix}5 & 7\\4 & 7\end{vmatrix} - 7\begin{vmatrix}3 & 7\\1 & 7\end{vmatrix} + 11\begin{vmatrix}3 & 5\\1 & 4\end{vmatrix}$$

$$= 3(35 - 28) - 7(21 - 7) + 11(12 - 5)$$

$$= 3(7) - 7(14) + 11(7)$$

$$= 21 - 98 + 77$$

$$= 0\ (As\ required)$$

## Follow up Exercise

## Question

Show that the following lines are coplanar:

(i) $\boldsymbol{r} = \begin{pmatrix}-3\\1\\5\end{pmatrix} + s\begin{pmatrix}-3\\1\\5\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}-1\\2\\5\end{pmatrix} + s\begin{pmatrix}-1\\2\\5\end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix}2\\1\\-3\end{pmatrix} + s\begin{pmatrix}3\\2\\5\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}4\\\frac{16}{7}\\0\end{pmatrix} + s\begin{pmatrix}2\\3\\1\end{pmatrix}$

(iii) $\boldsymbol{r} = \begin{pmatrix}5\\2\\-1\end{pmatrix} + s\begin{pmatrix}1\\-2\\-3\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}2\\0\\4\end{pmatrix} + s\begin{pmatrix}1\\2\\-1\end{pmatrix}$

(iv) $\boldsymbol{r} = \begin{pmatrix}-2\\1\\0\end{pmatrix} + s\begin{pmatrix}3\\2\\1\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}1\\-1\\8\end{pmatrix} + s\begin{pmatrix}1\\2\\-2\end{pmatrix}$

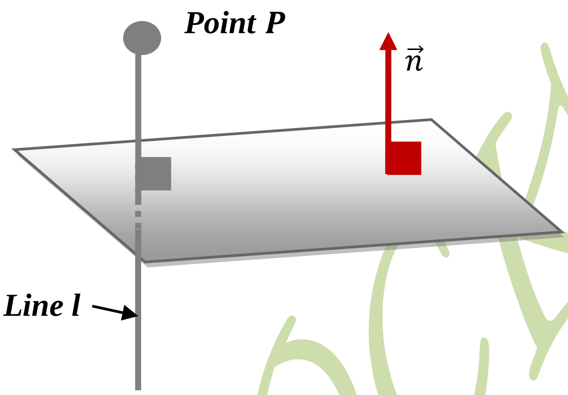
(v) $\boldsymbol{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + s\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}2\\2\\1\end{pmatrix} + s\begin{pmatrix}-3\\2\\-1\end{pmatrix}$

# Equation of a line perpendicular to the Plane

When finding the equation of the line that is perpendicular to the plane take note of the following:

- The normal vector ($\vec{n}$) of the plane is the direction vector of the line which passes through the point, say $P$.
- Suppose the normal vector is given by $\vec{n} = < a, b, c >$ and the line passes through the point $P(x, y, z)$ then the equation is given by

$$r = \overrightarrow{OP} + t\vec{n} \Rightarrow r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$




## Solved Problems

### Question 1

Find the equation of the line passing through $(2,1,-1)$ and which is perpendicular to the plane $3x - 7y + 2z = 8$

### Solution

The normal vector is given by $\vec{n} = < 3, -7, 2 >$ and the line passes through the point $P(2,1,-1)$

$\therefore$ The equation is given by $r = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix}$

## Question 2

**ZIMSEC June 2019 Paper 2**

Find the equation of the line perpendicular to the plane $\pi$: $r.\begin{pmatrix}4\\-1\\3\end{pmatrix} = 1$ through $P(-5,1,3)$.

## Solution

The normal vector is given by $\vec{n} =< 4, -1, 3 >$ and the line passes through the point $P(-5, 1, 3)$

$\therefore$ The equation is given by $r = \begin{pmatrix}-5\\1\\3\end{pmatrix} + t\begin{pmatrix}4\\-1\\3\end{pmatrix}$.

## Follow Up Questions

## Question 1

Find the equation of the line perpendicular to the plane $\pi$: $r.\begin{pmatrix}1\\0\\3\end{pmatrix} = -1$ through $P(2, 0, -1)$.

## Question 2

Find the equation of the line perpendicular to the plane $\pi$: $r.\begin{pmatrix}3\\-1\\4\end{pmatrix} = 2$ through $P(5, -1, -3)$.

## Question 3

Find the equation of the line perpendicular to the plane $\pi$: $r.\begin{pmatrix}2\\1\\-3\end{pmatrix} = 6$ through $P(-1, 1, -1)$.

## Question 4

Find the equation of the line passing through $(1, 0, 3)$ and which is perpendicular to the plane $x - 2z = 4$

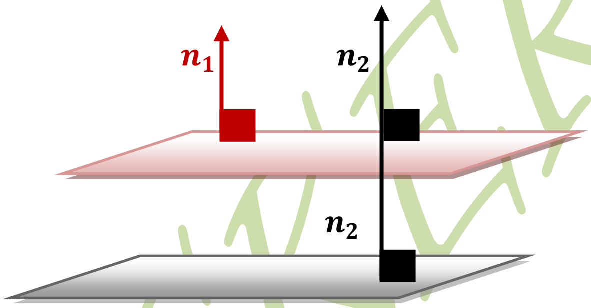
## Question 5

Find the equation of the line passing through $(-1,-1,2)$ and which is perpendicular to the plane $x+y-2z=-1$

## Question 6

Find the equation of the line passing through $(2,0,3)$ and which is perpendicular to the plane $5x-2y+z=2$

## Determining whether two Planes are Parallel




Two planes are parallel if

- their normal vectors are multiples of each other,
- the angle between the normals $n_1$ and $n_2$ is zero.

## Illustration

Show that the planes $\pi_1: 6x-15y-3z+9=0$ and $\pi_2: 2x-5y-z-1=0$ are parallel.

$\boldsymbol{n}_1 = \begin{pmatrix} 6 \\ -15 \\ -3 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ $[2,-5,-1]$.

$$\Rightarrow \boldsymbol{n}_1 = \begin{pmatrix} 6 \\ -15 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} = 3\boldsymbol{n}_2.$$

## Solved Problems

## Question1

Are the planes defined by $x+2y-3z=4$ and $2x+4y-6z=1$ parallel?

Solution

A normal vector to the first plane is $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ while a normal vector to the second plane is $\boldsymbol{b} == \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$.

$$\Rightarrow \boldsymbol{b} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 2\boldsymbol{a}$$

Since $\boldsymbol{b}$ is a scalar multiple of $\boldsymbol{a}$ (namely $\boldsymbol{b} = 2\boldsymbol{a}$) then the normal vectors are parallel, which implies the original planes are parallel.

Question 2

Find a point $P_0$ and the perpendicular vector $\boldsymbol{n}$ to the plane

Solution

The equation of a plane is $2x + 4y - z = 3$

The components of the normal vector $\boldsymbol{n}$ are the coefficients that multiply the variables $x, y$ and $z$.

Hence $\boldsymbol{n} = < 2;\ 4;\ -1 >$

A point $P_0$ on the plane is found by considering the intersection of the plane with one of the coordinate axis.

We can set $y = 0, z = 0$ and find $x$ from the equation of the plane:

$\Rightarrow 2x = 3$

$\Rightarrow x = 3/2.$

$\therefore P_0 = (3/2, 0, 0)$

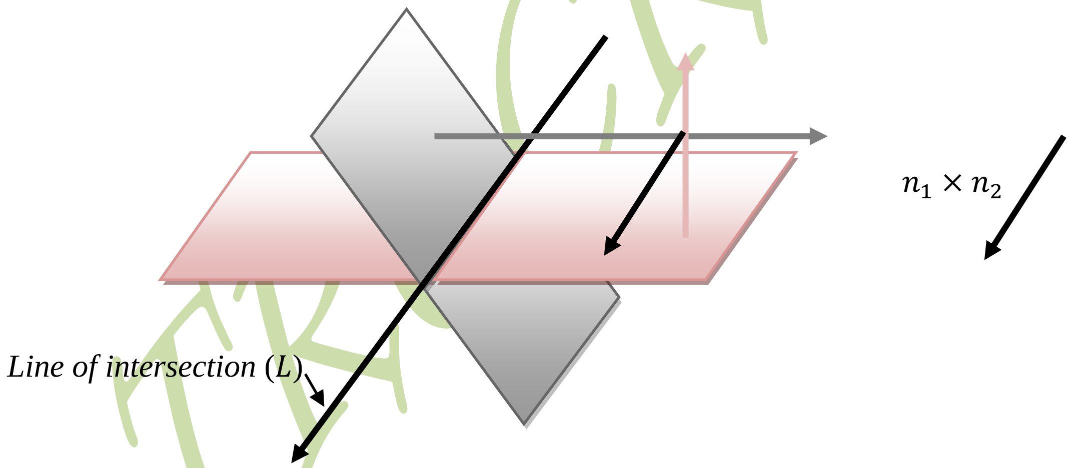
## Intersection of two planes

If two planes are not parallel, they must intersect in a line and in non-parallel lines, normals are not multiples of each other.

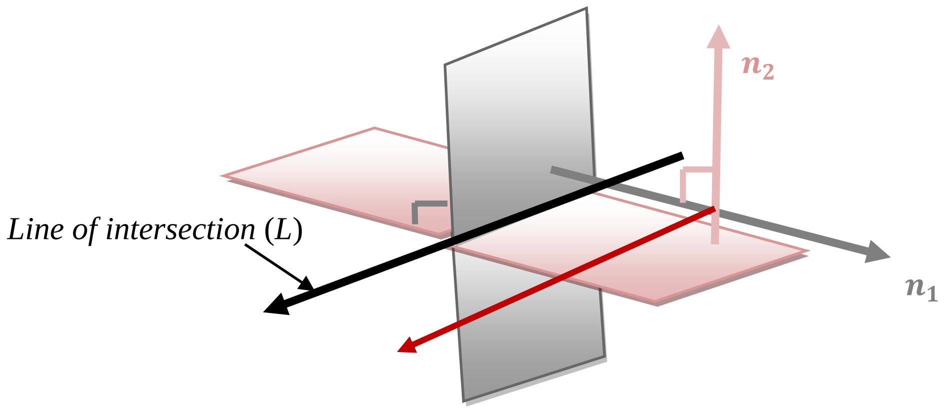
### NOTE

- The Normals of planes can identify relationships between the planes.
  - If the Normals of the two planes are scalar multiples of each other, the planes are parallel and can have either NO solution or an Infinite number of solutions.
  - If the Normals are not parallel, then the planes intersect and have an infinite number of points in common.
- This is an important concept because it opens the door for using parameters to solve problems.

## Line of intersection of two planes




- From the diagram the vector $n_1 \times n_2$ is parallel to the line of intersection of two planes. That is the line of intersection of two planes is perpendicular to both normals.




- If the normals $n_1$ and $n_2$ are perpendicular to each other that means they are at $90°$ angle and their dot product is equal to zero.

▪ We can employ the following two methods to find this line of intersection

## Method 1: Algebraic Approach

Let $P_1$ and $P_2$ are two planes with normal vectors $\vec{n}_1$ and $\vec{n}_2$ respectively.

