Vector 2

Compiled by: Nyasha P. Tarakino (Trockers)

+263772978155/+263717267175

ntarakino@gmail.com

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SYLLABUS (6042) REQUIREMENTS

- find vector and Cartesian equations of a straight line in 3D
- find cross product of given vectors
- use cross product to find area of plane shapes
- find vector, parametric and Cartesian equations of a plane
- find the angle between a line and a plane, two lines and two planes
- determine whether two lines are parallel, intersecting or skewed
- determine whether two lines intersect
- find the perpendicular distance from a point to a straight line and the coordinates of the foot of the perpendicular
- determine whether a line lies in, is parallel to or intersects a plane
- find the point of intersection of a line and a plane
- find the line of intersection of two planes
- find the perpendicular distance from a point to a plane
- solve problems involving lines and planes

VECTORS IN $\mathbb{R}^3$

Definition

- A vector quantity is defined by magnitude and direction, e.g. force, velocity, acceleration.
- A vector in $3D$-space is given by a three tuple (a, b, c), indicating the change in the A, B and C directions from the origin.
- A vector v in $\mathbb{R}^3$ is an ordered triple of real numbers.
- The standard basis vectors for $\mathbb{R}^3$ is given by:

$$i := < 1;\ 0;\ 0 >$$

$$j := < 0;\ 1;\ 0 >$$

$$k := < 0;\ 0;\ 1 >$$

Vector equation of a line

- It is an equation that represents all possible points on the line.
- We describe a line in $\mathbb{R}^3$ with parametric equations of the form:

$$r_1 = a_1 + tb_1; \quad r_2 = a_2 + tb_2; \quad r_3 = a_3 + tb_3:\ t\boldsymbol{\epsilon}\mathbb{R}$$

- The values a_1, a_2, and a_3 correspond to a point on the line (which we can see by plugging in $t = 0$ to the equations), while the values b_1, b_2;, and b_3 correspond to the direction the line travels in the r_1, r_2, and r_3 directions (respectively).
- This means that the $\boldsymbol{b}$ values combine to form our direction vector! So, instead of using three parametric equations, we combine them to form the single vector equation of a line in $\mathbb{R}^3$: $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$: $t\boldsymbol{\epsilon}\mathbb{R}$ where $\boldsymbol{a}$ is a point on the line, and $\boldsymbol{b}$ is a direction vector for the line.

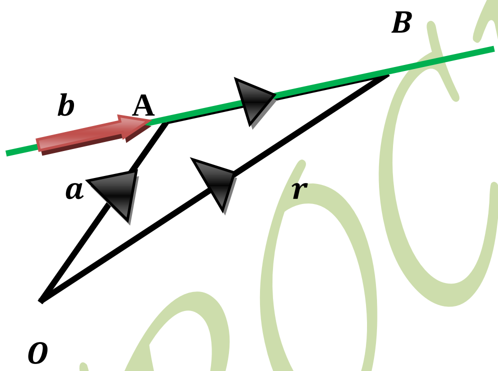
- Therefore, the equation is given by:

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}: t\epsilon\mathbb{R}$$

or

$$\boldsymbol{r} = \overrightarrow{\boldsymbol{OA}} + t\overrightarrow{\boldsymbol{AB}}: t\epsilon\mathbb{R}$$

- The point P can be put anywhere along the line by varying the value of t.
- t is called the parameter.
- The vector $\boldsymbol{a}$ or $\overrightarrow{\boldsymbol{OA}}$ is the position vector of a given point on the line.
- The vector $\boldsymbol{b}$ or $\overrightarrow{\boldsymbol{AB}}$ is called the direction vector; it specifies the direction of the line.
- NB: To write the vector equation of a line, we need a point $P(x_0, y_0, z_0)$ on the line and a vector $\vec{v} = (a, b, c)$ that is parallel to the line.



- The arrow on $\overrightarrow{\boldsymbol{AB}}$, $\overrightarrow{\boldsymbol{OA}}$ and $\overrightarrow{\boldsymbol{OB}}$ is called the **position vector** of the point, **b, a** and **r** respectively..

Example

Write the vector equation of a lone which passes through $P(3;\ 5;\ 1)$ with direction vector $\vec{\boldsymbol{d}} = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$.

Solution

The equation is given by:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$$

Example

Write the vector equation of a lone which passes through $(2,-1,3)$ and parallel to the vector $3\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$.

Solution

The equation is given by:

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \quad \text{or} \quad \overrightarrow{\boldsymbol{OP}} = (2,-1,3) + t(3\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})$$

Example

Find the vector equation for the line through the points $A(3,6,2)$ and $B(6,2,-5)$.

Solution

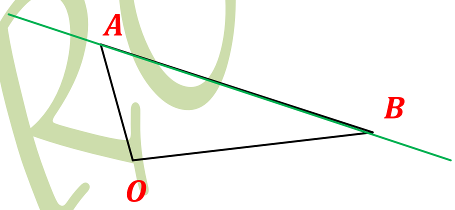
Steps

(i) Draw a picture if necessary.

(ii) Accuracy doesn't matter!

(iii)To find the vector equation we need:

- The equation of the vector $\overrightarrow{AB}$.
- One point on the line AB: We can choose either A or B for vector AB.



NB: Given $\overrightarrow{\boldsymbol{AB}}$: $\overrightarrow{\boldsymbol{OB}}$ is the head and $\overrightarrow{\boldsymbol{OA}}$ thus $\overrightarrow{\boldsymbol{OB}} = \boldsymbol{Head} - \boldsymbol{Tail} = \overrightarrow{\boldsymbol{OB}} - \overrightarrow{\boldsymbol{OA}}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$$

Now choosing $A(3,6,2)$ to be the point on $\overrightarrow{AB}$.

$$\boldsymbol{r} = \overrightarrow{\boldsymbol{OA}} + \boldsymbol{t}\overrightarrow{\boldsymbol{AB}}$$

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} + \boldsymbol{t}\begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$$

The equation $\boldsymbol{r}$ can also be written as:

$$\boldsymbol{r} = (3\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}) + t(3\boldsymbol{i} - 4\boldsymbol{j} - 7\boldsymbol{k}).$$

Parametric equation

- The parametric equation of a line is derived from the vector equation of a line.

Definition

- The parametric equation of a line that contains the point $P(x_0, y_0, z_0)$ and is parallel to the vector $\vec{r} = (a, b, c)$ is:

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

point $P(x_0, y_0, z_0)$ on the line and a vector $\vec{r} = (a, b, c)$ that is parallel to the line.

Example

Find a parametric equation for the line which contains the point $(1, 2, 0)$ and has direction vector, $(1, 2, 1)$.

Solution

The equation is given by:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 + \mathrm{t} \\ 2 + 2\mathrm{t} \\ \mathrm{t} \end{pmatrix}$$

Thus the parametric equation is given by:

$$x = 1 + t$$
$$y = 2 + 2t$$
$$z = t$$

Example

Let L which passes through the points $P(1, 1, 1)$ and $Q(3, 2, 1)$.

Find a vector which is parallel to the line. Find the vector-equation and parametric equation of the line.

Solution

The vector parallel to the line is given by:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Now choosing $P(1,1,1)$ to be the point on $\overrightarrow{PQ}$.

$$\boldsymbol{r} = \overrightarrow{\boldsymbol{OP}} + t\overrightarrow{\boldsymbol{AP}}$$

The vector-equation of the line is given by:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

The parametric equation of the line is given by:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 + 2t \\ 1 + t \\ 1 \end{pmatrix}$$

Thus the parametric equation is given by:

$$x = 1 + 2t$$
$$y = 1 + t$$
$$z = 1$$

Cartesian equation

- The Cartesian form of the vector equation is obtained by eliminating the parameter from the parametric equations.

- The vector equation is given by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- Making t the subject we obtain the Cartesian equation of a curve.
- The Cartesian equation is given by:

$$t = \frac{y - y_0}{b} = \frac{x - x_0}{a} = \frac{z - z_0}{c}$$

Examples

Determine the vector equation of the straight line passing through the point with position vector $i - 3j + k$ and parallel to the vector, $2i + 3j - 4k$. Express the vector equation of the straight line in standard Cartesian form.

Solution

The vector parallel to the line is given by:

$$\boldsymbol{r} = \overrightarrow{\boldsymbol{OP}} + t\overrightarrow{\boldsymbol{AP}}$$

The vector-equation of the line is given by:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

The parametric equation of the line is given by:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 + 2t \\ -3 + 3t \\ 1 - 4t \end{pmatrix}$$

Thus the parametric equation is given by:

$$x = 1 + 2t$$
$$y = -3 + 3t$$
$$z = 1 - 4t$$

Now making t the subject of the formula in all equations:

$$x = 1 + 2t$$

$$x - 1 = 2t$$

$$\frac{x-1}{2} = t \qquad \textbf{(i)}$$

$$y = -3 + 3t$$

$$y + 3 = 2t$$

$$\frac{y+3}{2} = t \qquad \textbf{(ii)}$$

$$z = 1 - 4t$$

$$z - 1 = -4t$$

$$\frac{1-z}{4} = t \qquad \textbf{(iii)}$$

All three equations are equal to t and so equal to one another, therefore the Cartesian equation is given by:

$$\frac{x-1}{2} = \frac{y+3}{2} = \frac{1-z}{4}.$$

Follow up questions

Question 1

Determine the vector equation of the straight line passing through the two points $P_1(3, -1, 5)$ and $P_2(-1, -4, 2)$.

Answer: $r = 3i - j + 5k - t(4i + 3j + 3k)$.

<u>Question 2</u>

Determine the vector equation of the straight line passing through the two points $A(3,4, -7)$ and $B(1, -1, 6)$.

Answer: $r = 3i - 4j - 7k - t(-2i - 5j + 13k)$.

<u>Question 3</u>

.Find the parametric and Cartesian equations for the line $\boldsymbol{r} = 5\boldsymbol{i} + 2\boldsymbol{k} + \lambda(4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$

Answer: The parametric equations are: $x = 5 + 4\lambda, y = \lambda, z = 2 + 4\lambda$

The Cartesian equation is given by:

$$\frac{x-5}{4} = -y = \frac{z-2}{4}.$$

<u>Question 4</u>

A line passes through points $A(2, -1, 5)$ and $B(3, 6, -4)$.

a) Write a vector equation of the line.

b) Write parametric equations for the line.

c) Determine if the point $C(0, -15, 9)$ lies on the line.

<u>Answers</u>

a) A vector equation is $[x, y, z] = [2, -1, 5] + t[1, 7, -9]$

b) The corresponding parametric equations are

$x = 2 + t$

$y = -1 + 7t$

$z = 5 - 9t$

b) Substitute the coordinates of $C(0, -15, 9)$ into the parametric equations and solve for t.

$0 = 2 + t \quad \Rightarrow t = -2$

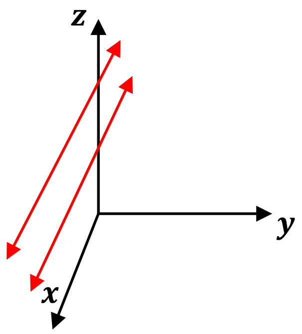
$-15 = -1 + 7t \quad \Rightarrow t = -2$

$9 = 5 - 9t \quad \Rightarrow t = -4/9$

The t −values are not equal, so the point does not lie on the line.

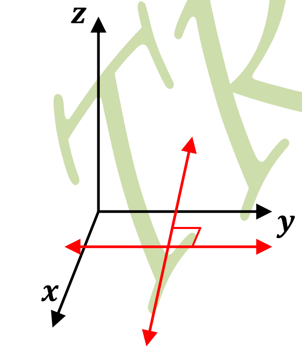
Perpendicular, Parallel and Skew Lines

Parallel Lines



Let l_1 and l_2 be two lines in $\mathbb{R}^3$, with direction vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, and let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The lines l_1 and l_2 are **parallel** whenever $\boldsymbol{a}$ and $\boldsymbol{b}$, are parallel.

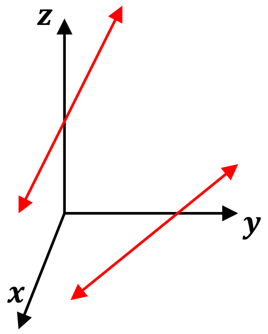
Perpendicular/Intersecting Lines



Let l_1 and l_2 be two lines in $\mathbb{R}^3$, with direction vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, and let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The lines l_1 and l_2 are **perpendicular/orthogonal** whenever $\boldsymbol{a}$ and $\boldsymbol{b}$, are perpendicular/orthogonal. When finding the point of intersection of two lines expressed in vector form, we equate both vectors, l_1 and l_2, and then find the values of the parameters and substitute in either equation.

Skew Lines

Two lines in $\mathbb{R}^3$ are skew if they are not parallel and do not intersect.



Note: Skew lines lie in parallel planes.

WORKED EXAMPLES

Example 1

Show that the lines $l_1 = (\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}) + t(3\boldsymbol{i}-\boldsymbol{j})$ and $l_2 = (4\boldsymbol{i}-\boldsymbol{k}) + u\,(2\boldsymbol{i} + 3\boldsymbol{k})$ intersect. Also, find their point of intersection.

Solution

Two lines intersect if $l_1 = l_2$.

$(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}) + t(3\boldsymbol{i}-\boldsymbol{j}) = (4\boldsymbol{i}-\boldsymbol{k}) + u\,(2\boldsymbol{i} + 3\boldsymbol{k})$

$$\begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}4\\0\\-1\end{pmatrix} + u\begin{pmatrix}2\\0\\3\end{pmatrix}$$

$$\begin{pmatrix}1+3t\\1-t\\-1\end{pmatrix} = \begin{pmatrix}4+2u\\0\\-1+3u\end{pmatrix}$$

$1+3t = 4+2u \quad (1)$

$1-t = 0 \quad (2) \qquad \Rightarrow t = 1$

$-1 = -1+3u \quad (3) \qquad \Rightarrow u = 0$

To check if the values of s and t are true, we substitute them in (1)

$1 + 3t = 4 + 2u \quad (1)$

LHS: $1 + 3t = 1 + 3(1) = 4$

LHS: $4 + 2u \quad = 4 + 0 \quad = 4$

Since $LHS = RHS = 4 \therefore t = 1$ and $u = 0$.

To find the point of intersection, we substitute t in l_1 or u in l_2.

$$l_2 = \begin{pmatrix} 4+2u \\ 0 \\ -1+3u \end{pmatrix} = \begin{pmatrix} 4+0 \\ 0 \\ -1+0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$$

$\therefore$ The point of intersection is $(4, 0, -1)$.

Example 2

Show that the lines $l_1 : \frac{x-3}{2} = \frac{y-5}{6} = \frac{z+1}{4}$ and $l_2 : \frac{x+1}{1} = \frac{y}{3} = \frac{z-8}{2}$ are parallel to each other.

Solution

We write l_1 and l_2 in vector form.

$$l_1 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

$$l_2 = \begin{pmatrix} -1 \\ 0 \\ 8 \end{pmatrix} + u\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Now the direction vector of l_1 is $\boldsymbol{a}$ and of l_2 is $\boldsymbol{b}$.

$\boldsymbol{a} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, then l_1 and l_2 are parallel since $\boldsymbol{a} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 2\boldsymbol{b}$.

Example 3

Show that lines $L_1 = \frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3}$ and $L_2 = \frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{2}$ are skew.

Solution

Write the equation in parametric form.

$L_1 : x = 2t + 4;\ y = 4t - 5;\ z = -3t + 1$

$L_2 : x = s + 2;\ y = 3s - 1;\ z = 2s$

The lines are not parallel since the vectors $\boldsymbol{v_1} = < 2;\ 4; -3i >$ and $\boldsymbol{v_2} = < 1;\ 3;\ 2i >$ are not parallel. Next we try to find intersection point by equating $x, y,$ and z.

$2t + 4 = s + 2$ (1)

$4t - 5 = 3s - 1$ (2)

$-3t + 1 = 2s$ (3)

(1) gives $s = 2t + 2$. Substituting into (2) gives $4t - 5 = 3(2t + 2) - 1$) $t = -5$. Then $s = -8$. However, this contradicts with (3). So there is no solution for s and t. Since the two lines are neither parallel nor intersecting, they are skew lines.

Follow up questions

Question 1

Show that the lines $r = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k} + (\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$;
$r = 5\boldsymbol{i} - 2\boldsymbol{j} + \mu(3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$ are intersecting. Hence, find their point of intersection.

Answer: $(\mathbf{-1, -6, -12})$

Question 2

Determine whether the lines $\frac{x+5}{2} = \frac{2-y}{5} = \frac{1-z}{-1}$ and $\frac{x}{1} = \frac{2y+1}{4} = \frac{1-z}{-3}$

are parallel, perpendicular or skewed to each other.

Question 3

Show that the lines $r = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + t(3\boldsymbol{i} - \boldsymbol{j})$ and $r = 4\boldsymbol{i} - \boldsymbol{k} + u(2\boldsymbol{i} + \boldsymbol{k})$ intersect. Also find their point of intersection.

Answer: $(\mathbf{4, 0, -1})$

Question 4

Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{z-4}{3} = \frac{z-6}{5}$ intersect. Also, find their point of intersection.

Answer: $(\mathbf{1/2, -1/2, -3/2})$

Dot Product

Algebraic Definition

The dot product 2 vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ $\vec{a}.\vec{b}$ is defined to be the scalar

$$\vec{a}\,.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Geometric Definition

The dot product 2 vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ $\vec{a}.\vec{b}$ is defined to be the scalar

$$\vec{a}\,.\vec{b} = |\vec{a}\,|.\left|\vec{b}\right|cos\theta,$$

where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Proposition

Let $\vec{a}$ and $\vec{b}$ be non-zero vectors. The vectors, $\vec{a}$ and $\vec{b}$, are perpendicular to each other if and only if $\vec{a}.\vec{b} = 0$.

WORKED EXAMPLES

Example 1

Given that the vectors $\vec{a} = (-1,2,3)$ and $\vec{b} = (3,1,4)$, find the dot product of $\vec{a}$ and $\vec{b}$. to each other.

Solution

$\vec{a}.\vec{b} = (-1,2,3).(3,1,4) = -3 + 2 + 12 = 11.$

Example 2

Determine if the vectors $\vec{a} \ = \ (5, -5, 0)$ and $\vec{b} \ = \ (2, 2, 3)$ are perpendicular to each other.

Solution

$\vec{a}.\vec{b} = (5, -5, 3).(2, 2, 3) = 10 - 10 + 0 = 0$

Since $\vec{a}.\vec{b} \ = \ 0$ therefore $\vec{a}$ and $\vec{b}$, are perpendicular to each.

Determinants of Matrices

DETERMINANTS – 2×2 MATRICES

Suppose A is any (2×2) Matrix such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of A can be written as:

$|A|$ or $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

WORKED EXAMPLES

Example

Given that $A = \begin{pmatrix} 6 & -1 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$;find the determinant of A and the determinant of B.

Solution

(i) $\begin{vmatrix} 6 & -1 \\ 4 & 1 \end{vmatrix} = ad - bc = 6(1) - (-1 \times 4) = 6 + 4 = 10.$

(ii) $\begin{vmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{vmatrix} = ad - bc = cos\theta(cos\theta) - (sin\theta \times -sin\theta) = cos^2\theta + sin^2\theta \equiv 1$

DETERMINANTS – 3 × 3 MATRICES

Suppose A is any (3×3) Matrix such that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then the determinant of A can be written as $|A|$ or $\det(A)$ or $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\therefore \det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$

WORKED EXAMPLE

Example

Given that $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}$; find the determinant of A .

Solution

$\det(A) = 3\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

$= 3(2 - 2) - 1(-4 - 2) + 1(2 - -1)$

$= 3(0) - 1(-6) + 1(3)$

$= 0 + 6 + 3$

$= 9$

Cross Product

Geometric definition

Given 2 vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ the cross product $\vec{a} \times \vec{b}$ can also be defined as follows

$$\vec{a} \times \vec{b} = (|\vec{a}|.|\vec{b}| \sin\theta)\,\vec{n}$$

where:

$\vec{n}$ is a unit vector which is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$, determined by the right-hand-rule. i.e. place your right hand on $\vec{a}$ and curl it towards $\vec{b}$, $\vec{n}$ now points in the direction of your thumb.

Algebraic definition

Given 2 vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ the cross product $\vec{a} \times \vec{b}$ can also be defined as follows:

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$ are unit vectors along the X, Y, Z axes respectively. However while calculating the determinant they only have symbolic value.

ADDITIONAL NOTES

(a) The cross product operates on 2 vectors and produces a new vector.

(b) The two important properties of cross products are:

- The direction of the cross product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. In particular it points in the direction of your thumb if the fingers of your right hand curl from $\vec{a}$ to $\vec{b}$.
- The magnitude of $\vec{a} \times \vec{b}$ is given by $= (|\vec{a}|.|\vec{b}|\sin\theta)$ if θ is the angle between $\vec{a}$ and $\vec{b}$. It is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

WORKED EXAMPLE

Example

Consider the two vectors $\vec{a} = (2, 1, 0)$ and $\vec{b} = (-1, 1, 0)$ in $\mathbb{R}^3$.

Find

(a) the cross-product of the two vectors,

(b) verify that it is indeed perpendicular to both $\vec{a}$ and $\vec{b}$,

(c) the magnitude of $\vec{a} \times \vec{b}$.'