▪ Find the direction vector of the line

<u>Steps</u>

- The two planes are parallel if and only if their normal vectors are scalar multiples of each other otherwise they intersect.
- Non-parallel planes intersect but instead of intersecting at a single point they form a line from the set of points where they intersect
- Let's call the line $L$ with direction vector $\vec{d}$.
- $L$ is contained in $P_1$ and $\vec{n}_1$ must be orthogonal/perpendicular to $\vec{d}$.
- Similarly, $L$ is contained in $P_2$ and $\vec{n}_2$ must be orthogonal to $\vec{d}$ as well.
- $\vec{d}$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ thus $\vec{d}$ is parallel to the cross product of $\vec{n}_1$ and $\vec{n}_2$ i.e.

$$\vec{d} = \vec{n}_1 \times \vec{n}_2$$

▪ Find the point on the line

<u>Steps</u>

- Let one of the variables be 0

NB: (i) We have 3 unknowns and two equations

(ii) You can use any number but 0 is much convenient

- Solve the equations simultaneously to get the values of the remaining variables

  NB: We now have 2 unknowns and two equations

- Write the point in coordinate form i.e. $(x,\ y,\ z)$
- Express $x, y$ and $z$ in the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

- Write the equation in the required form i.e. as a vector equation, parametric equations or Cartesian equations

## Method 2: Parameter Approach

- Choose any variable and express it in term of $t$
- Substitute the variable in (i) with $t$ in the two equations
- Attempt to solve these equations simultaneously by expressing the remaining variables explicitly in terms of $t$ only
- Write the equation in the required form i.e. as a vector equation, parametric equations or Cartesian equations

## Solved Problems

### Question 1

Find a vector equation of the line of intersections of the two planes $x - 5y + 3z = 11$ and $-3x + 2y - 2z = -7$.

### Solution

### Method 1

First we read off the normal vectors of the planes:

The normal vector $\vec{n}_1$ of $x - 5y + 3z = 11$ is $\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$ and

The normal vector $\vec{n}_2$ of $-3x + 2y - 2z = -7$ is $\begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$

Next, we find the direction vector $\vec{d}$ for the line of intersection, by computing

$$\vec{d} = \vec{n}_1 \times \vec{n}_2 = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$$

$$= \begin{vmatrix} i & j & k \\ 1 & -5 & 3 \\ -3 & 2 & -2 \end{vmatrix}$$

$$= 4\boldsymbol{i} - 7\boldsymbol{j} - 13\boldsymbol{k}$$

$$= \begin{pmatrix} 4 \\ -7 \\ -13 \end{pmatrix}.$$

Now, we have to find $P$ {a point on the line}

Let $z = 0$:

$\Rightarrow x - 5y + 0 = 11$ (i)

$-3x + 2y + 0 = -7$ (ii)

From (i)

$x = 5y + 11$ (iii)

Plugging equation (iii) into equation (ii)

$-3(5y + 11) + 2y = -7$

$\Rightarrow y = -2.$

Now:

$x = 5y + 11$ (iii)

$\Rightarrow x = -10 + 11$

$\therefore x = 1$

$\Rightarrow P(1; -2; 0)$ is a point on both planes.

NB: We can plug $P$ in to the given equations of the plane to double check:

$(1) - 5(-2) + 3(0) = 11$ and $-3(1) + 2(-2) - 2(0) = -7.$

So, the point on the line is $(1; -2; 0)$ and the direction vector $\begin{pmatrix} 4 \\ -7 \\ -13 \end{pmatrix}$

Now:

The vector equation of the line is

$$r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ -7 \\ -13 \end{pmatrix}, t \in \mathbb{R}$$

## Method 2

Let $z = t$

$x - 5y + 3t = 11$ (i)

$-3x + 2y - 2t = -7$ (ii)

Now using equation (i):

$x - 5y + 3t = 11$

$x = 11 + 5y - 3t$ (iii)

Substituting equation (iii) into (ii)

$-3x + 2y - 2t = -7$

$-3(11 + 5y - 3t) + 2y - 2t = -7$

$-33 - 15y + 9t + 2y - 2t = -7$

$13y = 7 - 33 + 7t$

$13y = 7t - 26$

$y = \frac{7}{13}t - 2$ (iv)

Substituting equation (iv) into (iii)

$x = 11 + 5y - 3t$

$x = 11 + 5\left(\frac{7}{13}t - 2\right) - 3t$

$x = 11 + \frac{35}{13}t - 10 - 3t$

$x = 11 - 10 + \frac{35}{13}t - 3t$

$x = 1 - \frac{4}{13}t$ (v)

Now:

The parametric equations are:

$x = 1 - \frac{4}{13}t$

$y = -2 + \frac{7}{13}t$

$z = t$

The vector equation is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{4}{13} \\ \frac{7}{13} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \left(-\frac{1}{13}\right)t\begin{pmatrix} 4 \\ -7 \\ -13 \end{pmatrix}$$

NB: Since $\left(-\frac{1}{13}\right)t$ is just a constant thus we can replace it by any parameter say $\lambda$

$$\therefore \boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -7 \\ -13 \end{pmatrix}$$

## Question 2

Determine the vector equation, and hence the Cartesian equations (in standard form), of the line of intersection of the planes whose vector equations are

$$\boldsymbol{r}\,.\,\boldsymbol{n}_1 = 2 \text{ and } \boldsymbol{r}\,.\,\boldsymbol{n}_2 = 17,$$

where

$\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{n}_2 = 4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$.

## Solution

Firstly:

$$\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}.$$

Secondly:

The Cartesian equations of the two planes are

$$x + y + z = 2 \text{ and } 4x + y + 2z = 17;$$

Letting z = 0:

$x + y = 2$ (i)

$4x + y = 17$ (ii)

Equation (ii) – Equation (i)

$3x = 15$

$\Rightarrow \; x = 5$

$x + y = 2$ (i)

$\Rightarrow 5 + y = 2$

$\therefore y = -3$

Thirdly:

The point of intersection is $(5, -3, 0)$ and its position vector is given by $5i - 3j$.

Now:

The vector equation of the line of intersection is

$r = 5i - 3j + t(i + 2j - 3k).$

Finally:

The parametric equations are: $x = 5 + t, \; y = -3 + 2t$ and $z = -3t$

$\Rightarrow$The Cartesian equation is

$$\frac{x-5}{1} = \frac{y+3}{2} = \frac{z}{-3}$$

## Question 3

Find the parametric equations of the line of intersection of the two planes

$$x + y + z = 1$$
$$x - 2y + 3z = 2$$

## Solution

Let $z = t$

$x + y + t = 1$ (i)

$x - 2y + 3t = 2$ (ii)

Now using equation (i):

$x + y + t = 1$

$x = 1 - y - t$ (iii)

Substituting equation (iii) into (ii)

$x - 2y + 3t = 2$

$1 - y - t - 2y + 3t = 2$

$3y = 1 - 2 + 2t$

$3y = 2t - 1$

$y = \frac{2}{3}t - \frac{1}{3}$ (iv)

Substituting equation (iv) into (iii)

$x = 1 - y - t$

$x = 1 - \left(\frac{2}{3}t - \frac{1}{3}\right) - t$

$x = 1 - \frac{2}{3}t + \frac{1}{3} - t$

$x = 1 + \frac{1}{3} - \frac{2}{3}t - t$

$x = \frac{4}{3} - \frac{5}{3}t$ (v)

Now:

The parametric equations are:

$x = \frac{4}{3} - \frac{5}{3}t; \quad y = -\frac{1}{3} + \frac{2}{3}t; \quad z = t$

## Question 4

Write the parametric and symmetric equations of the line of intersection of the planes $2x - y + z = 5$ and $x - y - z = 1$.

## Solution

The planes have normal vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ respectively.

Let $L$ denote the line of intersection.

The direction vector $v$ is parallel to $L$

$$\vec{d} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= i\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} - j\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}$$

$$= 2i + 3j - k$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

Find a point $P$ on $L$ by letting $x$ or $y$ or $z = 0$.

Now let $z = 0$

$2x - y = 5$ (i)

$x - y = 1$ (ii)

Subtracting equation (ii) from (i)

$\Rightarrow x = 4$

Now:

$x - y = 1$ (ii)

$4 - y = 1$

$y = 5.$

$\therefore$ The point is $(4;5;0)$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

The parametric equations of the line are

$$x = 4 + 2t$$
$$y = 5 + 3t$$
$$z = -t$$

The Cartesians equations of the line are

$$\frac{x-4}{2} = \frac{y-5}{3} = \frac{z}{-1}$$

## Question 5

Determine the equation of the line that passes through the point $P(5,2,3)$ and is parallel to the line of intersection of the two planes:

$$\pi_1 : x + 2y - z = 6$$
$$\pi_2 : y + 2z = 1$$

## Solution

Since we have the point, find the direction vector (slope)

**Method One**

$\overrightarrow{n_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{n_2} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$$\vec{d} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= i\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 5\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

$$= \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

∴The equation is $\boldsymbol{r} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

**Method Two**

Find the equation of the line of intersection using the parameter approach

Let $z = t$

$x + 2y - t = 6$ (i)

$y + 2t = 1$ (ii)

Now using equation (ii):

$y + 2t = 1$

$y = 1 - 2t$ (iii)

Substituting equation (iii) into (i)

$x + 2y - t = 6$

$x + 2(1 - 2t) - t = 6$

$x + 2 - 4t - t = 6$

$x = 5t + 6 - 2$

$x = 5t + 4$ (iv)

Now:

The parametric equations are:

$x = 4 + 5t$

$y = 1 - 2t$

$z = t$

The Vector Equation of the line of intersection:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

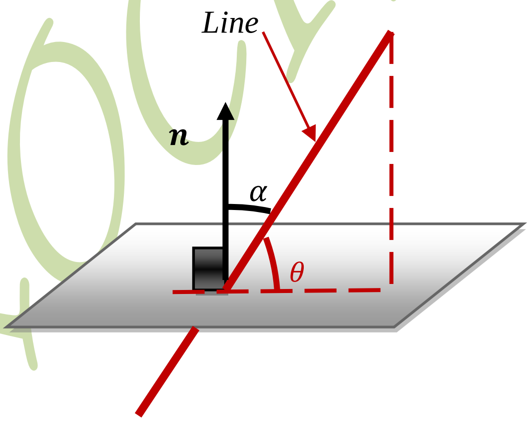
$$\Rightarrow \vec{r} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Use the direction vector of the line of intersection and point $P(5,2,3)$ to determine the desired line.

$$\boldsymbol{r} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

## Angle between a line and a plane

The angle between the line and a plane is the complement between the line and the normal to the plane




- The angle is defined to be the angle between the direction vector of the line and the normal vector of the plane.
- The acute angle $\theta$ between the line $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ and plane $r \,.\, \boldsymbol{n} = d$ is given by:

$$\cos\alpha = \left|\frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}\right|$$

$$\therefore \theta = 90° - \alpha$$

Alternatively it is calculated as follows:

$Since\ Sin(90° - \theta) \equiv Cos\theta$

Then:

$$sin\theta = \left|\frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}\right|$$

Solved Problem

Question

Find the angle between the plane with equation $r.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$ and the line with equation

$r = \begin{pmatrix}2\\0\\5\end{pmatrix} + t\begin{pmatrix}3\\12\\4\end{pmatrix}$, where $t$ is the parameter.