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} - j\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k[2-(-1)]$$

$$= 3k$$

(b) $\vec{n} = \vec{a} \times \vec{b} = (0, 0, 3)$

$\vec{n}.\vec{a} = (2, 1, 0).(0, 0, 3) = 0 + 0 + 0 = 0.$

$\vec{n}.\vec{b} = (-1, 1, 0).(0, 0, 3) = 0 + 0 + 0 = 0.$

Since both the dot products are zero, we can conclude that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

(c) The magnitude of $\vec{a} \times \vec{b}$ is given b:

$$\vec{a} \times \vec{b} = |\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + 3^2} = \sqrt{9} = 3$$

Follow up questions

Question 1

Consider the two vectors $\vec{a} = (2, 1, -4)$ and $\vec{b} = (3, -2, 5)$ in $\mathbb{R}^3$. Find the cross-product of the two vectors.

Answer: $(-3, -22, -7)$

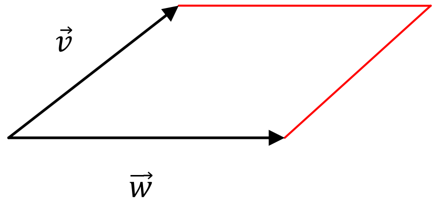
Question 2

Find the cross-product of the two vectors $\vec{v} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\vec{w} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

Answer: $11\boldsymbol{i} - 7\boldsymbol{j} + 5\boldsymbol{k}$

Using cross product to find area of plane shapes

Area of parallelogram

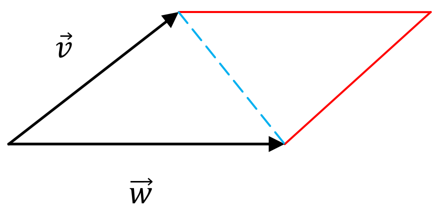


The area of a parallelogram with edges $\vec{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ and $\vec{w} = w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k}$ is:

$Area = |\vec{v} \times \vec{w}|$ where $\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

NB: $\vec{v}$ and $\vec{w}$ are non-zero vectors and they are parallel.

Area of triangle



The area of triangle is twice the area of parallelogram.

$$Area = \frac{1}{2}|\vec{v} \times \vec{w}|$$

WORKED EXAMPLE

Example

Find the area of the parallelogram with edges $\vec{v} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\vec{w} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$.

Suggestion

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix} = (2+9)\boldsymbol{i} - (4+3)\boldsymbol{j} + (6-1)\boldsymbol{k}$$

$$= 11\boldsymbol{i} - 7\boldsymbol{j} + 5\boldsymbol{k}$$

The area of the parallelogram with edges $\vec{v}$ and $\vec{w}$ is the magnitude of the vector $\vec{v} \times \vec{w}$.

$$Area = ||\vec{v} \times \vec{w}|| = \sqrt{11^2 + (-7)^2 + 5^2} = \sqrt{195}$$

Follow up questions

Question 1

Find the area of the parallelogram with edges $\vec{a} = (2, 1, -4)$ and $\vec{b} = (3, -2, 5)$ in $\mathbb{R}^3$.

Answer: $\sqrt{542}$

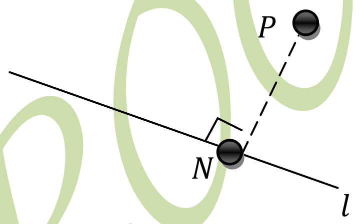
Question 2

Find the area of the parallelogram with edges $\vec{a} = (2, 1, 0)$ and $\vec{b} = (-1, 1, 0)$ in $\mathbb{R}^3$.

Answer: 3

The perpendicular distance from a point to a straight line

The scalar product plays a crucial role when determining the distance between a point and a line.

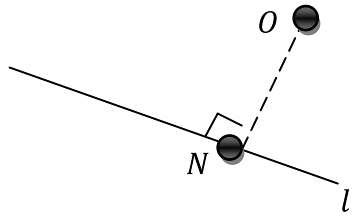


Steps

- First we need to find the coordinates of N
- Find vector $\overrightarrow{NP}$
- Since N lies on the line l, let the value of the scalar $= s$ or any other letter which different from the original scalar in the equation.
- In relation to the origin find $\overrightarrow{NP}$
- Find the dot product between $\overrightarrow{NP}$ and the direction vector of the line l to find the value of s.
- Substitute that value into $\overrightarrow{ON}$ and $\overrightarrow{NP}$.
- Calculate the required modulus.

NB: To find the coordinates of the foot of the perpendicular we simply substitute the value of t into $\overrightarrow{ON}$

The perpendicular distance from the origin to a straight line



Steps

- Let point N be the foot of the perpendicular from O.
- Rewrite the vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{pmatrix}$
- State that vector $\boldsymbol{ON} = \begin{pmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{pmatrix}$
- Since $\boldsymbol{ON}$ is perpendicular to l, find the value of the scalar t by equating the dot product between $\boldsymbol{ON}$ and the direction vector of the line l to zero as follows:

$$\boldsymbol{ON}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

- Substitute this numerical value of t in $\boldsymbol{ON} = \begin{pmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{pmatrix}$ to find the coordinates of the foot of the perpendicular
- To find the required distance then find the magnitude of $\boldsymbol{ON}$ as follows:

$$|\boldsymbol{ON}| = \sqrt{(x_0 + at)^2 + (y_0 + bt)^2 + (z_0 + ct)^2}$$

NB: Remember that when evaluating $|ON|$, you should have replaced t by its numerical value.

WORKED EXAMPLES

Example

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(7,-1,6)$ to the line l with equation

$$\boldsymbol{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}1\\-2\\4\end{pmatrix}.$$

Solution

a) Finding the coordinates of N:
Let $t=s$

$$\Rightarrow \overrightarrow{ON}=\begin{pmatrix}2\\1\\3\end{pmatrix}+s\begin{pmatrix}1\\-2\\4\end{pmatrix}=\begin{pmatrix}2+s\\1-2s\\3+4s\end{pmatrix}$$

Finding vector $\overrightarrow{NP}$

$$\Rightarrow \overrightarrow{NP}=\overrightarrow{OP}-\overrightarrow{ON}$$

$$=\begin{pmatrix}7\\-1\\6\end{pmatrix}-\begin{pmatrix}2+s\\1-2s\\3+4s\end{pmatrix}$$
$$=\begin{pmatrix}5-s\\-2+2s\\3-4s\end{pmatrix}$$

Since $\overrightarrow{NP}$ is perpendicular to line l then:

$$\Rightarrow \overrightarrow{NP}.\begin{pmatrix}1\\-2\\4\end{pmatrix}=0$$

$$\begin{pmatrix}5-s\\-2+2s\\3-4s\end{pmatrix}.\begin{pmatrix}1\\-2\\4\end{pmatrix}=0$$

$$5-s+4-4s+12-16s=0$$
$$21-21s=0$$
$$21=21s$$
$$\therefore s=1$$

To find the coordinates of the foot of the perpendicular we simply substitute the value of s into $\overrightarrow{ON}$.

$$\Rightarrow \overrightarrow{ON}=\begin{pmatrix}2+s\\1-2s\\3+4s\end{pmatrix}=\begin{pmatrix}3\\-1\\7\end{pmatrix}$$

$\therefore$ the coordinates are $(3, -1, 7)$.

b) Substituting $s = 1$ into $\overrightarrow{NP}$.

$$\overrightarrow{NP} = \begin{pmatrix} 5 - s \\ -2 + 2s \\ 3 - 4s \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$$

$$\therefore |\overrightarrow{NP}| = \sqrt{4^2 + 0^2 + (-1)^2} = \sqrt{16 + 1} = \sqrt{17}$$

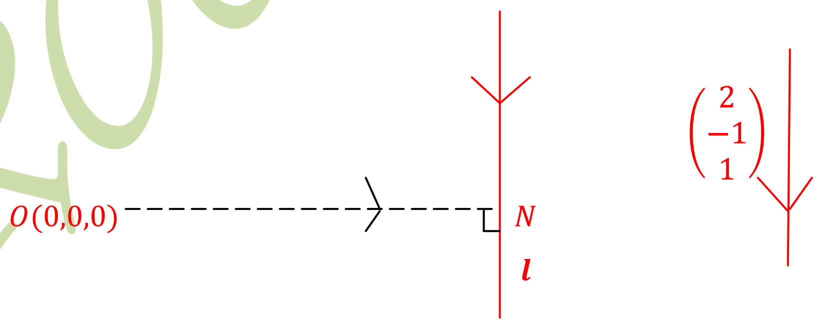
Example

The line l has the equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, where t is a parameter.

Find the perpendicular distance from the origin to l.

Suggested Solution

Let point N be the foot of the perpendicular from O.



The coordinates of N are $(1 + 2t;\ -t;\ 1 + t)$

Now

$$\boldsymbol{ON} = \begin{pmatrix} 1 + 2t \\ -t \\ 1 + t \end{pmatrix},$$

Now

$$\boldsymbol{ON}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$$

$$\begin{pmatrix}1+2t\\-t\\1+t\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$$

$$2+4t+t+1+t=0$$

$$6t=-3$$

$$\therefore t=\frac{-3}{6}=\frac{-1}{2}$$

Now substituting $t=\frac{-1}{2}$ into $\boldsymbol{ON}$

$$\boldsymbol{ON}=\begin{pmatrix}2+3\left(\frac{-1}{2}\right)\\12\left(\frac{-1}{2}\right)\\5+4\left(\frac{-1}{2}\right)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\-6\\3\end{pmatrix}$$

Now

$$|\boldsymbol{ON}|=\sqrt{\left(\frac{1}{2}\right)^{2}+(-6)^2+(3)^2}$$

$$=\sqrt{\frac{181}{4}}=\frac{\sqrt{181}}{2}.$$

Follow Up Questions

Question 1

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(11,-5,-3)$ to the line l with equation

$$r=\begin{pmatrix}1\\5\\0\end{pmatrix}+t\begin{pmatrix}-3\\1\\4\end{pmatrix}.$$

Answers: a) $(7,3,-8)$. b) $\sqrt{105}$

Question 2

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(-2, 11, 5)$ to the line l with equation

$$r = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}.$$

Answers: a) $(-2,6,7)$. b) $\sqrt{29}$

Question 3

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the point $P(8, 4, -1)$ to the line l with equation

$$r = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}.$$

Answers: a)$(2,7,-3)$. b) 7

Question 4

Find
a) the coordinates of the foot of the perpendicular

b) the shortest distance from the origin to the line l with equation

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix}$$

Answers: a)$\left(\frac{20}{13}, \frac{-24}{13}, \frac{57}{13}\right)$. b) 5

The angle between two lines

- The angle is defined to be the angle between the direction vectors of the two lines.
- If θ is the acute angle between the lines $\boldsymbol{r} = \boldsymbol{a}_1 + t\boldsymbol{b}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + t\boldsymbol{b}_2$ then θ is given by:

$$\cos\theta = \frac{\boldsymbol{b}_1 . \boldsymbol{b}_2}{|\boldsymbol{b}_1||\boldsymbol{b}_2|}$$

WORKED EXAMPLES

Example

Find the angle between the skew lines $l_1: x = 2 + 3t, y = 4 - t, z = 3 + 2t$ and $l_2: x = 1 - t, y = 5 + 2t, z = 6 + 3t$.

Solution

The vector $b_1 = (3, -1, 2)$ is parallel to line l_1 and $b_2 = (-1, 2, 3)$ is parallel to line l_1. The angle between the lines is equal to the angle between the vectors b_1 and b_2. Let θ be this angle.

$$\cos\theta = \frac{\boldsymbol{b}_1 . \boldsymbol{b}_2}{|\boldsymbol{b}_1||\boldsymbol{b}_2|}$$

$$\cos\theta = \frac{\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{3^2 + (-1)^2 + 2^2}\sqrt{(-1)^2 + 2^2 + 3^2}}$$

$$\cos\theta = \frac{-3 - 2 + 6}{\sqrt{14}\sqrt{14}}$$

$$\cos\theta = \frac{1}{14}$$

$$\theta = \cos^{-1}\left(\frac{1}{14}\right) = 85.90395624° = 86°$$

Example

Find the angle between the line with equation $r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, where t and s are parameters.

$$\cos\theta = \frac{\boldsymbol{b}_1 . \boldsymbol{b}_2}{|\boldsymbol{b}_1||\boldsymbol{b}_2|}$$

$$\cos\theta = \frac{\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{(-1)^2 + (-1)^2 + 3^2}\sqrt{1^2 + 2^2 + (-2)^2}}$$

$$\cos\theta = \frac{-9}{3\sqrt{11}}$$

$$\cos\theta = \frac{-3}{\sqrt{11}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{11}}\right) = 154.7505982° = 150°$$

Follow Up Questions

Question 1

Find the angle between the line with equation $r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where t and s are parameters.

Answer: 53.6°

Question 2

Find the angle between the line with equation $r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 12 \\ 3 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, where t and s are parameters.

Answer: 49.8°

Question 3

Find the angle between the line with equation $r = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$, where t and s are parameters.

Answer: 126°

Question 4

Find the angle between the line with equation $r = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + s\begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix}$ and the line with equation $r = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$, where t and s are parameters.

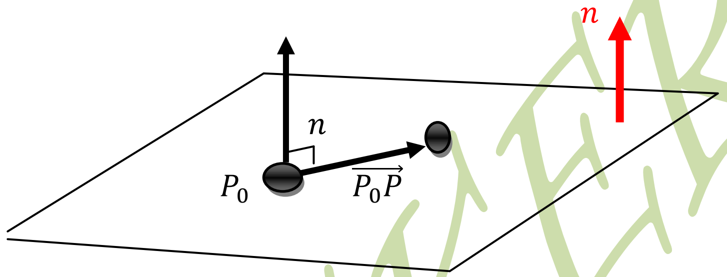
Answer: 8.72°

PLANES IN $\mathbb{R}^3$

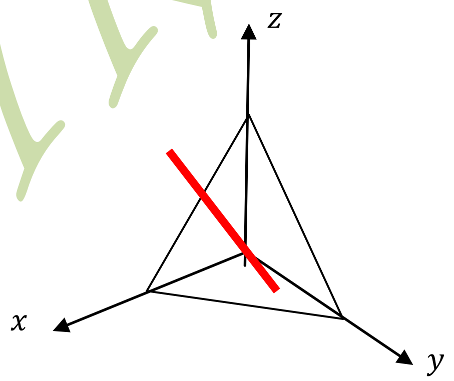
Definition

The plane by a point P_0 perpendicular to a non-zero vector $\boldsymbol{n}$, called the normal vector, is the set of points P solution of the equation

$$\overrightarrow{P_0P}.\boldsymbol{n} = 0$$



- A plane can be thought of as the collection of all lines perpendicular/orthogonal to a given line.
- The equation of a plane in $\mathbb{R}^3$ has the form: $ax + by + cz = d$, where not all of a, b, and c can be zero.
- If $ax + by + cz = d$ defines a plane, then $\boldsymbol{v} < a_1; a_2; a_3 >$ is normal to the plane.



If point $P_0 = (x_0;\ y_0;\ z_0)$ lies in the plane and vector $\boldsymbol{a} = < a_1; a_2; a_3 >$ is **normal** to the plane (i.e., orthogonal/perpendicular to every line in the plane) and if point $P = (x;\ y;\ z)$ is an arbitrary point in the plane, then

- $\overrightarrow{P_0P}$ is orthogonal to **a**,
- $< x - x_0;\ y - y_0; z - z_0 > . < a_1; a_2; a_3 >$

The equation of the plane is

$$a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0$$

Vector, Parametric and Cartesian equations of a plane

- A plane can be uniquely defined by three non-collinear points or by a point and two non-parallel direction vectors.
- For a scalar equation, any point (x, y, z) that satisfies the equation lies on the plane.
- For vector and parametric equations, any combination of values of the parameters t and s will produce a point on the plane.
- The x −intercept of a plane is found by setting $y = z = 0$ and solving for x. Similarly, the $y-$ and z −intercepts are found by setting $x = z = 0$ and $x = y = 0$, respectively.
- The equation of a plane that cuts the co-ordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

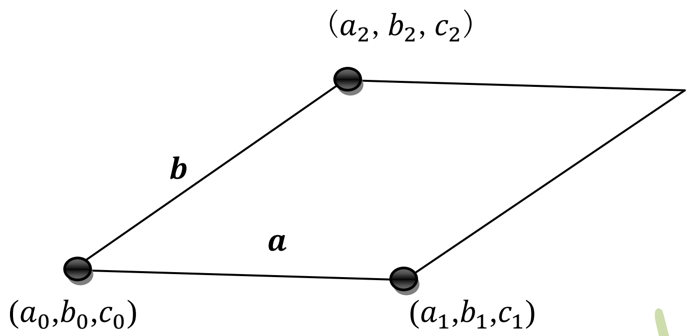
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- The Vector equation of any plane that passes through the intersection of planes $\boldsymbol{r} . \boldsymbol{n}_1 = d_1$ and $\boldsymbol{r} . \boldsymbol{n}_2 = d_2$ is $(\boldsymbol{r} . \boldsymbol{n}_1 - d_1) + t(\boldsymbol{r} . \boldsymbol{n}_2 - d_2) = 0$, where t is any non-zero constant.
- The Cartesian equation of any plane that passes through the intersection of two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is

$$(A_1x + B_1y + C_1z + D_1) + t\,(A_2x + B_2y + C_2z + D_2) = 0.$$

The Plane which contains three non-collinear points

- Sometimes you need to find the equation of a plane which contains three points. Consider the following picture.



- We have to find the normal $\boldsymbol{n}$ and it is obtained by taking $\boldsymbol{a} \times \boldsymbol{b}$ where:

$$\boldsymbol{a} = (a_1 - a_0, b_1 - b_0, c_1 - c_0) \text{ and } \boldsymbol{b} = (a_2 - a_0, b_2 - b_0, c_2 - c_0).$$

NB: $\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b}$ is the cross product between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

- Then the equation is given by:
  $(x - a_0, y - b_0, z - c_0).\boldsymbol{n} = 0$

- This can be simplified to:

$$(\boldsymbol{r}.\boldsymbol{a}).[(\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{c} - \boldsymbol{a})] = 0,$$

  where $(\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{c} - \boldsymbol{a})$ is the cross product.

WORKED EXAMPLES

Example 1

Find the equation of the plane containing the points $P = (1;\ 3;\ 2)$, $Q = (3;\ -1;\ 6)$, and $R = (5;\ 2;\ 0)$.

Solution

First we must find a vector orthogonal to the plane containing the three points.

Let $\boldsymbol{a} = \overrightarrow{PQ} = <2;\ -4;\ 4>$ and let $\boldsymbol{b} = \overrightarrow{PR} = <4;\ -1;\ 2>$, then using the cross product we have a vector perpendicular to the plane.

$$\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b} = < 12;\ 20;\ 14 >$$

The equation of the plane is

$$(x-1;\ y-3; z-2).(12;\ 20;\ 14)$$

$$12x - 12 + 20y - 60 + 14z - 28 = 0$$

$$12x + 20y + 14z = 0 + 60 + 28 + 12$$

$$6x + 10y + 7z = 50$$

Example 2

Find the equation of the plane through the point $(2;\ 4; -1)$ with normal vector $\boldsymbol{n} = < 2;\ 3;\ 4 >$.

Solution

The equation of the plane is

$$(x-2;\ y-4; z+1).(2;\ 3;\ 4)$$

$$2x - 4 + 3y - 12 + 4z + 4 = 0$$

$$2x + 3y + 4z = 0 + 12$$

$$2x + 3y + 4z = 12$$

Example 3

Find the equation of the plane with normal vector, $\mathbf{n} = < 1, 2, 3 >$ containing the point $(2, -1, 5)$

Solution

From the above, the equation of this plane is

$$(1, 2, 3) \cdot (\mathrm{x} - 2, \mathrm{y} + 1, \mathrm{z} - 5)$$

$$x - 2 + 2y + 2 + 3z - 15 = 0$$

$$x + 2y + 3z = 0 + 15$$

$$x + 2y + 3z = 15$$

Example 4

Consider the plane with direction vectors $\vec{a} = [8, -5, 4]$ and $\vec{b} = [1, -3, -2]$ through $P(3, 7, 0)$.

a) Write the vector and parametric equations of the plane.

b) Determine if the point $Q(-10, 8, -6)$ is on the plane.

c) Find the coordinates of two other points on the plane.

d) Find the x-intercept of the plane.

Answers

a) A vector equation of the plane is

$$[x, y, z] = [3, 7, 0] + t[8, -5, 4] + s[1, -3, -2]$$

NB: *Use any letters other than x, y and z for the parameters.*

Parametric equations for the plane are

$$x = 3 + 8t + s$$

$$y = 7 - 5t - 3s$$

$$z = 4t - 2s$$

b) If the point $(-10, 8, -6)$ is on the plane, then there exists a single set of t and s values that satisfy the equations.

$$-10 = 3 + 8t + s \ (1)$$

$$8 = 7 - 5t - 3s \ (2)$$

$$6 = 4t - 2s \ (3)$$

Solve (1) and (2) for t and s using elimination.

$$-39 = 24t + 3s \ (4)$$

$$\underline{1 = -5t - 3s \ (2)}$$

$$-38 = 19t \qquad (4) + (2)$$

$$\therefore t = -2$$

Substitute $t = -2$ into either equation 1 or 2.

$$-13 = 8(-2) + s$$

$$s = 3$$

Now check if $t = -2$ and $s = 3$ satisfy (3).

$$\text{RHS} = 4(-2) - 2(3)$$

$$= -14$$

$$\neq \text{LHS}$$

Since the values for t and s do not satisfy equation (3), the point $(-10, 8, -6)$ does not lie on the plane.

c) To find other points on the plane, use the vector equation and choose arbitrary values for the parameters t and s.