Solution

$$sin\theta = \frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}$$

$$sin\theta = \frac{\begin{pmatrix}3\\12\\4\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix}}{\sqrt{3^2+12^2+14^2}\sqrt{(-1)^2+2^2+2^2}}$$

$$sin\theta = \frac{-3+24+8}{\sqrt{169}\sqrt{9}}$$

$$\theta = sin^{-1}\left(\frac{29}{39}\right) = 48.03811117° = 48°$$

Alternatively we can use the following method:

$\theta = 90° - \alpha$

Now:

$$cos\alpha = \frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}$$

$$cos\alpha = \frac{\begin{pmatrix}3\\12\\4\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix}}{\sqrt{3^2+12^2+14^2}\sqrt{(-1)^2+2^2+2^2}}$$

$$cos\alpha = \frac{-3+24+8}{\sqrt{169}\sqrt{9}}$$

$$\alpha = \cos^{-1}\left(\frac{29}{39}\right) = 41.96188883°$$

Now:

$\theta = 90° - \alpha$

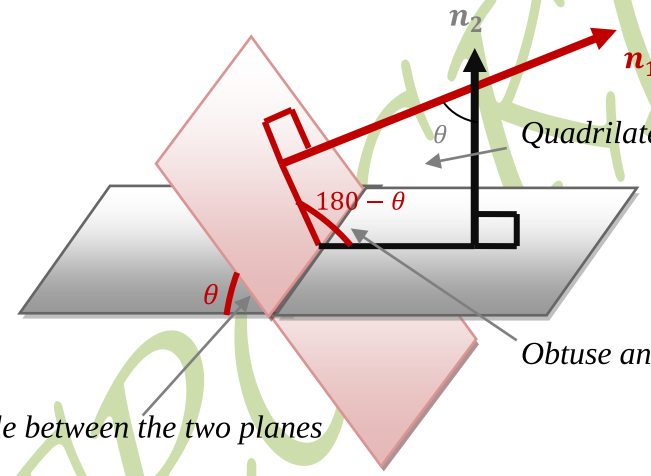
$\theta = 90° - 41.96188883°$

$\theta = 48.03811117°$

$\therefore \theta = 48°$

## Angle between two planes

The angle between the two planes is defined as the angle between their normals.




- From the diagram we can actually see that the angle between the two normals is $\theta$ (use the properties of angles of a quadrilateral.)
- The angle between the two normals is calculated using the fact that the dot product of vectors $\overrightarrow{n_1}$and $\overrightarrow{n_2}$ is equal the product of the magnitude of vector $\overrightarrow{n_1}$ and magnitude of vector $\overrightarrow{n_2}$ and cosine of angle $\theta$. This implies that $\overrightarrow{n_1}.\overrightarrow{n_2} = |\overrightarrow{n_1}|.|\overrightarrow{n_2}|\cos\theta$
- This implies that:

$$\cos\theta = \frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$$

- This further implies that the angle between the two planes is also the angle between the two normals.
- Thus, suppose two planes intersect, then the angle between the planes is defined to be the angle between their normal vectors.
- If the normal vectors of the two planes are $\overrightarrow{n_1}$and $\overrightarrow{n_2}$, then the angle is given by

$$\cos\theta = \left|\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}\right|$$

## Solved Problems

### Question 1

Find the angle between the plane with equation $r.\begin{pmatrix}1\\-2\\2\end{pmatrix} = 66$ and the plane with equation $r.\begin{pmatrix}4\\0\\3\end{pmatrix} = 27$.

### Solution

$$\cos\theta = \left|\frac{\begin{pmatrix}1\\-2\\2\end{pmatrix}.\begin{pmatrix}4\\0\\3\end{pmatrix}}{\sqrt{1^2+(-2)^2+2^2}\sqrt{4^2+0^2+3^2}}\right|$$

$$\cos\theta = \left|\frac{4+6}{\sqrt{9}\sqrt{25}}\right|$$

$$\cos\theta = \left|\frac{10}{15}\right|$$

$$\cos\theta = \frac{10}{15}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

$$= 48.189685104°$$

$$= 48°$$

## Question 2

Find the angle between the two planes with equations $2x - y + z = 5$ and $x + y - z = 1$ respectively.

## Solution

$$cos\ \theta = \left|\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}\right|$$

$$\overrightarrow{n_1} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} and\ \overrightarrow{n_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$cos\ \theta = \left|\frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{2^2 + (-1)^2 + 1^2}\sqrt{1^2 + 1^2 + (-1)^2}}\right|$$

$$cos\ \theta = \frac{0}{\sqrt{6}\sqrt{3}}$$

$$cos\ \theta = 0$$

$$\theta = cos^{-1}(0)$$

$$\theta = 90°$$

## Question 3

Find the angle between the planes

$$x + y + z = 1$$

$$x - 2y + 3z = 2$$

## Solution

The angle between the planes will be the angle between their normal vectors.

Let $\overrightarrow{n_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{n_2} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$$cos\ \theta = \left|\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}\right|$$

$$\cos\theta = \left|\frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\3\end{pmatrix}}{\sqrt{3}\sqrt{14}}\right|$$

$$\cos\theta = \left|\frac{2}{\sqrt{42}}\right|$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$$

$$\theta = 72.02471619°$$

$$\theta = 72°$$

## Follow Up Questions

### Question 1

Find the angle between the plane with equation $2x + 3y + z = 4$ and the line with equation $r = \begin{pmatrix}2\\-1\\0\end{pmatrix} + t\begin{pmatrix}-1\\2\\5\end{pmatrix}$, where $t$ is the parameter.

Answer: 26°

### Question 2

Find the angle between the two planes with equations $2x + 3y + 5z = 8$ and $5x + y - 4z = 12$ respectively.

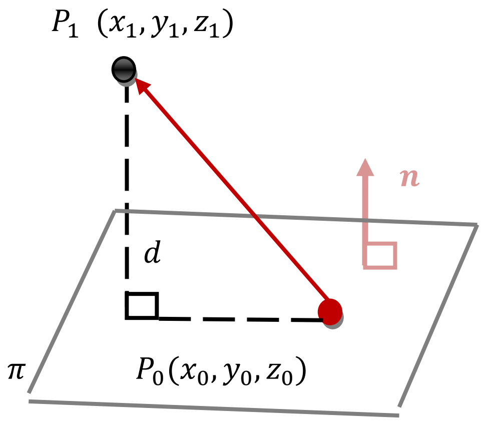
Answer: 79.9°

### Question 3

Find the angle between the two planes with equations $x + 3y + z = -4$ and $5x - y + z = 8$ respectively.

Answer: 80.0

# The perpendicular distance from a point to a plane




## Vector Form

- Let consider a plane $\pi$ with a normal vector $\vec{n}$ and a point $P_0(x_0, y_0, z_0)$ on this plane.

  NB: The point $P_0$ is found by letting any two variables equal to 0 and solve the equation
- The distance from a point $P_1(x_1, y_1, z_1)$ to the plane $\pi$ is given by the scalar projection of the vector $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$

$$d = \frac{|\overrightarrow{P_0P_1}.\vec{n}|}{|\vec{n}|}$$

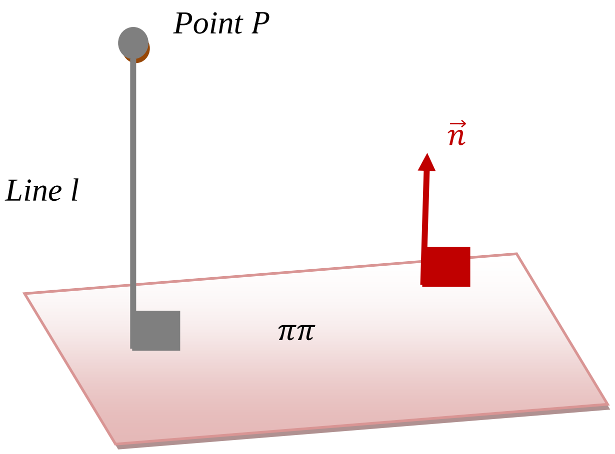
## Cartesian Form

If the plane π is given by the Cartesian equation $Ax + By + Cz + D = 0$, then the distance from a point $P_1(x_1, y_1, z_1)$ to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

NB: $|Ax_1 + By_1 + Cz_1 + D|$ means the absolute value of $Ax_1 + By_1 + Cz_1 + D$.

## Alternative Method




## Steps

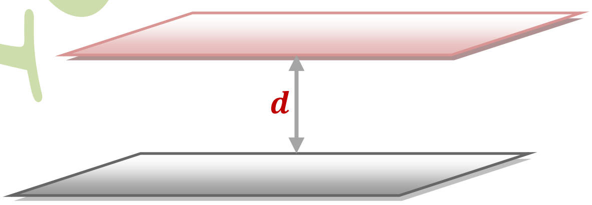
(i) Find the equation of the vector which is perpendicular to the plane: The normal vector of the plane is the direction vector of the line which passes through the point $P$. i.e.

$r = \overrightarrow{OP} + t\vec{n}$

(ii) Find the point of intersection between the line and the plane. NB: The point of intersection represents the coordinates of the foot of the perpendicular from a given point

(iii) Lastly find the length between P and the foot of this perpendicular line using coordinate geometry

## The distance between two Parallel Planes




## Steps

a) Find a specific point into one of these planes.

b) Find the distance between that specific point and the other plane using one of the methods above.

## Solved Problems

### Question 1

Determine the perpendicular distance, $p$, of the point $(2, -3, 4)$ from the plane whose cartesian equation is $x + 2y + 2z = 13$.

### Solution

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$d = \frac{|(1)(2) + (2)(-3) + (2)(4) - 13|}{\sqrt{(1)^2 + (2)^2 + (2)^2}}$$

$$d = \frac{9}{3}$$

$$\therefore d = 3.$$

### Question 2

Determine the perpendicular distance $p$ of the point (2,3,0) from the plane whose cartesian equation is $-x + 2y - 3z + 10 = 0$.

### Suggested Solution

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$d = \frac{10 + (-1)(2) + (2)(3) + (0)(-3)}{\sqrt{(-1)^2 + (2)^2 + (-3)^2}}$$

$$d = \frac{14}{\sqrt{14}}$$

$$\therefore d = \sqrt{14}$$

NB: *Three methods are employed on the following example*

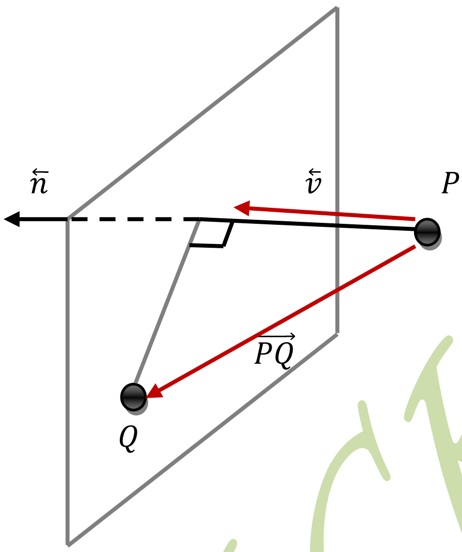
## Question 3

Find the distance between the point $P = (1;\ 0; -1)$ and the plane $5x + 4y + 3z = 1$.

## Solution

### Method 1

- Draw the point and the plane with an additional point $Q$ on the plane.




- The distance is the length of the vector $\vec{v}$ and $\vec{v}$ is the projection of $\overrightarrow{PQ}$ onto $\vec{n}$
- $\vec{n}$ is the normal vector to the plane, which is $\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$

Thus the distance is given by:

$$d = \frac{|\overrightarrow{PQ}.\vec{n}|}{|\vec{n}|}$$

### Finding $\overrightarrow{PQ}$

Set $x = y = 0$ and plug these values into the plane equation

$\Rightarrow z = \frac{1}{3}$

$\left(0;\ 0;\ \frac{1}{3}\right)$ is on the plane.

This means that $\overrightarrow{PQ} = \left(-1;\ 0;\ \frac{4}{3}\right)$.