Let $t = 1$ and $s = -1$.

$$[x, y, z] = [3, 7, 0] + (1)[8, -5, 4] + (-1)[1, -3, -2]$$
$$= [3, 7, 0] + [8, -5, 4] + [-1, 3, 2]$$
$$= [10, 5, 6]$$

Let $t = 2$ and $s = 1$.

$$[x, y, z] = [3, 7, 0] + (2)[8, -5, 4] + (1)[1, \_3, \_2]$$
$$= [3, 7, 0] + [16, -10, 8] + [1, -3, -2]$$
$$= [20, -6, 6]$$

The coordinates of two other points on the plane are $(10, 5, 6)$ and $(20, -6, 6)$.

d) To find the x-intercept, set $y = z = 0$ and solve for s and t.

$x = 3 + 8t + s$ (1)

$0 = 7 - 5t - 3s$ (2)

$0 = 4t - 2s$ (3)

Solve (2) and (3) for t and s using elimination to get t=7⁄11 and $t = 14/11$.

Now, substitute $t = 7/11$ and $t = 14/11$ to get $x = 103/11$.

ADDITIONAL NOTES

Two planes with normal vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are

(i) Parallel if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

(ii) Orthogonal/perpendicular if $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal/perpendicular.

WORKED EXAMPLES

Example 1

Are the planes defined by $x + 2y - 3z = 4$ and $2x + 4y - 6z = 1$ parallel?

Solution

A normal vector to the first plane is $\boldsymbol{a} = < 1;\ 2;\ -3 >$ while a normal vector to the second plane is $\boldsymbol{b} = < 2;\ 4;\ -6 >$.

Since $\boldsymbol{b}$ is a scalar multiple of $\boldsymbol{a}$ (namely $\boldsymbol{b} = 2\boldsymbol{a}$) then the normal vectors are parallel, which implies the original planes are parallel.

Example 2

Find a point P_0 and the perpendicular vector $\boldsymbol{n}$ to the plane $2x + 4y - z = 3$.

Solution

The equation of a plane is $a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0$

The components of the normal vector $\boldsymbol{n}$ are the coefficients that multiply the variables x, y and z. Hence, $\boldsymbol{n} = < 2;\ 4;\ -1 >$

A point P_0 on the plane is simple to find. Just look for the intersection of the plane with one of the coordinate axis. For example: set $y = 0, z = 0$ and find x from the equation of the plane: $2x = 3$, that is $x = 3/2$. Therefore, $P_0 = (3/2, 0, 0)$.

The angle a plane, two lines and two planes

Angle between a line and a plane

- The angle is defined to be the angle between the direction vector of the line and the normal vector of the plane.
- The acute angle θ between the line $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ and plane $r\,.\,\boldsymbol{n} = d$ is given by

$$\sin\theta = \frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}$$

WORKED EXAMPLE

Example

Find the angle between the plane with equation $r.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$ and the line with equation $r = \begin{pmatrix}2\\0\\5\end{pmatrix} + t\begin{pmatrix}3\\12\\4\end{pmatrix}$, where t is the parameter.

Solution

$$\sin\theta = \frac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}$$

$$\sin\theta = \frac{\begin{pmatrix}3\\12\\4\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix}}{\sqrt{3^2+12^2+14^2}\sqrt{(-1)^2+2^2+2^2}}$$

$$\sin\theta = \frac{\begin{pmatrix}3\\12\\4\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix}}{\sqrt{3^2+12^2+4^2}\sqrt{(-1)^2+2^2+2^2}}$$

$$\sin\theta = \frac{-3+24+8}{\sqrt{169}\sqrt{9}}$$

$$\theta = \sin^{-1}\left(\frac{29}{39}\right) = 48.03811117° = 48°$$

Angle between two planes

- Suppose two planes intersect, then the angle between the planes is defined to be the angle between their normal vectors.
- If the normal vectors of the two planes are $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, then the angle is given by:

$$\cos\theta = \frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}|.|\overrightarrow{n_2}|}$$

WORKED EXAMPLES

Example

Find the angle between the plane with equation $r.\begin{pmatrix}1\\-2\\2\end{pmatrix} = 66$ and the plane with equation $r.\begin{pmatrix}4\\0\\3\end{pmatrix} = 27$.

Solution

$$\cos\theta = \frac{\begin{pmatrix}1\\-2\\2\end{pmatrix}.\begin{pmatrix}4\\0\\3\end{pmatrix}}{\sqrt{1^2+(-2)^2+2^2}\sqrt{4^2+0^2+3^2}}$$

$$\cos\theta = \frac{8+6}{\sqrt{9}\sqrt{25}}$$

$$\cos\theta = \frac{14}{15}$$

$$\theta = \cos^{-1}\left(\frac{14}{15}\right) = 21.03946978° = 21°$$

Example

Find the angle between the two planes with equations $2x - y + z = 5$ and $x + y - z = 1$ respectively.

Solution

$$\cos\theta = \frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}|.|\overrightarrow{n_2}|}$$

$$\overrightarrow{n_1} = \begin{pmatrix}2\\-1\\1\end{pmatrix} \text{ and } \overrightarrow{n_2} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}2\\-1\\1\end{pmatrix}.\begin{pmatrix}1\\1\\-1\end{pmatrix}}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+1^2+(-1)^2}}$$

$$\cos\theta = \frac{0}{\sqrt{6}\sqrt{3}}$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1}(0) = 90°$$

Example

Find the angle between the planes

$$x + y + z = 1$$

$$x - 2y + 3z = 2$$

Solution

The angle between the planes will be the angle between their normal vectors.

Let $\mathbf{n_1} = < 1;\ 1;\ 1 >$ and $\mathbf{n_2} = < 1;\ -2;\ 3 >$, then

$$\mathbf{n_1}.\mathbf{n_2} = |\mathbf{n_1}|.|\mathbf{n_2}|\cos\theta$$

$$2 = \sqrt{42}\cos\theta$$

$$\theta = \cos\left(\frac{2}{\sqrt{42}}\right) \approx 72$$

Follow Up Questions

Question 1

Find the angle between the plane with equation $2x + 3y + z = 4$ and the line with equation $r = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, where t is the parameter.

Answer: 26°

Question 2

Find the angle between the two planes with equations $2x + 3y + 5z = 8$ and $5x + y - 4z = 12$ respectively.

Answer: 79.9°

Question 2

Find the angle between the two planes with equations $x + 3y + z = -4$ and $5x - y + z = 8$ respectively.

Answer: 80.0°

Determining whether a point lies in a plane

The point is said to lie in a plane if its coordinates satisfy the equation of a plane

WORKED EXAMPLE

Example

Show that the point $(5,2,-2)$ lies in the plane $2x+4y+5z=8$.

Solution

Substitute the point $(5,2,-2)$ to the LHS of the plane $2x+4y+5z=8$.

Now LHS: $2x+4y+5z=2(5)+4(2)+5(-2)=10+8-10=8\equiv RHS$

Since $LHS\equiv RHS$ $\therefore$ the point $(5,2,-2)$ lies in the plane $2x+4y+5z=8$.

Follow Up Questions

Question 1

Show that the point $(5,2,-2)$ lies in the plane $2x+6y-z=24$.

Question 2

Show that the point $(2,0,-1)$ lies in the plane $3x+y+z=5$.

Question 3

Show that the point $(1,1,3)$ lies in the plane $x-3y+2z=4$.

Question 4

Show that the point $(-2,3,0)$ lies in the plane $-5x - 3y - 2z = 1$.

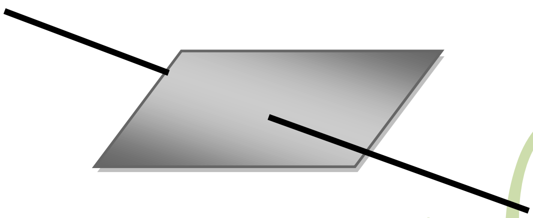
Question 5

Show that the point $(1,3,2)$ lies in the plane $x + 2y + z = 9$.

Determining whether a line lies in a plane, is parallel to or intersects a plane

Case 1

The line and the plane are not parallel so they intersect in one point



A line is said to be perpendicular to a plane when a value of the parameter can be estimated. Alternatively, when the direction vector of the line is parallel to the normal vector of the plane, then line is said to be perpendicular to the plane

WORKED EXAMPLE

Example

Show that the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is perpendicular to the plane $5x + y - z = 1$.

Solution

Show that the line $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and so for any point on the line

$$x = 2 + t, \quad y = 3 + 2t \quad \text{and} \quad z = 4 - t$$

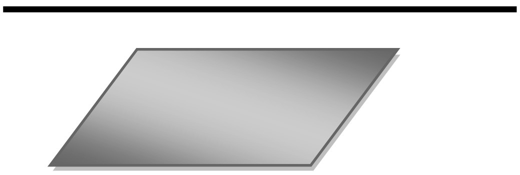
Substituting these into the equation of the plane $5x + y - z = 1$ gives

$$5(2+t)+(3+2t)-(4-t)=1$$
$$8t=-8$$
$$\therefore t=-1$$

Since there is a constant value of the parameter t, thus the line and the plane are perpendicular.

Case 2

The line and the plane are parallel so they do not intersect



When a line is parallel to a plane, its direction vector is perpendicular to the plane's normal vector i.e.

$$\boldsymbol{d}.\hat{\boldsymbol{n}}=\mathbf{0}$$

WORKED EXAMPLE

Example

Show that the line $\boldsymbol{r}=\begin{pmatrix}2\\0\\-1\end{pmatrix}+t\begin{pmatrix}3\\1\\-2\end{pmatrix}$ is parallel to the plane $2x+4y+5z=8$.

Solution

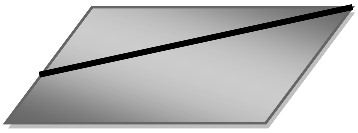
The direction vector of the line, $\boldsymbol{d}$, is $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and normal vector of the plane, $\hat{\boldsymbol{n}}$, is $\begin{pmatrix}2\\4\\5\end{pmatrix}$

$$\boldsymbol{d}.\hat{\boldsymbol{n}}=\begin{pmatrix}3\\1\\-2\end{pmatrix}.\begin{pmatrix}2\\4\\5\end{pmatrix}=6+4-10=0.$$

Since $\boldsymbol{d}.\hat{\boldsymbol{n}}=0$, the line is parallel to the plane.

Case 3

The line and the plane are parallel and the line lies in the plane.



To prove that a line lies in a plane, you need to show that the line and the plane are parallel and that any point on the line also lies in the plane.

WORKED EXAMPLE

Example

Does the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ lie in the plane $2x + 4y + 5z = 8$?

Solution

The direction vector of the line, $\boldsymbol{d}$, is $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and normal vector of the plane, $\widehat{\boldsymbol{n}}$, is $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

$$\boldsymbol{d}.\widehat{\boldsymbol{n}} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}.\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = 6 + 4 - 10 = 0.$$

Since $\boldsymbol{d}.\widehat{\boldsymbol{n}} = 0$, the line is parallel to the plane.

Now find a point on the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ by setting $t = 1$.