Now

$$d = \frac{|\overrightarrow{PQ}.\vec{n}|}{|\vec{n}|}$$

$$d = \frac{\left|\begin{pmatrix} -1 \\ 0 \\ \frac{4}{3} \end{pmatrix} . \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}\right|}{\sqrt{5^2 + 4^2 + 3^2}}$$

$$d = \frac{|-5 + 4|}{\sqrt{50}}$$

$$d = \frac{|-1|}{5\sqrt{2}}$$

$$d = \frac{1}{5\sqrt{2}}$$

$$\therefore d = \frac{\sqrt{2}}{10}$$

Method 2

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

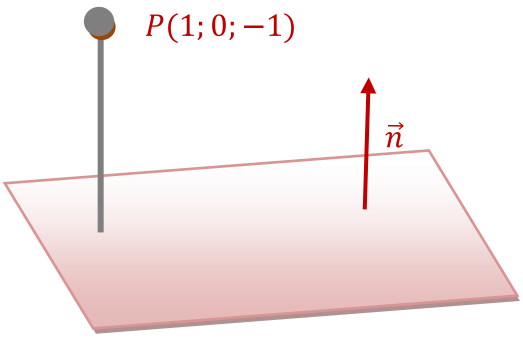
$$d = \frac{|(5)(1) + (4)(0) + (3)(-1) - 1|}{\sqrt{(5)^2 + (4)^2 + (3)^2}}$$

$$d = \frac{|1|}{\sqrt{50}}$$

$$d = \frac{1}{5\sqrt{2}}$$

$$\therefore d = \frac{\sqrt{2}}{10}$$

## Method 3




The equation of the line (in grey) is given by:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1+5t \\ 4t \\ -1+3t \end{pmatrix} \qquad \text{(i)}$$

Now substituting equation (i) into $5x + 4y + 3z = 1$ to find the value of the parameter $t$.

$\Rightarrow 5(1+5t) + 4(4t) + 3(-1+3t) = 1$

$\Rightarrow 5 + 25t + 16t - 3 + 9t = 1$

$\Rightarrow 50t = -1$

$\therefore t = -\frac{1}{50}$.

Now $\boldsymbol{r} = \begin{pmatrix} 1+5t \\ 4t \\ -1+3t \end{pmatrix} \qquad \text{(i)}$

$$\Rightarrow \boldsymbol{r} = \begin{bmatrix} 1+5\left(-\frac{1}{50}\right) \\ 4\left(-\frac{1}{50}\right) \\ -1+3\left(-\frac{1}{50}\right) \end{bmatrix}$$

$$\therefore \boldsymbol{r} = \begin{pmatrix} \frac{9}{10} \\ -\frac{2}{25} \\ -\frac{53}{50} \end{pmatrix}$$

The coordinates of the foot are $\left(\frac{9}{10};\ -\frac{2}{25}; -\frac{53}{50}\right)$.

Now finding the distance between $P(1;0;-1)$ and the foot $\left(\frac{9}{10};\ -\frac{2}{25};-\frac{53}{50}\right)$.

$$d=\sqrt{\left(1-\frac{9}{10}\right)^2+\left(0+\frac{2}{25}\right)^2+\left(-1+\frac{53}{50}\right)^2}$$

$$\Rightarrow d=\sqrt{\frac{1}{100}+\frac{4}{625}+\frac{9}{2500}}$$

$$\Rightarrow d=\sqrt{\frac{1}{50}}$$

$$\Rightarrow d=\frac{1\times\sqrt{50}}{\sqrt{50}\times\sqrt{50}}$$

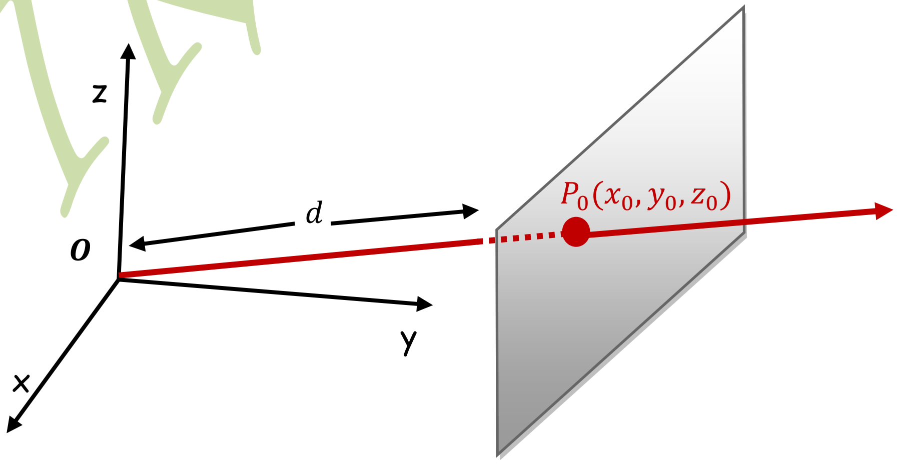
$$\Rightarrow d=\frac{\sqrt{50}}{50}$$

$$\Rightarrow d=\frac{5\sqrt{2}}{50}$$

$$\therefore d=\frac{\sqrt{2}}{10}$$

NB: *When to finding the coordinates of the foot of the perpendicular we use method 3*

## The shortest distance from the origin to a plane



## Cartesian Form

If the plane $\pi$ is given by the Cartesian equation $Ax + By + Cz + D = 0$, then the distance from the origin to $O(0,0,0)$ to the plane is given by:

$$d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$$

NB: $|D|$ means the absolute value of $D$.

## Vector Form

- Let consider a plane $\pi$ with a normal vector $\vec{n}$ and a point $P_0(x_0, y_0, z_0)$ on this plane.

  NB: The point $P_0$ is found by letting any two variables equal to 0 and solve the equation
- The distance from a point $O(0,\ 0,\ 0)$ to the plane $\pi$ is given by the scalar projection of the vector $\overrightarrow{OP_0}$ onto the normal vector $\vec{n}$

$$d = \frac{|\overrightarrow{OP_0}.\vec{n}|}{|\vec{n}|}$$

## Solved Problem

### Question

Show that the distance from the origin to the plane with equation $\boldsymbol{r}.(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 13$ is $\frac{13\sqrt{6}}{6}$.

### Solution

$$d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$d = \frac{|-13|}{\sqrt{2^2 + 1^2 + (-1)^2}}$$

$$d = \frac{13}{\sqrt{4 + 1 + 1}}$$

$$d = \frac{13}{\sqrt{6}}$$

$$d = \frac{13\sqrt{6}}{6} \quad (as\ required)$$

## Follow up questions

### Question 1

Determine the vector equation and hence the cartesian equation of the plane, passing through the point with position vector $\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$, and perpendicular to the vector $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$.

Answer: $r\,.(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = 10$, or $2x + y - 3z = 10$.

### Question 2

Determine the cartesian equation of the plane passing through the three points: $(1, -1, 2)$, $(3, -2, -1)$ and $(-1, 4, 0)$.

Answer: $17x + 10y + 8z = 23$.

### Question 3

Determine the point of intersection of the plane, whose vector equation is

$\boldsymbol{r}\,.(5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = -3$, and the straight line passing through the point $(2, 1, -3)$, which is parallel to the vector $\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$.

Answer: $(0, -1, 5)$.

### Question 4

Determine the vector equation, and hence the cartesian equations (in standard form), of the line of intersection of the planes, whose vector equations are $\boldsymbol{r}\,.\boldsymbol{n}_1 = 14$ and $\boldsymbol{r}\,.\boldsymbol{n}_2 = -1$, where $\boldsymbol{n}_1 = -4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{n}_2 = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$.

Answer: $r = -2\boldsymbol{i} + 3\boldsymbol{j} + t(7\boldsymbol{i} + 10\boldsymbol{j} - 8\boldsymbol{k})$ or

$$\frac{x+2}{7} = \frac{y-3}{10} = \frac{z}{-8} (= t)$$

## Question 5

Determine, in surd form, the perpendicular distance of the point $(-5, -2, 8)$ from the plane whose cartesian equation is $2x - y + 3z = 17$.

Answer: $p = \frac{1}{\sqrt{14}}$

## Question 6

Find the distance from the point $(1, 4, 1)$ to the plane $2x + 3y - z = -1$.

Answer: $p = \sqrt{14}$

## Question 7

Find the distance between the point $P\ (1;\ 0; -1)$ and the plane $5x + 4y + 3z = 1$.

Answer: $p = \frac{\sqrt{2}}{10}$

## Question 8

Find the distance between the point $(2;\ 8;\ 5)$ and the plane $x - 2y - 2z = 1$.

Answer: $p = \frac{25}{3}$

## Question 9

Find the distance between the point $P\ (2, 1, 3)$ and the plane $2x + y + 2z = 3$.

Answer: $p = \frac{8}{3}$

## Question 10

Show that the distance from the origin to the plane with equation $3x - y + 2z = 17$ is $\frac{17\sqrt{14}}{14}$.

## Question 11

Show that the distance from the point $P\ (-2, -3, 1)$ to the plane with equation

$4x + 2y + 4z = 1$ is $\frac{11}{6}$.

## SOLVED PRACTICE QUESTIONS

### Question 1

(a) (i) Show that the point of intersection, $A$, of the line $l_1$: $\frac{x-1}{-2} = \frac{y-3}{4} = \frac{z}{1}$ and the plane $\pi_1$: $\boldsymbol{r}.\begin{pmatrix}3\\-2\\5\end{pmatrix} = 15$ is $A(5;\ -5;\ -2)$.

(ii) Given that the points $A, B$ and $C$ are in the same plane (coplanar points) and $B$ and $C$ have position vectors $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $-\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, respectively.

Find the vector equation of the plane, $\pi_2$ which contains the points $A, B$ and $C$.

(b) Find the perpendicular distance from $B$ to $l_1$.

(c) Another plane $\pi_3$ has an equation $x - 3y + 2z = 2$.

(i) Show that $C$ lies on the plane $\pi_3$.

(ii) Given that $l_2$ is a line which passes through $D(-1; 2; 1)$ and is parallel to the line which passes through $A$ and $B$, find the

1. Cartesian equation of $l_2$ ,
2. angle between $l_2$ and $\pi_3$.

### Suggested Solution

(a) (i) $l_1$: $\frac{x-1}{-2} = \frac{y-3}{4} = \frac{z}{1} \Rightarrow l_1$: $\boldsymbol{r} = \begin{pmatrix}1\\3\\0\end{pmatrix} + t\begin{pmatrix}-2\\4\\1\end{pmatrix} = \begin{pmatrix}1-2t\\3+4t\\t\end{pmatrix}$ (i)

$\pi_1$: $\boldsymbol{r}.\begin{pmatrix}3\\-2\\5\end{pmatrix} = 15$ (ii)

Substituting equation (i) into equation (ii)

$$\begin{pmatrix}1-2t\\3+4t\\t\end{pmatrix}.\begin{pmatrix}3\\-2\\5\end{pmatrix} = 15$$

$\Rightarrow 3 - 6t - 6 - 8t + 5t = 15$

$\Rightarrow -9t = 18$

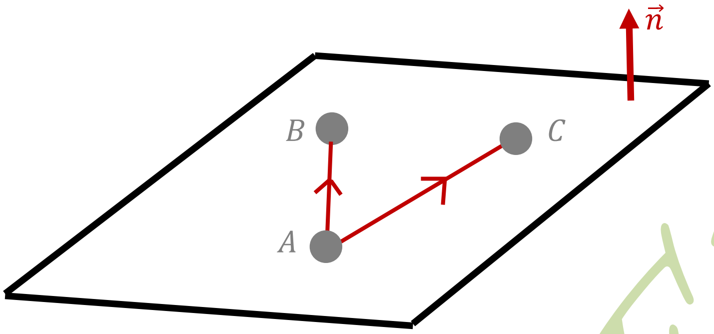
$\Rightarrow t = -2$

Now to get the point of intersection we substitute the numerical value of $t$ into equation (i):

$$\boldsymbol{r} = \begin{pmatrix} 1-2(-2) \\ 3+4(-2) \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -2 \end{pmatrix}$$

$\therefore$ The point of intersection is $A(5;\ -5;\ -2)$.

(ii) $A(5;\ -5;\ -2)$, $B(2;\ -1;\ 1)$ and $C(-1;\ 1;\ 3)$




$$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} \text{ and } \boldsymbol{AC} = \boldsymbol{OC} - \boldsymbol{OA} = \begin{pmatrix} -6 \\ 6 \\ 5 \end{pmatrix}$$

Now

$$\boldsymbol{n} = \boldsymbol{AB} \times \boldsymbol{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3 & 4 & 3 \\ -6 & 6 & 5 \end{vmatrix} = 2\boldsymbol{i} - 3\boldsymbol{i} + 6\boldsymbol{k}$$

Now:

$$\begin{pmatrix} x-5 \\ y+5 \\ x+2 \end{pmatrix} . \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 0$$

$$2x - 10 - 3y - 15 + 6x + 12 = 0$$

$$2x - 3y + 6x = 13$$

$$\Rightarrow \boldsymbol{r}. \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 13$$

Alternatively we can use the following method:

$$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} \text{ and } \boldsymbol{AC} = \boldsymbol{OC} - \boldsymbol{OA} = \begin{pmatrix} -6 \\ 6 \\ 5 \end{pmatrix}$$

The equation is given by:

$$\boldsymbol{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\boldsymbol{r} = \begin{pmatrix} 5 \\ -5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 6 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 6 \\ 5 \end{pmatrix}$$

$$\begin{aligned} & x = 5 - 3\lambda - 6\mu \quad \text{(i)} \\ \Rightarrow\ & y = -5 + 4\lambda + 6\mu \quad \text{(ii)} \\ & z = -2 + 3\lambda + 5\mu \quad \text{(iii)} \end{aligned}$$

Now we have to eliminate $\lambda$ and $\mu$ from the above equations:

Equation (i) + equation (iii)

$x + z = 3 - \mu$

$\Rightarrow \ \mu = 3 - x - z \quad$ (iv)

Equation (i) + equation (ii)

$x + y = \lambda \quad$ (v)

Substituting equations (iv) and (v) in either (i), (ii)or (iii)

Now:

$x = 5 - 3\lambda - 6\mu \quad$ (i)

$\Rightarrow x = 5 - 3(x + y) - 6(3 - x - z)$

$\Rightarrow x = 5 - 3x - 3y - 18 + 6x + 6z$

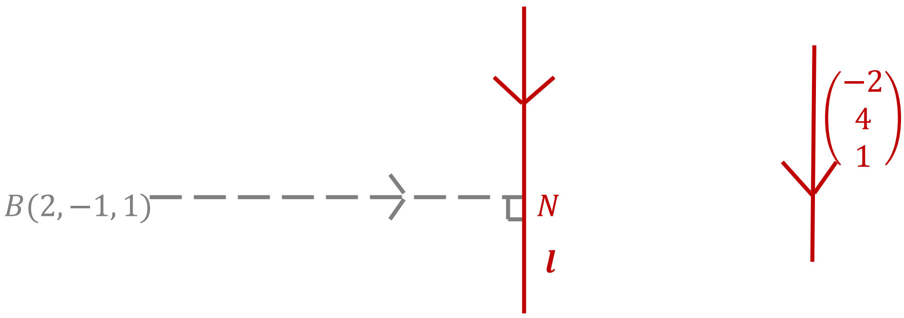
$\Rightarrow -2x + 3y - 6z = -13$

$\Rightarrow 2x - 3y + 6z = 13$

$$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 13$$

(b) Let point $N$ be the foot of the perpendicular from $B$.

$B(2;\ -1;\ 1)$ and the line is $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$



The coordinates of $N$ are $(1-2t;\ 3+4t;\ t)$

Now

$$\boldsymbol{BN} = \begin{pmatrix} -1-2t \\ 4+4t \\ -1+t \end{pmatrix}$$

Now

$$\boldsymbol{BN}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1-2t \\ 4+4t \\ -1+t \end{pmatrix}.\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = 0$$

$$2+4t+16+16t-1+t=0$$

$$21t=-17$$

$$\therefore t=-\frac{17}{21}$$

Now substituting $t=-\frac{17}{21}$ into $\boldsymbol{BN}$

$$\boldsymbol{BN} = \begin{pmatrix} -1-2\left(-\frac{17}{21}\right) \\ 4+4\left(-\frac{17}{21}\right) \\ -1+\left(-\frac{17}{21}\right) \end{pmatrix} = \begin{pmatrix} \frac{13}{21} \\ \frac{16}{21} \\ -\frac{38}{21} \end{pmatrix}$$

Now

$$|\boldsymbol{BN}| = \sqrt{\left(\frac{13}{21}\right)^2+\left(\frac{16}{21}\right)^2+\left(-\frac{38}{21}\right)^2}$$

$$=\sqrt{\frac{169}{441}+\frac{256}{441}+\frac{1444}{441}}=\sqrt{\frac{1869}{441}}$$

$$=\sqrt{\frac{89}{21}}.$$

(c) Another plane $\pi_3$ has an equation $x-3y+2z=2$.

(iii)Substituting $C(-1;\ 1;\ 3)$ into plane $\pi_3$:

$x-3y+2z=2$

LHS:

$x-3y+2z=-1-3(1)+2(3)=-1-3+6=2$

Since LHS $= RHS$ $\therefore$ $C(-1;\ 1;\ 3)$ lies on plane $\pi_3$.

(iv)Given that $l_2$ is a line which passes through $D(-1;2;1)$ and is parallel to the line which passes through $A$ and $B$, find the

1. Cartesian equation of $l_2$ :

$A(5;\ -5;\ -2)$ and $B(2;-1;1)$

$$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}$$

Now the vector equation is $\boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}$

$\therefore$ The Cartesian equation is $\frac{x+1}{-3} = \frac{y-2}{4} = \frac{z-1}{3}$

2. Angle between $l_2$ and $\pi_3$.

$$sin\theta = \left|\frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}\right|$$

$$sin\theta = \left|\frac{\begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}.\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}{\sqrt{(-3)^2+4^2+3^2}\sqrt{(1)^2+(-3)^2+2^2}}\right|$$

$$sin\theta = \left|\frac{\begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}.\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}{\sqrt{9+16+9}\sqrt{1+9+4}}\right|$$

$$sin\theta = \left|\frac{-3-12+6}{\sqrt{34}\sqrt{14}}\right|$$

$$\theta = sin^{-1}\left(\frac{9}{\sqrt{34}\sqrt{14}}\right)$$

$= 24.36287885°$

$= 24°$

Alternatively we use the following method:

$\theta = 90° - \alpha$
Now:

$$cos\alpha = \left|\frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}\right|$$

$$cos\alpha = \left|\frac{\begin{pmatrix}-3\\4\\3\end{pmatrix}.\begin{pmatrix}1\\-3\\2\end{pmatrix}}{\sqrt{(-3)^2+4^2+3^2}\sqrt{(1)^2+(-3)^2+2^2}}\right|$$

$$cos\alpha = \left|\frac{\begin{pmatrix}-3\\4\\3\end{pmatrix}.\begin{pmatrix}1\\-3\\2\end{pmatrix}}{\sqrt{9+16+9}\sqrt{1+9+4}}\right|$$

$$cos\alpha = \left|\frac{-3-12+6}{\sqrt{34}\sqrt{14}}\right|$$

$$\alpha = cos^{-1}\left(\frac{9}{\sqrt{34}\sqrt{14}}\right)$$

$$= 65.63712115°$$

Now:

$\theta = 90° - \alpha$

$\theta = 90° - 65.63712115$

$\theta = 24.36287885°$

$\therefore \theta = 24°$

## Question 2

a) Show that the cartesian equation of the plane, $\pi_1$, that contains the points $P(1;\ 0;\ 2)$, $Q(-1; 1;\ 2)$ and $R(5;\ 0;\ 3)$ is given by $x + 2y - 4z = -7$.

b) Find

(i) the parametric equations of the line, $l_1$, that passes through $P$ and $Q$,

(ii) the vector equation of the line of intersection of the plane $\pi_1$ and $\pi_2$ with equation $\boldsymbol{r}.\begin{pmatrix}2\\-1\\1\end{pmatrix} = 3$. Denote it by $l_2$.

c) Another line, $l_3$, passes through the point $(2;\ -1; 0)$ and is perpendicular the plane $\pi_2$. Find the vector equation of $l_3$.

d) Show that the cartesian equation of the plane, $\pi_3$, consisting of all points which are equidistant from $Q$ and $U(3; 5;\ -2)$ is $x + y - z = 4$.

e) Find the angle between

(i) $l_2$ and $\pi_3$,

(ii) $\pi_1$ and $\pi_2$.

f) Calculate the

(i) perpendicular distance from $Q$ to $\pi_2$,

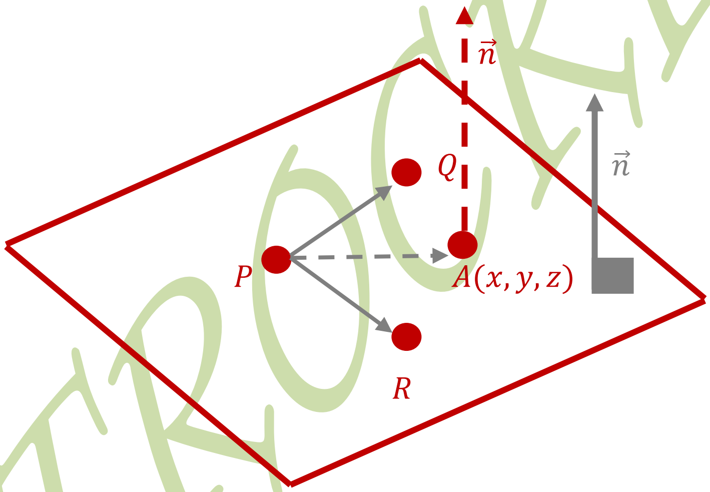
(ii) shortest distance from the origin to $l_2$.

g) Show that $V(-3;\ 0;\ -7)$ lies on the plane $\pi_3$.

h) Find the equation of the plane which contains $V$ and the line $l_2$.

Suggested Solution

a)






- Find any two vectors in the plane:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix} \quad \text{and}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix}5\\0\\3\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}4\\0\\1\end{pmatrix}$$

- Find the normal vector by taking the cross of these two vectors

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{vmatrix} = i\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} - j\begin{vmatrix}-2 & 0\\4 & 1\end{vmatrix} + k\begin{vmatrix}-2 & 1\\4 & 0\end{vmatrix}$$

$$= i + 2j - 4k$$

$$\therefore \vec{n} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

- Find the equation of this plane by letting the arbitrary point be $P(x, y, z)$
- Find $\overrightarrow{AP}$.
- Find the dot product of $\overrightarrow{AP}$ and the normal vector $\vec{n}$ and we set it to zero since they are perpendicular.

Now:

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} x-1 \\ y \\ z-2 \end{pmatrix}$$

Then:

$$\overrightarrow{AP}.\vec{n} = 0$$

$$\Rightarrow \begin{pmatrix} x-1 \\ y \\ z-2 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 0$$

$$\Rightarrow 1(x-1) + 2(y) - 4(z-2) = 0$$

$$\Rightarrow x - 1 + 2y - 4z + 8 = 0$$

$$\Rightarrow x + 2y - 4z = -7 \quad (\text{as required})$$

b) (i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Now:

$$\boldsymbol{r} = \overrightarrow{OP} + t\overrightarrow{PQ}$$

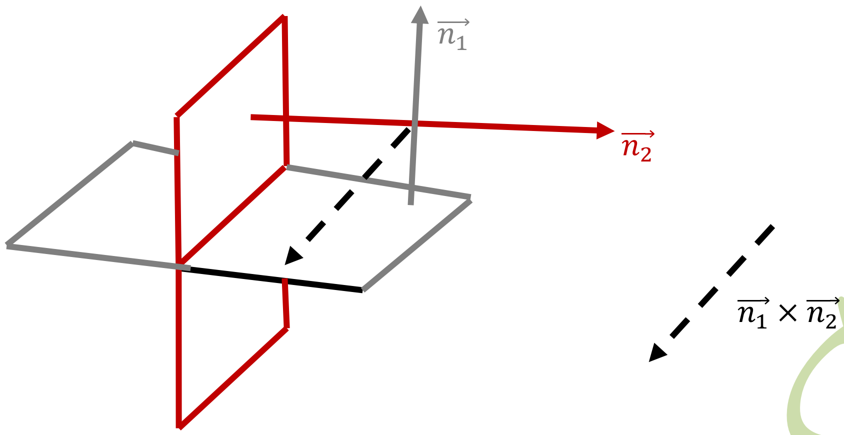
$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Now:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow x = 1 - 2t; \quad y = t; \quad z = 2$$

(ii)




The direction vector of the line is the cross between the normal vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$.