So the point $(5,2,-2)$ lies on the line

Now check if the point satisfies the equation of the plane $2x + 4y + 5z = 8$.

Substitute the point $(5,2,-2)$ to the LHS of the plane $2x + 4y + 5z = 8$.

Now LHS: $2x + 4y + 5z = 2(5) + 4(2) + 5(-2) = 10 + 8 - 10 = 8 \equiv RHS$

Since $LHS \equiv RHS$ $\therefore$ the point $(5,2,-2)$ lies in the plane $2x + 4y + 5z = 8$.

Since the line and the plane are parallel and the point $(5,2,-2)$ lies both on the line and in the plane, therefore the line must lie in the plane,

FURTHER EXAMPLES

Example

Given a line L and a plane P: $L : x = 15 - 3t, y = 6 - 5t, z = 2 + 3t$ $P : 2x + 3y + 4z = 20.$ Determine if L and P intersect or are parallel.

Solution

The normal vector of P is $n = (2,3,4)$ and the line L is parallel $to\ v = (-3,-5,3)$. As the dot product $n \cdot v = (2)(-3) + (3)(-5) + 4(3) = -6 - 15 + 12 \neq 0$, L and P are not parallel to each other.

We need to see where the two objects intersect. Substitute the parametric equations of L into the equation of P and solve the resulting equation for t:

$2(15 - 3t) + 3(6 - 5t) + 4(2 + 3t) = 20 \Rightarrow -9t = -36 \Rightarrow t = 4.$

Substitute $\mathrm{t} = 4$ in the parametric equations of L to get the point of intersection at $(3,-14,14)$.

Example

Given a line L and a plane P:

$$L : x = 15 + 7t, y = 10 + 12t, z = 5 - 4t$$

$$P : 12x - 5y + 6z = 10.$$

Determine if L and P intersect or are parallel.

Solution

The normal vector of P is $\boldsymbol{n} = (12,-5,6)$ and the line L is parallel to $\boldsymbol{v} = (7,12,-4)$.

As the dot product $\boldsymbol{n} \cdot \boldsymbol{v} = (7)(12) + (-5)(12) + 6(-4) = 84 - 60 - 24 = 0$, L and P are parallel to each other.

We need to see if the two objects intersect. Since L is parallel to P, if L intersects P, then any point of L should be in P. Thus we only have to check if, when $t = 0$, the point $(15,10,5)$ is on P.

Note that $12(15) - 5(10) + 6(6) = 180 - 50 + 36 \neq 10$, and so L and P do not intersect.

Finding the point of intersection of a line and a plane

- The vector equation of a straight line passing through the fixed point, with position vector $\boldsymbol{r}_1$, and parallel to the fixed vector $\boldsymbol{a}$, is

$$r = \boldsymbol{r}_1 + t\boldsymbol{a}.$$

- For the point of intersection of this line with the plane, whose vector equation is

$$\boldsymbol{r}\,.\boldsymbol{n} = d,$$

  the value of t must be such that

$$(\boldsymbol{r}_1 + t\boldsymbol{a}).\boldsymbol{n} = d,$$

  which is an equation from which the appropriate value of t and, hence, the point of intersection may be found.

WORKED EXAMPLES

Example

Find the intersection of the line $x = t, y = 2t, z = 3t$, and the plane $x + y + z = 1$.

Solution:

Substitute the line into the plane: $t + 2t + 3t = 1 \Rightarrow t = \frac{1}{6}$

Put t back to the line: $x = \frac{1}{6}, y = \frac{2}{6} = \frac{1}{3}, z = \frac{3}{6} = \frac{1}{2}$

Hence the intersection point is $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$.

Example

Determine the point of intersection of the plane, whose vector equation is $r\,.(\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}) = 7$,and the straight line passing through the point, $(4, -1, 3)$, which is parallel to the vector $2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$.

Solution

We need to obtain the value of the parameter, t, such that

$$(4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} + t[2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}]).(\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}) = 7.$$

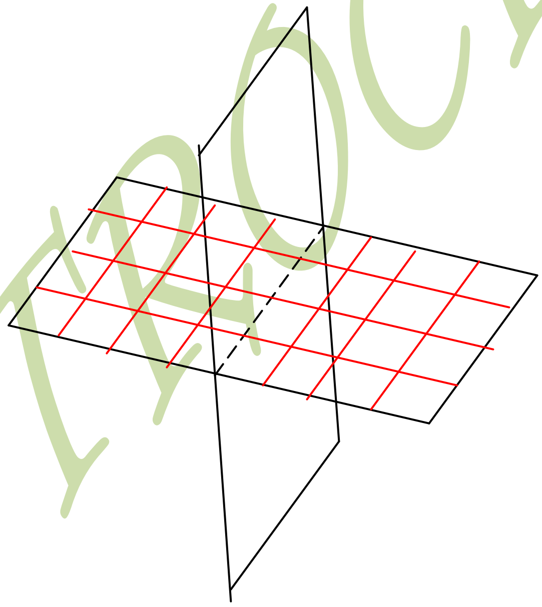
That is,

$$(4 + 2t)(1) + (-1 - 2t)(-3) + (3 + 5t)(-1) = 7 \text{ or } 4 + 3t = 7,$$

which gives $t = 1$; and, hence, the point of intersection is

$$(4 + 2, -1 - 2, 3 + 5) = (6, -3, 8).$$

Intersection of two planes



- The Normals of planes can identify relationships between the planes.
- If the Normals of the two planes are scalar multiples of each other, the planes are parallel and can have either NO solution or an Infinite number of solutions.

- If the Normals are not parallel, then the planes intersect and have an infinite number of points in common.
- This is an important concept because it opens the door for using parameters to solve problems.

Line of intersection of two planes

- Suppose P_1 and P_2 are two planes, with normal vectors $\vec{n}_1$ and $\vec{n}_2$, then two planes: are parallel (or the same) if and only if their normal vectors are scalar multiples of each other
- If two planes are not parallel, they intersect but instead of intersecting at a single point, the set of points where they intersect form a line.
- Let's call the line L with direction vector $\vec{d}$.
- Then since L is contained in P_1, we know that $\vec{n}_1$ must be orthogonal to $\vec{d}$. And, similarly, L is contained in P_2, we know that $\vec{n}_2$ must be orthogonal to $\vec{d}$ as well.
- That means that to $\vec{d}$, we need to find a vector that is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$.

WORKED EXAMPLES

Example

Find a vector equation of the line of intersections of the two planes $x - 5y + 3z = 11$ and $-3x + 2y - 2z = -7$.

Solution

First we read off the normal vectors of the planes:

The normal vector $\vec{n}_1$ of $x - 5y + 3z = 11$ is $\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$ and

The normal vector $\vec{n}_2$ of $-3x + 2y - 2z = -7$ is $\begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}$

Next, we find the direction vector $\vec{d}$ for the line of intersection, by computing

$$\vec{d} = \vec{n}_1 \times \vec{n}_2 = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \times \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} (-5)\times(-2) - (3\times 2) \\ 3\times(-3) - 1(-2) \\ 1\times(-2) - (-5)\times(-3) \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -13 \end{bmatrix}.$$

Now, we have to find L{a point on the line}

That is, we need to find a point P that is on both planes.

Let $z = 0$, and this implies that $x - 5y + 0 = 11$ and $-3x + 2y + 0 = -7$.

But $x - 5y + 0 = 11$ means that $x = 5y + 11$.

Plugging this fact into the second equation gives us

$-3(5y + 11) + 2y + 0 = -7 \Rightarrow y = -2$.

And so $x = 1$, and we see that $P(1; -2;\ 0)$ is a point on both planes.

NB: We can plug P in to the given equations of the plane to double check:

$(1) - 5(-2) + 3(0) = 11$ and $-3(1) + 2(-2) - 2(0) = -7$.

So, we have a point $(1; -2;\ 0)$ on the line, and a direction vector $\begin{bmatrix} 4 \\ -7 \\ -13 \end{bmatrix}$ for the line, so a vector equation of the line is

$$r = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -7 \\ -13 \end{bmatrix}, t \in \mathbb{R}$$

Example

Determine the vector equation, and hence the Cartesian equations (in standard form), of the line of intersection of the planes whose vector equations are

$$\boldsymbol{r} \,.\, \boldsymbol{n}_1 = 2 \text{ and } \boldsymbol{r} \,.\, \boldsymbol{n}_2 = 17,$$

where

$\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{n}_2 = 4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$.

Solution

Firstly,

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}.$$

Secondly, the Cartesian equations of the two planes are

$$x + y + z = 2 \text{ and } 4x + y + 2z = 17;$$

and, when z = 0, these become

$$x + y = 2 \text{ and } 4x + y = 17,$$

which, as simultaneous linear equations, have a common solution of $x = 5, y = -3$.

Thirdly, therefore, a point on the line of intersection is (5,–3, 0), which has position vector $5i - 3j$.

Hence, the vector equation of the line of intersection is

$r = 5i - 3j + t(i + 2j - 3k)$.

Finally, since $x = 5 + t, y = -3 + 2t$ and $z = -3t$ the line of intersection is represented, in standard Cartesian form, by

$$\frac{x-5}{1} = \frac{y+3}{2} = \frac{z}{-3} (= t)$$

Example

Find the line of intersection of the two planes

$$x + y + z = 1$$
$$x - 2y + 3z = 2$$

Solution

Eliminate x from the two equations and then treat z as the parameter.

$$1 - y - z = x = 2 + 2y - 3z$$

$$y = -\frac{1}{3} + \frac{2}{3}z$$

Parametric Form:

$$x = \frac{4}{3} - \frac{5}{3}t; \quad y = -\frac{1}{3} + \frac{2}{3}t; \quad z = t$$

Example

Write the parametric and symmetric equations of the line of intersection of the planes $2x - y + z = 5$ and $x + y - z = 1$.

Solution

Solution: The planes have normal vectors $\boldsymbol{a} = (2, -1, 1)$ and $\boldsymbol{b} = (1, 1, -1)$, respectively. Let L denote the line of intersection.

Then $v = \boldsymbol{a} \times \boldsymbol{b} = [1 - 1, -(-2 - 1), 2 - (-1)] = (0, 3, 3)$ is parallel to L.

We only need to find a point P on L.

To find P, solve the system of equations of the planes:

$$2x - y + z = 5 \text{ and } x + y - z = 1.$$

We consider P to be the point of L on the plane $z = 0$. Thus substitute $z = 0$ in the system above to get

$$2x - y = 5 \text{ and } x + y = 1 \Rightarrow x = 2, y = -1.$$

Hence we get $P(2, -1, 0)$, and so the equations of the line are

$$x = 2$$

$$y = -1 + 3t \quad \text{and} \quad x = 2 \Rightarrow y + 1 = z$$

$$z = 3t$$

Example

Determine the equation of the line that passes through the point $P(5,2,3)$ and is parallel to the line of intersection of the two planes:

$$\pi_1: x + 2y - z = 6$$

$$\pi_2: y + 2z = 1$$

Solution

To find the equation of a line, we need a point and a direction vector (slope).