The equation of the planes are $\boldsymbol{r}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 3$ and $\boldsymbol{r}.\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = -7$.

Now:

$\overrightarrow{n_1} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{n_2} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

$$\vec{d} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & -4 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= i\begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= -2i - 9j - 5k$$

$\therefore \vec{d} = \begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix}$

Any point on the line of intersection is found by letting $x = 0, y = 0$ or $z = 0$.

$\Rightarrow 2x - y + z = 3$ and $x + 2y - 4z = -7$

Let $z = 0$:

$\Rightarrow 2x - y = 3$ (i)

$x + 2y = -7$ (ii)

(ii) $\times 2 \Rightarrow$ $2x + 4y = -14$ (iii)

(iii) – (i) gives:

$5y = -17$

$\Rightarrow y = -\frac{17}{5}$.

Now $x + 2y = -7$ (ii)

$\Rightarrow x = -2y - 7 = -2\left(-\frac{17}{5}\right) - 7$

$$= \frac{34}{5} - 7$$

$$= \frac{34 - 35}{5}$$

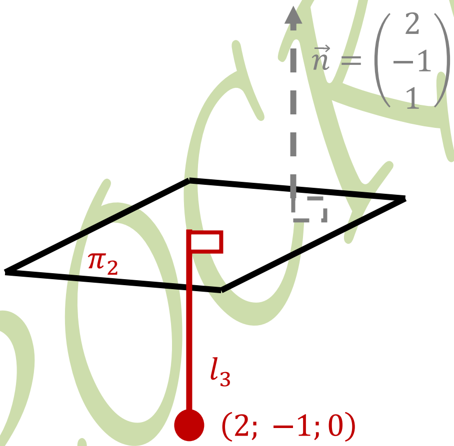
$$= -\frac{1}{5}.$$

$\therefore$The point is $A\left(-\frac{1}{5}; -\frac{17}{5}; 0\right)$

The equation is given by $\boldsymbol{r} = \overrightarrow{OA} + t\vec{d}$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{17}{5} \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix}$$

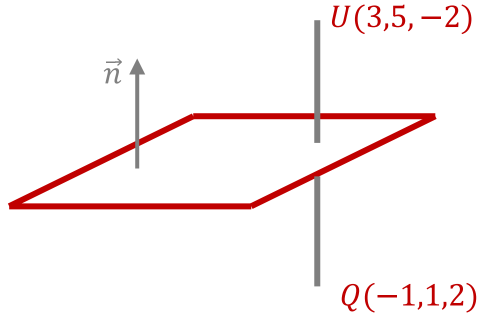
c)




The normal vector of the plane is the direction vector of this line. Thus:

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

d)




The normal vector is found by considering $\overrightarrow{UQ}$. Now:

$$\vec{n} = \overrightarrow{UQ} = \overrightarrow{OQ} - \overrightarrow{OU} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$$

Also the point in the plain is calculated by simply finding the midpoint of $U$ and $Q$.Now:

$$Point\ on\ plain = Midpoint\ (UQ) = \left(\frac{3 \pm 1}{2}; \frac{5+1}{2}; \frac{-2+2}{2}\right) = (1; 3; 0)$$

Now the equation is given by:

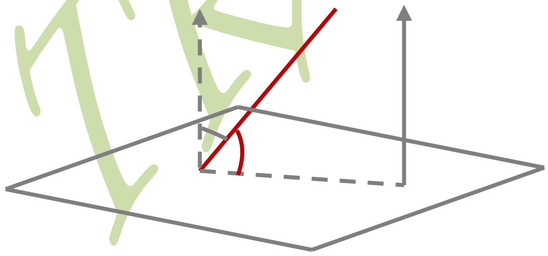
$$\begin{pmatrix} x-1 \\ y-3 \\ z \end{pmatrix} . \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix} = 0$$

$\Rightarrow -4x + 4 - 4y + 12 + 4z = 0$

$\Rightarrow -4x - 4y + 4z = -16$

$\Rightarrow x + y - z = 4$ (as required)

e) Angle between $l_2$ and $\pi_3$,



**Method 1**

$$Sin\ \theta = \frac{\vec{n}.\vec{d}}{|\vec{n}||\vec{d}|}$$

$$Sin\ \theta = \frac{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} . \begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix}}{\sqrt{(1)^2 + (1)^2 + (-1)^2}\sqrt{(-2)^2 + (-9)^2 + (-5)^2}}$$

$$Sin\ \theta = \frac{-6}{\sqrt{3}.\sqrt{110}}$$

$$\Rightarrow\ \theta = Sin^{-1}\left|\left(\frac{-6}{\sqrt{3}\sqrt{110}}\right)\right|$$

$$\Rightarrow\ \theta = 19.28632541°$$

$$\therefore\ \theta = 19°$$

## Method 2

$$Cos\ \alpha = \frac{\vec{n}.\vec{d}}{|\vec{n}||\vec{d}|}$$

$$Cos\ \alpha = \frac{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} . \begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix}}{\sqrt{(1)^2 + (1)^2 + (-1)^2}\sqrt{(-2)^2 + (-9)^2 + (-5)^2}}$$

$$Cos\ \alpha = \frac{-6}{\sqrt{3}\sqrt{110}}$$

$$\Rightarrow\ \alpha = Cos^{-1}\left|\left(\frac{-6}{\sqrt{3}\sqrt{110}}\right)\right|$$

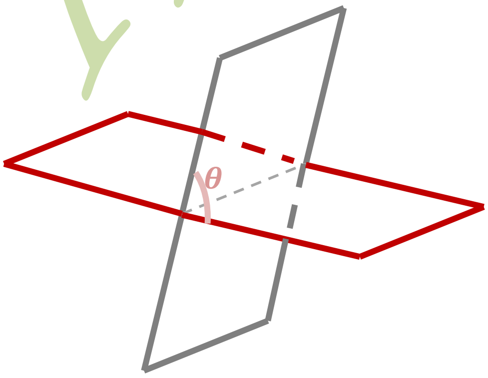
$$\Rightarrow\ \alpha = 70.71367459°$$

$$\Rightarrow\ \theta = 90° - \alpha$$

$$\Rightarrow 90° - 70.71367459° = 19.28632541°$$

$$\therefore\ \theta = 19°$$

(ii) $\pi_1$ and $\pi_2$.



$$Cos\ \theta = \frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$$

$$Cos\theta = \frac{\begin{pmatrix}1\\2\\-4\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}}{\sqrt{(1)^2+(2)^2+(-4)^2}\sqrt{(2)^2+(-1)^2+(1)^2}}$$

$$Cos\theta = \frac{-4}{\sqrt{21}\sqrt{6}}$$

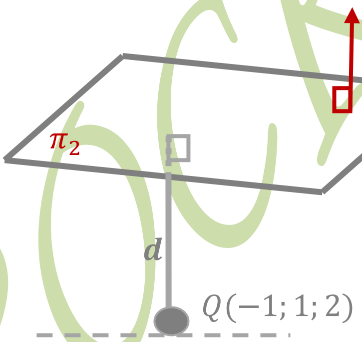
$$\Rightarrow\ \theta = Cos^{-1}\left(\frac{-4}{\sqrt{21}\sqrt{6}}\right)$$

$$\Rightarrow\ \theta = 110.8761026°$$

$$\therefore\ \theta = 111°$$

f) (i) perpendicular distance from $Q$ to $\pi_2$,

$$\pi_2: \boldsymbol{r}.\begin{pmatrix}2\\-1\\1\end{pmatrix} = 3$$




Let the distance be $d$.

$$d = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

$$d = \frac{|2(-1)+(-1)(1)+1(2)+(-3)|}{\sqrt{(2)^2+(-1)^2+(1)^2}}$$

$$d = \frac{|-2-1+2-3|}{\sqrt{6}}$$

$$d = \frac{|-4|}{\sqrt{6}}$$

$$d = \frac{4}{\sqrt{6}}$$

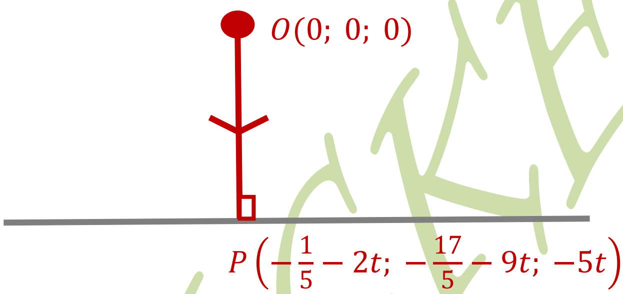
$d = \frac{4\sqrt{6}}{\sqrt{6}\sqrt{6}}$

$d = \frac{4\sqrt{6}}{6}$

$\therefore d = \frac{2\sqrt{6}}{3}$

(ii) shortest distance from the origin to $l_2$.

$l_2: \boldsymbol{r} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{17}{5} \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix}$




Now:

$\overrightarrow{OP} = \begin{pmatrix} -\frac{1}{5} - 2t \\ -\frac{17}{5} - 9t \\ -5t \end{pmatrix}$

Also $\overrightarrow{OP}.\begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix} = 0$ since they are perpendicular:

$\Rightarrow \begin{pmatrix} -\frac{1}{5} - 2t \\ -\frac{17}{5} - 9t \\ -5t \end{pmatrix}.\begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix} = 0$

$\Rightarrow \frac{2}{5} + 4t + \frac{153}{5} + 81t + 25t = 0$

$\Rightarrow 110t = -31$

$\therefore t = -\frac{31}{110}$

$$\Rightarrow \overrightarrow{OP} = \begin{pmatrix} -\frac{1}{5} - 2\left(-\frac{31}{110}\right) \\ -\frac{17}{5} - 9\left(-\frac{31}{110}\right) \\ -5\left(-\frac{31}{110}\right) \end{pmatrix} = \begin{pmatrix} \frac{4}{11} \\ -\frac{19}{22} \\ \frac{31}{22} \end{pmatrix}$$

Now the required shortest distance is $|\overrightarrow{OP}|$

$$\Rightarrow |\overrightarrow{OP}| = \sqrt{\left(\frac{4}{11}\right)^2 + \left(-\frac{19}{22}\right)^2 + \left(\frac{31}{22}\right)^2}$$

$$\Rightarrow |\overrightarrow{OP}| = \sqrt{\frac{63}{22}} = \frac{3\sqrt{7}}{\sqrt{22}} = \frac{3\sqrt{154}}{22}$$

$\therefore$The required distance is $\frac{3\sqrt{154}}{22}$.

g) Showing that $V(-3;\ 0;\ -7)$ lies on the plane $\pi_3$.