We need the direction vector of the line that lies at the intersection of the two planes.

Method One

The cross product of the two planar normals yields a vector that is perpendicular to both normals and is also the direction vector of the line of intersection.

$$\overrightarrow{n_1} = [1,2,-1]$$

$$\overrightarrow{n_2} = [0,1,2]$$

$$\vec{m} = \overrightarrow{n_1} \times \overrightarrow{n_2} = 5\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

The equation is $\boldsymbol{r} = \begin{pmatrix}5\\2\\3\end{pmatrix} + s\begin{pmatrix}5\\-2\\1\end{pmatrix}$

Method Two

Find the equation of the line of intersection using the parameter approach.

$$\pi_1 : x + 2y - z = 6$$

$$\pi_2 : y + 2z = 1$$

Let $\boldsymbol{z} = \boldsymbol{s}$ and sub into (2)

$$y + 2z = 1$$

$$y + 2s = 1$$

$$y = -2s + 1$$

Parametric equations are:

$$x = 5s - 4$$

$$y = -2s + 1$$

$$z = s$$

Sub z and **y** into (1)

$$x + 2y - z = 6$$

$$x + 2(-2s + 1) - s = 6$$

$$x - 4s + 2 - s = 6$$

$$x - 4s + 2 - s = 6$$

$$x = 5s + 4$$

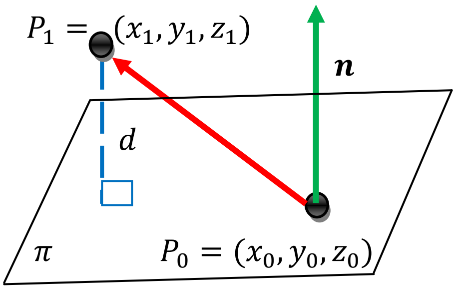
Vector Equation of the line of intersection:

$$\vec{r} = [4, -2, 0] + s[5, -2, 1]$$

Use the direction vector and point to determine the desired line.

$$\boldsymbol{r} = \begin{pmatrix}5\\2\\3\end{pmatrix} + s\begin{pmatrix}5\\-2\\1\end{pmatrix}.$$

The distance from a point to a plane



Vector Form

Let consider a plane π with a *normal vector* n r and a point $P_0(x_0, y_0, z_0)$ on this plane. The *distance* from a point $P_1(x_1, y_1, z_1)$to the plane π is given by the *scalar projection* of the vector $\boldsymbol{P}_0\,\boldsymbol{P}_1$ onto the normal vector $\boldsymbol{n}$:

$$d = \frac{||\boldsymbol{P}_0\,\boldsymbol{P}_1 . \boldsymbol{n}||}{||\boldsymbol{n}||}$$

Cartesian Form

If the plane π is given by the *Cartesian equation* : $Ax + By + Cz + D = 0$, then the *distance* from a point $P_1(x_1, y_1, z_1)$to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

NB: $|Ax_1 + By_1 + Cz_1 + D|$ means the absolute value of $Ax_1 + By_1 + Cz_1 + D$.

Two Parallel Planes

To get the *distance* between *two parallel planes*:

a) Find a specific point into one of these planes.

b) Find the distance between that specific point and the other plane using one of the formulas above.

WORKED EXAMPLES

Example

Determine the perpendicular distance, p, of the point $(2, -3, 4)$ from the plane whose cartesian equation is $x + 2y + 2z = 13$.

Solution

From the Cartesian formula

$$d = \frac{|(1)(2) + (2)(-3) + (2)(4) - 13|}{\sqrt{(1)^2 + (2)^2 + (2)^2}}$$

$$d = \frac{9}{3}$$

$$d = 3.$$

Example

Determine the perpendicular distance, p, of the point $(2,3,0)$ from the plane whose cartesian equation is $-x + 2y - 3z + 10 = 0$.

Solution

From the Cartesian formula

$$d = \frac{10 + (-1)(2) + (2)(3) + (0)(-3)}{\sqrt{(-1)^2 + (2)^2 + (-3)^2}}$$

$$d = \frac{14}{\sqrt{14}}$$

$$d = \sqrt{14}.$$

NB: We want to use the both methods to solve the following question

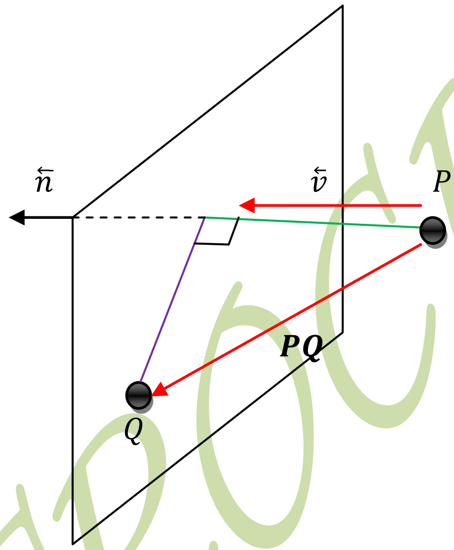
Example

Find the distance between the point $P = (1;\ 0; -1)$ and the plane $5x + 4y + 3z = 1$.

Solution

First Method

We draw the point and the plane with an additional point Q on the plane.



From the picture, we see that the distance is the length of the vector $\vec{v}$ and that $\vec{v}$ is the projection of $\overrightarrow{PQ}$ onto $\vec{n}$; here, $\vec{n}$ is the normal vector to the plane, which we can tell from the equation is $< 5;\ 4;\ 3 >$.

Thus, the distance is given by:

$$d = \frac{|\overrightarrow{PQ}.\vec{n}|}{||\vec{n}||}$$

Finding $\overrightarrow{PQ}$

In this case, to find a point on the plane, we can set $x = y = 0$; plugging this into the plane equation gives $z = \frac{1}{3}$ so $\left(0;\ 0;\ \frac{1}{3}\right)$ is on the plane. This means that $\overrightarrow{PQ} = \left(-1;\ 0;\ \frac{4}{3}\right)$.

Now

$$d = \frac{|\overrightarrow{PQ}.\vec{n}|}{||\vec{n}||}$$

$$d = \frac{\left|\begin{pmatrix}-1\\0\\\frac{4}{3}\end{pmatrix}.\begin{pmatrix}5\\4\\3\end{pmatrix}\right|}{\sqrt{5^2+4^2+3^2}}$$

$$d = \frac{|-5+4|}{\sqrt{50}}$$

$$d = \frac{|-1|}{5\sqrt{2}}$$

$$d = \frac{1}{5\sqrt{2}}$$

$$\therefore d = \frac{\sqrt{2}}{10}$$

Method 2

Alternatively we can use method 2 which is very simple.

From the Cartesian formula

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}$$

$$d = \frac{|(5)(1)+(5)(0)+(3)(-1)-1|}{\sqrt{(5)^2+(4)^2+(3)^2}}$$

$$d = \frac{|1|}{\sqrt{50}}$$

$$d = \frac{1}{5\sqrt{2}}$$

$$\therefore d = \frac{\sqrt{2}}{10}$$

Follow up questions

1. Determine the vector equation and hence the cartesian equation of the plane, passing through the point with position vector $\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$, and perpendicular to the vector $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$.
2. Determine the cartesian equation of the plane passing through the three points $(1, -1, 2)$, $(3, -2, -1)$ and $(-1, 4, 0)$.
3. Determine the point of intersection of the plane, whose vector equation is $\boldsymbol{r}\,.\,(5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = -3$, and the straight line passing through the point $(2, 1, -3)$, which is parallel to the vector $\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$.
4. Determine the vector equation, and hence the cartesian equations (in standard form), of the line of intersection of the planes, whose vector equations are $\boldsymbol{r}\,.\,\boldsymbol{n}_1 = 14$ and $\boldsymbol{r}\,.\,\boldsymbol{n}_2 = -1$, where $\boldsymbol{n}_1 = -4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{n}_2 = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$.
5. Determine, in surd form, the perpendicular distance of the point $(-5, -2, 8)$ from the plane whose cartesian equation is $2x - y + 3z = 17$.
6. Find the distance from the point $(1, 4, 1)$ to the plane $2x + 3y - z = -1$.
7. Find the distance between the point $P = (1;\ 0; -1)$ and the plane $5x + 4y + 3z = 1$.
8. Find the distance between the point $(2;\ 8;\ 5)$ and the plane $x - 2y - 2z = 1$.
9. Find the distance between the point $P = (2, 1, 3)$ and the plane $2x + y + 2z = 3$.

Answers

1. $r\,.\,(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = 10$, or $2x + y - 3z = 10$.
2. $17x + 10y + 8z = 23$.
3. $(0, -1, 5)$.
4. $r = -2\boldsymbol{i} + 3\boldsymbol{j} + t(7\boldsymbol{i} + 10\boldsymbol{j} - 8\boldsymbol{k})$ or

   $$\frac{x+2}{7} = \frac{y-3}{10} = \frac{z}{-8} (= t)$$
5. $p = \frac{1}{\sqrt{14}}$

6. $p = \sqrt{14}$

7. $p = \frac{\sqrt{2}}{10}$

8. $p = \frac{25}{3}$

9. $p = \frac{8}{3}$

Solved Past Examinations Questions

ZIMSEC NOVEMBER 2006 PAPER 2

The plane π and the line l have equations

$\boldsymbol{r}.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$ and $\boldsymbol{r} = \begin{pmatrix}2\\0\\5\end{pmatrix} + t\begin{pmatrix}3\\12\\4\end{pmatrix}$ respectively, where t is a parameter.

Find

(i) the position vector of the point of intersection of π and l, **[4]**

(ii) the angle between π and l, **[3]**

(iii) the perpendicular distance from the origin to l. **[5]**

Suggested Solution

(i) $\boldsymbol{r}.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$ (i) and $\boldsymbol{r} = \begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix}$ (ii)

substituting (i) into (ii)

$$\begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix}.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 66$$

$$-2-3t+24t+10+8t = 66$$

$$29t = 58$$

$\therefore t = 2$

Now $\boldsymbol{r} = \begin{pmatrix} 2+3t \\ 12t \\ 5+4t \end{pmatrix} = \begin{pmatrix} 2+3(2) \\ 12(2) \\ 5+4(2) \end{pmatrix} = \begin{pmatrix} 8 \\ 24 \\ 13 \end{pmatrix}$

$\therefore$ The position vector is $= \begin{pmatrix} 8 \\ 24 \\ 13 \end{pmatrix}$

NB: Suppose you are asked to find the point of intersection, you should write it as $(8; 24; 13)$

(ii) $\sin\theta = \dfrac{\boldsymbol{b}.\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}$

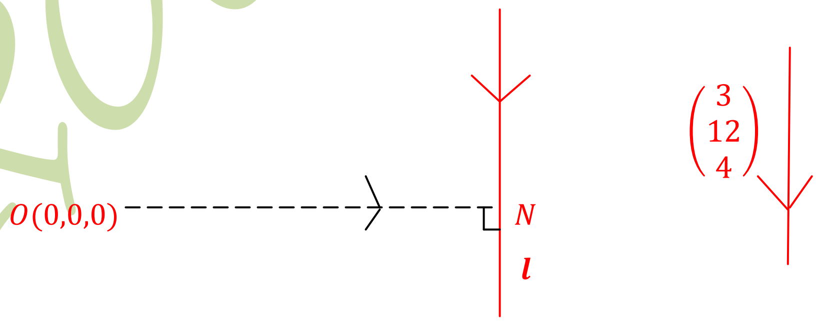
$$\sin\theta = \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix}}{\sqrt{(-1)^2+2^2+2^2}\sqrt{(3)^2+12^2+4^2}}$$

$$\sin\theta = \frac{-3+24+8}{\sqrt{9}\sqrt{169}}$$

$$\sin\theta = \frac{29}{3 \times 13}$$

$$\theta = \sin^{-1}\left(\frac{29}{39}\right) = 48.03811117° = 48°$$

(iii) Let point N be the foot of the perpendicular from O.