$\pi_3: x + y - z = 4$

$LHS: x + y - z$ (i)

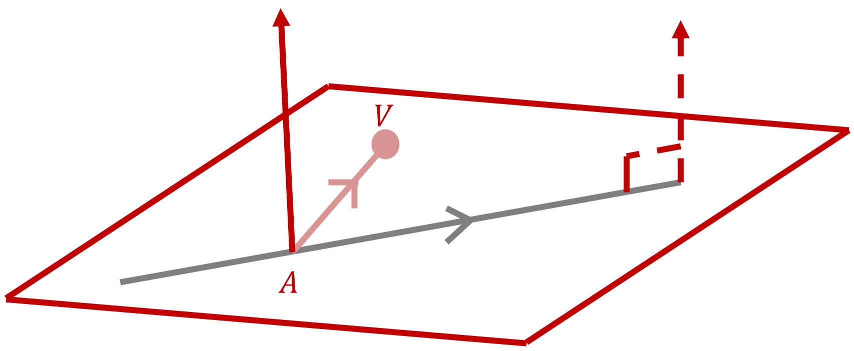
Substituting $V(-3;\ 0;\ -7)$ into (i)

$\Rightarrow x + y - z = -3 + 0 - (-7) = -3 + 7 = 4 \equiv RHS$

Since the equation of the plane is satisfied, therefore $V(-3;\ 0;\ -7)$ lies on the plane $\pi_3$.

h) The equation of the plane which contains $V$ and the line $l_2$.

$$l_2: \boldsymbol{r} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{17}{5} \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix}$$



Let $A$ be the point on the line which is also contained in the plain.

$A\left(-\frac{1}{5};\ -\frac{17}{5};\ 0\right)$

Now:

$$\overrightarrow{AV} = \overrightarrow{OV} - \overrightarrow{OA} = \begin{pmatrix} -3 \\ 0 \\ -7 \end{pmatrix} - \begin{pmatrix} -\frac{1}{5} \\ -\frac{17}{5} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{14}{5} \\ \frac{17}{5} \\ -7 \end{pmatrix}$$

NB: The cross of any two vectors in the plane gives the normal vector.

$$\vec{n} = \overrightarrow{AV} \times \begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix} = \begin{vmatrix} i & j & k \\ -\frac{14}{5} & \frac{17}{5} & -7 \\ -2 & -9 & -5 \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{17}{5} & -7 \\ -9 & -5 \end{vmatrix} - j\begin{vmatrix} -\frac{14}{5} & -7 \\ -2 & -5 \end{vmatrix} + k\begin{vmatrix} -\frac{14}{5} & \frac{17}{5} \\ -2 & -9 \end{vmatrix}$$

$$= -1071i + 32k$$

Now the equation is given by:

$$\begin{pmatrix} x+3 \\ y \\ z+7 \end{pmatrix} . \begin{pmatrix} -1071 \\ 0 \\ 32 \end{pmatrix} = 0$$

$\Rightarrow -1071x - 3213 + 32z + 224 = 0$

$\therefore$The equation of the plane which contains $V$ and the line $l_2$ is $-1071x + 32z = 2989$

# SOLVED PAST EXAMINATION QUESTIONS

## Question 1

**ZIMSEC NOVEMBER 2006 PAPER 2**

The plane $\pi$ and the line $l$ have equations

$\boldsymbol{r}.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$ and $\boldsymbol{r} = \begin{pmatrix}2\\0\\5\end{pmatrix} + t\begin{pmatrix}3\\12\\4\end{pmatrix}$ respectively, where $t$ is a parameter.

Find

(i) the position vector of the point of intersection of $\pi$ and $l$, **[4]**

(ii) the angle between $\pi$ and $l$, **[3]**

(iii) the perpendicular distance from the origin to $l$. **[5]**

## Suggested Solution

(i) $\boldsymbol{r}.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$ (i) and $\boldsymbol{r} = \begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix}$ (ii)

substituting (i) into (ii)

$$\begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$$

$$-2-3t+24t+10+8t = 66$$

$$29t = 58$$

$$\therefore t = 2$$

Now $\boldsymbol{r} = \begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix} = \begin{pmatrix}2+3(2)\\12(2)\\5+4(2)\end{pmatrix} = \begin{pmatrix}8\\24\\13\end{pmatrix}$

$\therefore$ The position vector is $= \begin{pmatrix}8\\24\\13\end{pmatrix}$

NB: The point of intersection is written in coordinate form as (8; 24; 13)

(ii) $sin\theta = \dfrac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}$

$$sin\theta = \frac{\begin{pmatrix}-1\\2\\2\end{pmatrix}.\begin{pmatrix}3\\12\\4\end{pmatrix}}{\sqrt{(-1)^2+2^2+2^2}\sqrt{(3)^2+12^2+4^2}}$$

$$sin\theta = \frac{-3+24+8}{\sqrt{9}\sqrt{169}}$$

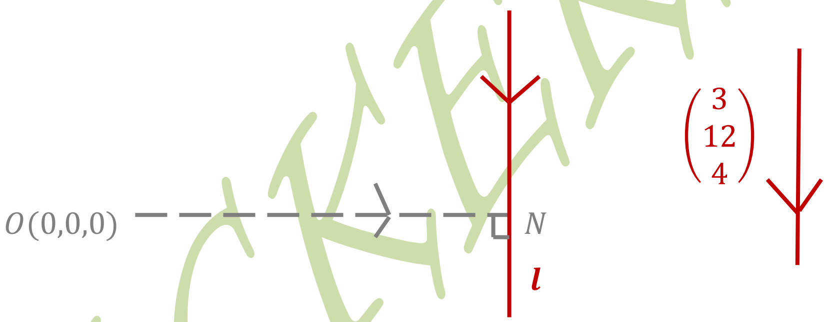
$$sin\theta = \frac{29}{3\times 13}$$

$$\theta = sin^{-1}\left(\frac{29}{39}\right)$$

$$\theta = 48.03811117°$$

$$\therefore \theta = 48°$$

(iii) Let point $N$ be the foot of the perpendicular from $O$.




The coordinates of $N$ are $(2+3t, 12t, 5+4t)$

Now

$$\boldsymbol{ON} = \begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix}$$

Taking the dot product of $\boldsymbol{ON}$ and the direction vector of the line

$$\boldsymbol{ON}.\begin{pmatrix}3\\12\\4\end{pmatrix} = 0$$

$$\begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix}.\begin{pmatrix}3\\12\\4\end{pmatrix} = 0$$

$$6+9t+144t+20+16t = 0$$

$$169t = -26$$

$$\therefore t = \frac{-26}{169} = \frac{-2}{13}$$

Now substituting $t = -\frac{2}{13}$ into $\boldsymbol{ON}$

$$\boldsymbol{ON} = \begin{pmatrix} 2+3\left(\frac{-2}{13}\right) \\ 12\left(\frac{-2}{13}\right) \\ 5+4\left(\frac{-2}{13}\right) \end{pmatrix} = \begin{pmatrix} \frac{20}{13} \\ \frac{-24}{13} \\ \frac{57}{13} \end{pmatrix}$$

Therefore the coordinates of the foot of the perpendicular are $\left(\frac{20}{13}; \frac{-24}{13}; \frac{57}{13}\right)$.

Hence the required distance is:

$$|\boldsymbol{ON}| = \sqrt{\left(\frac{20}{13}\right)^2 + \left(\frac{-24}{13}\right)^2 + \left(\frac{57}{13}\right)^2}$$

$$= \sqrt{25}$$

$$= 5.$$

## Question 2

**ZIMSEC JUNE 2017 PAPER 2**

The position vectors of points $A$ and B are $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $7\boldsymbol{i} - 9\boldsymbol{j}$, respectively.

Line $l$ has equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and the plane $P$ has equation $\boldsymbol{r}.\begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} = 8$

(a) Show that

(i) A lies in the plane $P$, **[2]**

(ii) $\overrightarrow{BA}$ is perpendicular to the plane $P$. **[3]**

(b) Calculate $O\hat{B}A$ where $O$ is the origin. **[3]**

(c) Find the acute angle between the line $l$ and the plane $P$. **[3]**

(d) Find the perpendicular distance from point $B$ to plane $P$. **[2]**

## Suggested Solution

a) (i) If $A$ lies on the plane the its coordinates bust satisfy the plane

Substitute the point $A(3, -1, 2)$ to the LHS of the plane $2x - 4y - z = 8$.

Now:

LHS: $2x - 4y - z = 2(3) - 4(-1) - 2$

$= 10 - 2$

$= 8 \equiv RHS$

Since $LHS \equiv RHS$ $\therefore$ the point $(3,-1,2)$ lies in the plane $2x - 4y - z = 8$.

Alternatively:

Substitute the point $A(3,-1,2)$ to the LHS of the plane $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix} = 8$.

Now:

$$\text{LHS: } \boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix} = \begin{pmatrix}3\\-1\\2\end{pmatrix}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}$$

$$= 6 + 4 - 2$$

$$= 8 \equiv RHS$$

Since $LHS \equiv RHS$ $\therefore$ the point $(3,-1,2$ lies in the plane $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix} = 8$

a) (ii) $\boldsymbol{BA} = \boldsymbol{OA} - \boldsymbol{OB} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} - (7\boldsymbol{i} - 9\boldsymbol{j})$

$$= -4\boldsymbol{i} + 8\boldsymbol{j} + 2\boldsymbol{k}$$

$$= \begin{pmatrix}-4\\8\\2\end{pmatrix}$$

$$= -2\begin{pmatrix}2\\-4\\-1\end{pmatrix}$$

Since $\boldsymbol{BA}$ is parallel to the direction vector of the plane $P$ $\therefore$ $\boldsymbol{BA}$ is perpendicular to the plane $P$.

(b) $cos(O\hat{B}A) = \dfrac{\overrightarrow{BO}.\overrightarrow{BA}}{|\overrightarrow{BO}||BA|}$

$$cos(O\hat{B}A) = \frac{\begin{pmatrix}-7\\9\\0\end{pmatrix}\begin{pmatrix}-4\\8\\2\end{pmatrix}}{\sqrt{(-7)^2 + (9)^2 + 0^2}\sqrt{(-4)^2 + 8^2 + 2^2}}$$

$$cos(O\hat{B}A) = \frac{28 + 72}{\sqrt{130}\sqrt{84}}$$

$$cos(O\hat{B}A) = \frac{100}{\sqrt{130}\sqrt{84}}$$

$$O\hat{B}A = cos^{-1}\left(\frac{100}{\sqrt{130}\sqrt{84}}\right)$$

$$= 16.87333855°$$

$$= 17°$$

(c) $sin\theta = \left|\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right|$

$$sin\theta = \left|\frac{\begin{pmatrix}2\\-1\\2\end{pmatrix}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}}{\sqrt{2^2+(-1)^2+2^2}\sqrt{2^2+(-4)^2+(-1)^2}}\right|$$

$$sin\theta = \frac{4+4-2}{\sqrt{9}\sqrt{21}}$$

$$sin\theta = \frac{6}{\sqrt{9}\sqrt{21}}$$

$$\theta = sin^{-1}\left(\frac{2}{\sqrt{21}}\right)$$

$$= 25.87669006°$$

$$= 26°$$

(d) Finding the distance between $B(7;-9;0)$ and plane $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix} = 8$ or $2x - 4y - z = 8$

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}$$

$$d = \frac{(2)(7)+(-9)(-4)+(-1)(0)-8}{\sqrt{(2)^2+(-4)^2+(-1)^2}}$$

$$d = \frac{42}{\sqrt{21}}$$

$$d = \frac{42\sqrt{21}}{21}$$

$$d = 2\sqrt{21}$$

## Question 3

**CEA Module FP3**

Tetra-Tents are designing a new model as shown in **Fig. 2** below.

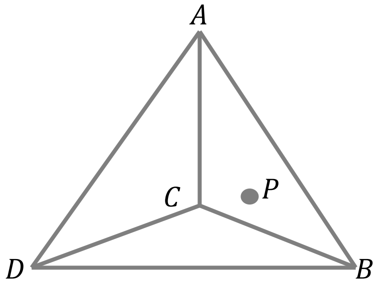



**Fig. 2**

They intend to attach a doorbell at position $P$.

The plane $ACD$ has equation $\boldsymbol{r}.\begin{pmatrix}2\\-3\\-1\end{pmatrix}=3$

The line $AB$ has equation has equation $\frac{x-1}{-1}=\frac{y+1}{1}=\frac{z-3}{-4}$

The point $P\left(2\frac{1}{3},-\frac{2}{3}\ 4\right)$ lies in the plane $ABC$

(i) Find the coordinates of the apex, $A$. **[4]**

(ii) Find an equation of the line $AC$. **[10]**

Suggested Solution

(i) The coordinates of $A$ are at the point of intersection of the plane $ACD$ and the line $AB$.

Express the Cartesian equation in vector form

$\boldsymbol{r}.\begin{pmatrix}2\\-3\\-1\end{pmatrix}=3$ (i)

$\boldsymbol{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+t\begin{pmatrix}-1\\1\\-4\end{pmatrix}$ (ii)

Substituting equation (i) into equation (ii):

$$\begin{pmatrix}1-t\\-1+t\\3-4t\end{pmatrix}.\begin{pmatrix}2\\-3\\-1\end{pmatrix}=3$$

$2 - 2t + 3 - 3t - 3 + 4t = 3$

$-t = 1$

$\therefore t = -1$

Substituting $t = -1$ into equation (ii):

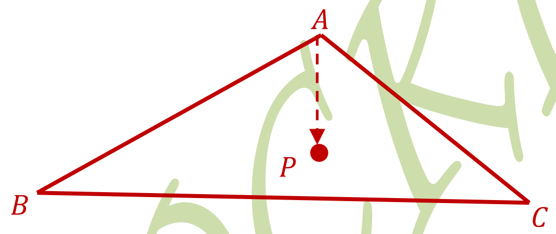
$$\boldsymbol{r} = \begin{pmatrix} 1 - t \\ -1 + t \\ 3 - 4t \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} 1 - (-1) \\ -1 + (-1) \\ 3 - 4(-1) \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ -2 \\ 7 \end{pmatrix}$$

$\therefore$ The coordinates of $A$ are $(2, -2, 7)$

(ii) We use the point $P$ and the line $AB$ to find the equation of the plane $ABC$.




$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$$

$$= \overrightarrow{OP} - \overrightarrow{OA}$$

$$= \begin{pmatrix} \frac{7}{3} \\ -\frac{2}{3} \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{4}{3} \\ -3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$$

To find the normal we take the cross $\overrightarrow{AB} \times \overrightarrow{AP}$

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AP} = \begin{vmatrix} i & j & k \\ 1 & 4 & -9 \\ -1 & 1 & -4 \end{vmatrix}$$

$$= i\begin{vmatrix} 4 & -9 \\ 1 & -4 \end{vmatrix} - j\begin{vmatrix} 1 & -9 \\ -1 & -4 \end{vmatrix} + k\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix}$$

$$= -7i + 13j + 5k$$

Now to find the equation of the plane we use $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

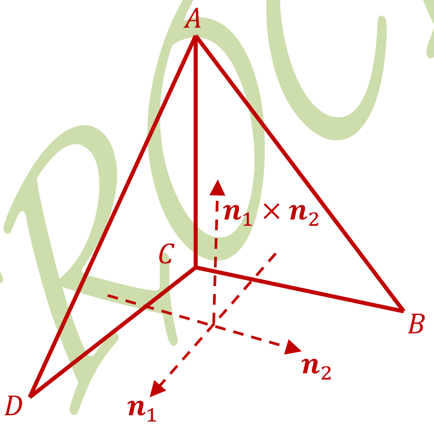
$$\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$$

$$\boldsymbol{r}.\begin{pmatrix} -7 \\ 13 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 7 \end{pmatrix}.\begin{pmatrix} -7 \\ 13 \\ 5 \end{pmatrix}$$

$$\boldsymbol{r}.\begin{pmatrix} -7 \\ 13 \\ 5 \end{pmatrix} = -14 - 26 + 35$$

$$\boldsymbol{r}.\begin{pmatrix} -7 \\ 13 \\ 5 \end{pmatrix} = -5$$

To find the equation of $AC$ we find the vector equation of the line of intersection of the plane $ABC$ and the plane $ACD$






The cross $n_1 \times n_2$ is parallel to $\overrightarrow{AC}$ thus it is the direction vector of the line

$$\boldsymbol{d} = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ -7 & 13 & 5 \\ 2 & -3 & -1 \end{vmatrix}$$

$$= i\begin{vmatrix} 13 & 5 \\ -3 & -1 \end{vmatrix} - j\begin{vmatrix} -7 & 5 \\ 2 & -1 \end{vmatrix} + k\begin{vmatrix} -7 & 13 \\ 2 & -3 \end{vmatrix}$$

$= 2i + 3j - 5k$

To find the point on the plane we let $z = 0$ and solve the two equations simultaneously

$$\boldsymbol{r}.\begin{pmatrix}-7\\13\\5\end{pmatrix} = -5 \Rightarrow -7x + 13y + 5z = -5$$

$$\boldsymbol{r}.\begin{pmatrix}2\\-3\\-1\end{pmatrix} = 3 \Rightarrow 2x - 3y - z = 3$$

Now:

$-7x + 13y = -5$ (i)

$2x - 3y = 3$ (ii)

Multiplying equation (i) by 2 and equation (ii) by 7

$-14x + 26y = -10$ (iii)

$14x - 21y = 21$ (iv)

Adding equation (iii) and equation (iv)

$5y = 11$

$$\Rightarrow y = \frac{11}{5}$$

$14x - 21y = 21$ (iv)

$$14x - 21\left(\frac{11}{5}\right) = 21$$

$$14x - \frac{231}{5} = 21$$

$$14x = 21 + \frac{231}{5}$$

$$14x = \frac{336}{5}$$

$$x = \frac{24}{5}$$

$\therefore$ The point on the line is $\left(\frac{24}{5}, \frac{11}{5}, 0\right)$

Now the equation of the line of $AC$ is given by:

$$\boldsymbol{r} = \begin{pmatrix}\frac{24}{5}\\\frac{11}{5}\\0\end{pmatrix} + t\begin{pmatrix}2\\3\\-5\end{pmatrix}$$

# PAST EXAMINATION QUESTIONS

**ZIMSEC JUNE 2007 PAPER 2**

The line $l$ has Cartesian equations

$$\frac{x-5}{2}=\frac{y-2}{-1}=\frac{z-2}{3}$$

Find

(i) the length of the perpendicular from the origin to $l$. **[5]**

(ii) an equation for the plane $\pi$ which contains $l$ and the origin, giving your answer in the form $\boldsymbol{r}.\boldsymbol{n}=D$. **[4]**

**ZIMSEC NOVEMBER 2008 PAPER 2**

A plane contains the points $A(1;2;3)$ and $B(4;5;6;)$ and passes through the origin.

(a) Find

(i) the equation of the plane OAB in the form $\boldsymbol{r}.\boldsymbol{n}=d$. **[5]**

(ii) the coordinates of the foot of the perpendicular from point $C(5;1;-4)$ to the line $AB$. **[4]**

(b) Calculate the angle between the lines $AB$ and OC. **[3]**

(c) Show that the line $BC$ is not contained in the plane $OAB$. **[3]**

**ZIMSEC JUNE 2016 PAPER 2**

Given the points $A(0,2,2), B(4,1,0)$ and $C(-2,0,3)$ find the

(i) value of $q$, given that the Cartesian equation of a perpendicular bisector of a line joining the points $A$ and $B$, passing through the point $D(1,q,0)$ is

$x-2=\frac{2y-3}{4}=z-1$ **[4]**

(ii) exact shortest distance of the line in part **(i)** to the origin, **[4]**

(iii) Cartesian equation of the plane containing points $A, B$ and $C$. **[4]**

**ZIMSEC NOVEMBER 2017 PAPER 2**

Relative to the origin $\boldsymbol{O}$, the position vectors of points $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $\begin{pmatrix}5\\3\\1\end{pmatrix}$ and $\begin{pmatrix}4\\c\\2\end{pmatrix}$ respectively

(a) Determine whether the point $\boldsymbol{P}$ lies on $l$ whose equation is $\boldsymbol{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\5\end{pmatrix}$, where $\lambda$ is the parameter. **[2]**

(b) (i) Given that line $\boldsymbol{PQ}$ intersects the line $l$, find the value of $c$. **[3]**

(ii) Hence, calculate the angle between the line $l$ and the $\boldsymbol{PQ}$ . **[3]**

(c) Find the position vector of the point $\boldsymbol{R}$ on $l$ such that line $\boldsymbol{PR}$ is perpendicular to line $l$. **[4]**

**ZIMSEC JUNE 2018 PAPER 2**

a) A plane has vector equation $\boldsymbol{r}.\begin{pmatrix}2\\-3\\1\end{pmatrix} = 1$ and the point $P$ has coordinates $(-9; 17; -2)$.

(i) Find the coordinates of the foot of the perpendicular from $P$ to the plane. **[3]**

(ii) Hence, or otherwise, find the coordinates of the image of $P$ when reflected in the plane. **[3]**

(b) The plane $\pi$ and the line $l$ have equations

$$\boldsymbol{r}.\begin{pmatrix}2\\-1\\-1\end{pmatrix} = 6 \text{ and } \frac{x-2}{1} = \frac{y-3}{2} = \frac{z+1}{-2} \text{ respectively.}$$

Find the

a) coordinates of the point of intersection between $\pi$ and $l$, **[4]**

b) angle between $\pi$ and $l$, **[3]**

c) shortest distance from the origin to $l$. **[4]**

## ZIMSEC 2018 NOVEMBER PAPER 2

(a) The line $L_1$ has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

and plane $\pi_1$ passes through the points $A, B$ and $C$ with coordinates $(2; -1; 3)$, $(4; 2; -5)$ and $(-1; 3; -2)$ respectively.

Find the

(i) Cartesian equation of $\pi_1$ , **[5]**

(ii) acute angle between the plane $\pi_1$ and the line $L_1$. **[3]**

(b) The plane $\pi_2$ has equation

$$\mathbf{r}.\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 6.$$

The line $L_2$ lies in the plane $\pi_2$ and is perpendicular to $L_1$. The line $L_2$ passes through the point $(4; 2; 1)$.

Find the

(i) vector equation of $L_2$, **[4]**

(ii) vector equation of the line of intersection of the planes $\pi_1$ and $\pi_2$. **[4]**

## ZIMSEC JUNE 2019 PAPER 2

The plane $\pi$ has vector equation $\mathbf{r}.\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = 1$. The points $P$ and $Q$ have coordinates $(-5; 1; 3)$ and $(2; 4; -1)$ respectively.

(i) Show that $Q$ lies on plane $\pi$. **[2]**

(ii) Find the

a) equation of the line perpendicular to $\pi$ through $P$, **[2]**

b) Cartesian equation of the line $PQ$. **[3]**

(iii) Calculate the

a) acute angle between $\pi$ and $l$. **[4]**

b) shortest distance from the $P$ to $\pi$. **[5]**

**ZIMSEC NOVEMBER 2019 PAPER 2**

Two planes $\pi_1$ and $\pi_2$ have equations $x + 2y - 2z = 2$ and $2x - 3y + 6z = 3$.

a) Show that the line $l$, with equation $\frac{x-2}{2} = y - 1 = \frac{z-1}{2}$ lies on $\pi_1$. **[3]**

b) Find the

(i) equation of the line of intersection between the two planes, **[6]**

(ii) point of intersection of line $l$ and plane $\pi_2$ **[3]**

(iii) the angle between the line $l$ and plane $\pi_2$ **[4]**

# ASANTE SANA

********THERE IS A LIGHT AT THE END OF EVERY TUNNEL ********

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

***ENJOY***

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