The coordinates of N are $(2+3t, 12t, 5+4t)$

Now

$$\boldsymbol{ON} = \begin{pmatrix} 2+3t \\ 12t \\ 5+4t \end{pmatrix}$$

Now

$$\boldsymbol{ON}.\begin{pmatrix}3\\12\\4\end{pmatrix}=0$$

$$\begin{pmatrix}2+3t\\12t\\5+4t\end{pmatrix}.\begin{pmatrix}3\\12\\4\end{pmatrix}=0$$

$$6+9t+144t+20+16t=0$$

$$169t=-26$$

$$\therefore t=\frac{-26}{169}=\frac{-2}{13}$$

Now substituting $t=\frac{-2}{13}$ into $\boldsymbol{ON}$

$$\boldsymbol{ON}=\begin{pmatrix}2+3\left(\frac{-2}{13}\right)\\12\left(\frac{-2}{13}\right)\\5+4\left(\frac{-2}{13}\right)\end{pmatrix}=\begin{pmatrix}\frac{20}{13}\\\frac{-24}{13}\\\frac{57}{13}\end{pmatrix}$$

Therefore the coordinates of the foot of the perpendicular are $\left(\frac{20}{13};\frac{-24}{13};\frac{57}{13}\right)$.

Hence the required distance is:

$$|\boldsymbol{ON}|=\sqrt{\left(\frac{20}{13}\right)^2+\left(\frac{-24}{13}\right)^2+\left(\frac{57}{13}\right)^2}$$

$$=\sqrt{25}$$

$$=5.$$

ZIMSEC JUNE 2017 PAPER 2

The position vectors of points A and B are $3\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and $7\boldsymbol{i}-9\boldsymbol{j}$, respectively.

Line l has equation $\boldsymbol{r}=\begin{pmatrix}2\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and the plane P has equation $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}=8$

(a) Show that

(i) A lies in the plane P, **[2]**

(ii) $\overrightarrow{BA}$ is perpendicular to the plane P. **[3]**

(b) Calculate $O\hat{B}A$ where O is the origin. **[3]**

(c) Find the acute angle between the line l and the plane P. **[3]**

(d) Find the perpendicular distance from point B to plane P. **[2]**

Suggested Solution

a) (i) If A lies on the plane the its coordinates bust satisfy the plane

Substitute the point $A(3,-1,2)$ to the LHS of the plane $2x-4y-z=8$.

Now LHS: $2x-4y-z=2(3)-4(-1)-2=10-2=8\equiv RHS$

Since $LHS\equiv RHS$ $\therefore$ the point $(3,-1,2)$ lies in the plane $2x-4y-z=8$.

Or alternatively

Substitute the point $A(3,-1,2)$ to the LHS of the plane $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}=8$.

Now LHS: $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}=\begin{pmatrix}3\\-1\\2\end{pmatrix}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}=6+4-2=8\equiv RHS$

Since $LHS\equiv RHS$ $\therefore$ the point (3, −1,2 lies in the plane $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}=8$

a) (ii) $\boldsymbol{BA}=\ \boldsymbol{OA}-\boldsymbol{OB}=\ 3\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}-(7\boldsymbol{i}-9\boldsymbol{j})=-4\boldsymbol{i}+8\boldsymbol{j}+2\boldsymbol{k}=\begin{pmatrix}-4\\8\\2\end{pmatrix}=-2\begin{pmatrix}2\\-4\\-1\end{pmatrix}$

Since $\boldsymbol{BA}$ is parallel to the direction vector of the plane P $\therefore$ $\boldsymbol{BA}$ is perpendicular to the plane P.

(b) $\cos(O\hat{B}A)=\dfrac{\boldsymbol{BO}.\boldsymbol{BA}}{|\boldsymbol{BO}||\boldsymbol{BA}|}$

$$\cos(O\hat{B}A)=\frac{\begin{pmatrix}-7\\9\\0\end{pmatrix}\begin{pmatrix}-4\\8\\2\end{pmatrix}}{\sqrt{(-7)^2+(9)^2+0^2}\sqrt{(-4)^2+8^2+2^2}}$$

$$\cos(O\hat{B}A)=\frac{28+72}{\sqrt{130}\sqrt{84}}$$

$$\cos(O\hat{B}A)=\frac{100}{\sqrt{130}\sqrt{84}}$$

$$O\hat{B}A=\cos^{-1}\left(\frac{100}{\sqrt{130}\sqrt{84}}\right)=16.87333855^\circ=17^\circ$$

(c) $\sin\theta = \frac{\boldsymbol{b.n}}{|\boldsymbol{b}||\boldsymbol{n}|}$

$$\sin\theta = \frac{\begin{pmatrix}2\\-1\\2\end{pmatrix}.\begin{pmatrix}2\\-4\\-1\end{pmatrix}}{\sqrt{2^2+(-1)^2+2^2}\sqrt{2^2+(-4)^2+(-1)^2}}$$

$$\sin\theta = \frac{4+4-2}{\sqrt{9}\sqrt{21}}$$

$$\sin\theta = \frac{4+4-2}{\sqrt{9}\sqrt{21}}$$

$$\theta = \sin^{-1}\left(\frac{2}{\sqrt{21}}\right) = 25.87669006° = 26°$$

(d) Finding the distance between $B(7;-9;0)$ and plane $\boldsymbol{r}.\begin{pmatrix}2\\-4\\-1\end{pmatrix} = 8$ or $2x-4y-z=8$

From the Cartesian formula

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}$$

$$d = \frac{(2)(7)+(-9)(-4)+(-1)(0)-8}{\sqrt{(2)^2+(-4)^2+(-1)^2}}$$

$$d = \frac{42}{\sqrt{21}}$$

$$d = \frac{42\sqrt{21}}{21}$$

$$d = 2\sqrt{21}$$

Practice Past Examinations Questions

ZIMSEC JUNE 2007 PAPER 2

The line l has Cartesian equations

$$\frac{x-5}{2}=\frac{y-2}{-1}=\frac{z-2}{3}$$

Find

(i) the length of the perpendicular from the origin to l. **[5]**

(ii) an equation for the plane π which contains l and the origin, giving your answer in the form $\boldsymbol{r}.\boldsymbol{n}=D$. **[4]**

ZIMSEC NOVEMBER 2008 PAPER 2

A plane contains the points $A(1;2;3)$ and $B(4;5;6;)$ and passes through the origin.

(a) Find

(i) the equation of the plane OAB in the form $\boldsymbol{r}.\boldsymbol{n}=d$. **[5]**

(ii) the coordinates of the foot of the perpendicular from point $C(5;1;-4)$ to the line AB. **[4]**

(b) Calculate the angle between the lines AB and OC. **[3]**

(c) Show that the line BC is not contained in the plane OAB. **[3]**

ZIMSEC JUNE 2016 PAPER 2

Given the points $A(0{,}2{,}2), B(4{,}1{,}0)$ and $C(-2{,}0{,}3)$ find the

(i) value of q, given that the Cartesian equation of a perpendicular bisector of a line joining the points A and B, passing through the point $D(1,q,0)$ is

$x-2=\frac{2y-3}{4}=z-1$, **[4]**

(ii) exact shortest distance of the line in part **(i)** to the origin, **[4]**

(iii) Cartesian equation of the plane containing points A, B and C. **[4]**

ZIMSEC NOVEMBER 2017 PAPER 2

Relative to the origin $\boldsymbol{O}$, the position vectors of points $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ c \\ 2 \end{pmatrix}$ respectively

(a) Determine whether the point $\boldsymbol{P}$ lies on l whose equation is $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$, where λ is the parameter. **[2]**

(b) (i) Given that line $\boldsymbol{PQ}$ intersects the line l, find the value of c. **[3]**

(ii) Hence, calculate the angle between the line l and the $\boldsymbol{PQ}$. **[3]**

(c) Find the position vector of the point $\boldsymbol{R}$ on l such that line $\boldsymbol{PR}$ is perpendicular to line l. **[4]**

<u>ZIMSEC JUNE 2018 PAPER 2</u>

(a) A plane has vector equation $\boldsymbol{r}.\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 1$ and the point P has coordinates $(-9; 17; -2)$.

(i) Find the coordinates of the foot of the perpendicular from P to the plane. **[3]**

(ii) Hence, or otherwise, find the coordinates of the image of P when reflected in the plane. **[3]**

(b) The plane π and the line l have equations

$$\boldsymbol{r}.\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 6 \text{ and } \frac{x-2}{1} = \frac{y-3}{2} = \frac{z+1}{-2} \text{ respectively.}$$

Find the

(i) coordinates of the point of intersection between π and l, **[4]**

(ii) angle between π and l, **[3]**

(iii) shortest distance from the origin to l. **[4]**

<u>ZIMSEC 2018 NOVEMBER PAPER 2</u>

(a) The line L_1 has equation

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

and plane π_1 passes through the points A, B and C with coordinates $(2; -1; 3)$, $(4; 2; -5)$ and $(-1; 3; -2)$ respectively.

Find the

(i) Cartesian equation of π_1 , **[5]**

(ii) acute angle between the plane π_1 and the line L_1. **[3]**

(b) The plane π_2 has equation

$$\boldsymbol{r}.\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 6.$$

The line L_2 lies in the plane π_2 and is perpendicular to L_1. The line L_2 passes through the point The line $(4; 2; 1)$.

Find the

(i) vector equation of L_2, **[4]**

(ii) vector equation of the line of intersection of the planes π_1 and π_2. **[4]**

ASANTE SANA

*******THERE IS A LIGHT AT THE END OF EVERY TUNNEL *******

CONSTRUCTIVE COMMENTS ON THE FORM OF THE PRESENTATION, INCLUDING ANY OMISSIONS OR ERRORS, ARE WELCOME.

ENJOY

Nyasha P. Tarakino (Trockers)

+263772978155/+263717267175

ntarakino@gmail.com